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Numerical solution of Hammerstein Fredholm and Volterra integral equations of the second kind using block pulse functions and collocation method

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Abstract

In this work, we present a numerical method for solving nonlinear Fredholm and Volterra integral equations of the second kind which is based on the use of Block Pulse functions(BPfs) and collocation method. Numerical examples show efficiency of the method.

Key words: Hammerstein Fredholm and Volterra integral equations; Block Pulse functions; Collocation method.

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1 Introduction

Integral equations of the Hammerstein type have been one of the most important domains of applications of the ideas and methods of nonlinear functional analysis and in particular of the theory of nonlinear operators of monotone type. Various applied problems arrising in mathematical physics, mechanics and control theory leads to multivalued analogs of the Hammerstein integral equations[11]. In recent years, many different basis functions have been used to solve and reduce integral equations to a system of algebraic equations [1-3] and [6-10] . The aim of this work is to present a numerical method for solving nonlinear Fredholm and Volterra integeral equations of Hammerstein type using BPfs. For this purpose we define a k-set of Block-Pulse functions (BPfs) as:

$$B_i(t) = \begin{cases} 1, & \frac{i-1}{k} \leq t < \frac{i}{k}, \text{ for all } i = 1, 2, \ldots, k \\ 0, & \text{elsewhere} \end{cases} \tag{1.1}$$

The functions $B_r(t)$ are disjoint and orthogonal. That is,

$$B_i(t)B_j(t) = \begin{cases} 0, & i \neq j \\ B_i(t), & i = j \end{cases} \tag{1.2}$$

$$< B_i(t), B_j(t) > = \begin{cases} 0, & i \neq j \\ \frac{1}{k}, & i = j \end{cases} \tag{1.3}$$

2 Function Approximation

A function $u(t)$ defined over the interval $[0, 1)$ may be expanded as:

$$u(t) = \sum_{n=0}^{\infty} u_n B_n(t), \tag{2.1}$$

with $u_n = k < u(t), B_n(t) >$.

In practice, only the first k-term of (4) are considered, where k is a power of 2, that is,

$$u(t) \simeq u_k(t) = \sum_{n=1}^{k} u_n B_n(t), \tag{2.2}$$

with matrix form:

$$u(t) \simeq u_k(t) = \mathbf{u}^t \mathbf{B}(t) \tag{2.3}$$

where, $\mathbf{u} = [u_1, u_1, \ldots, u_k]^t$ and $\mathbf{B}(t) = [B_1(t), B_2(t), \ldots, B_k(t)]^t$. Similarly, $K(x,t) \in L^2[0,1)^2$ may be approximated as:

$$K(x,t) \simeq \sum_{i=1}^{k} \sum_{j=1}^{k} K_{ij} B_i(x) B_j(t)$$

or in matrix form

$$K(x,t) \simeq \mathbf{B}^t(x) \mathbf{K} \mathbf{B}(t) \tag{2.4}$$

where $\mathbf{K} = [K_{ij}]_{1 \le i,j \le k}$ and $K_{ij} = k^2 < B_i(x), < K(x,t), B_j(t) >>$.
From (1) we have
$0 \le t < \frac{1}{k}$ implies that $B_1(t) = 1$ and $B_i(t) = 0$ for $i = 2, \ldots, k$.
$\frac{1}{k} \le t < \frac{2}{k}$ implies that $B_2(t) = 1$ and $B_i(t) = 0$ for $i = 1, \ldots, k$ and $i \neq 2$.
$\vdots$
$\frac{k-1}{k} \le t < 1$ implies that $B_k(t) = 1$ and $B_i(t) = 0$ for $i = 1, \ldots, k-1$.
Using (2) leads to

$$\mathbf{B}(x)\mathbf{B}^t(x) = \begin{bmatrix} B_1(x) & & \emptyset \\ & \vdots & \\ \emptyset & & B_k(x) \end{bmatrix}$$

$$= B_1(x) \begin{bmatrix} 1 & & & \emptyset \\ & 0 & & \\ & & \vdots & \\ \emptyset & & & 0 \end{bmatrix} + B_2(x) \begin{bmatrix} 0 & & & & \emptyset \\ & 1 & & & \\ & & 0 & & \\ & & & \vdots & \\ \emptyset & & & & 0 \end{bmatrix} + \cdots + B_k(x) \begin{bmatrix} 0 & & & & \emptyset \\ & 0 & & & \\ & & \vdots & & \\ & & & 0 & \\ \emptyset & & & & 1 \end{bmatrix}$$

so

$$\int_0^1 \mathbf{B}(x)\mathbf{B}^t(x)dx = \int_0^1 B_1(x)dx \begin{bmatrix} 1 & & \emptyset \\ & 0 & \\ & & \vdots \\ \emptyset & & 0 \end{bmatrix} + \int_0^1 B_2(x)dx \begin{bmatrix} 0 & & & \emptyset \\ & 1 & & \\ & & 0 & \\ & & \vdots & \\ \emptyset & & & 0 \end{bmatrix} + \dots$$

$$+ \int_0^1 B_k(x)dx \begin{bmatrix} 0 & & \emptyset \\ & 0 & \\ & \vdots & \\ & & 0 \\ \emptyset & & 1 \end{bmatrix}$$

$$= \frac{1}{k}\mathbf{I},$$

where, $\mathbf{I}_{k\times k}$ is the identity matrix of order k.

2.1 Nonlinear Fredholm integral equations of Hammerstein type

Now consider the following nonlinear Fredholm integral equation of the second kind of Hammerstein type:

$$u(x) = \int_0^1 K(x,t)\phi[t,u(t)]dt + g(x), \tag{2.5}$$

where, $K \in L^2[0,1)^2$ and $g, \phi \in L^2[0,1)$ are known functions and $u(t)$ is the unknown function to be determined. if we define

$$W(t) = \phi[t,u(t)] \tag{2.6}$$

from (8) we obtain

$$W(t) = \phi[t, \int_0^1 K(t,x)W(x)dx + g(t)]. \tag{2.7}$$

We approximate $W(t)$ as:

$$W(t) \simeq W_k(t) = \sum_{n=1}^{k} w_n B_n(t) \\ = \mathbf{w}^t \mathbf{B}(t) \tag{2.8}$$

where $\mathbf{w} = [w_1, w_2, \ldots, w_k]^t$.

Now from equations (8) and (9) we have:

$$u(x) = \int_0^1 K(x,t)W(t)dt + g(x). \tag{2.9}$$

If we approximate equation (12) by

$$u_k(x) = \int_0^1 K(x,t)W_k(t)dt + g(x) \tag{2.10}$$

we have to approximate $W_k(t)$ as (11). By approximating functions $K(x,t)$ and $W(t)$, as before, in the matrix form we have:

$$K(x,t) \simeq \mathbf{B}^t(x)\mathbf{K}\mathbf{B}(t) \tag{2.11}$$

$$W(t) \simeq \mathbf{w}^t\mathbf{B}(t) \tag{2.12}$$

by substituting the approximations (14) and (15) into (10) we obtain:

$$\mathbf{w}^t\mathbf{B}(t) = \phi[t, \int_0^1 \mathbf{B}^t(t)\mathbf{K}\mathbf{B}(x)\mathbf{B}^t(x)\mathbf{w}dx + g(t)] \\ = \phi[t, \frac{1}{k}\mathbf{B}^t(t)\mathbf{K}\mathbf{w} + g(t)] \tag{2.13}$$

Evaluating (16) at the collocation points $t_j = \frac{j-0.5}{k}$, $j = 1, 2, \ldots, k$, leads to

$$\mathbf{w}^t\mathbf{B}(t_j) = \phi[t_j, \frac{1}{k}\mathbf{B}^t(t_j)\mathbf{K}\mathbf{w} + g(t_j)] \tag{2.14}$$

which is a nonlinear system of algebraic equations. Solving (17) gives column vector $\mathbf{w}$. Therefore from (11) we can approximate $W(t)$ by $W_k(t)$ and from (13) we get desired approximation $u_k(t)$ for $u(t)$.

Now consider the nonlinear Volterra integral equation of the second kind of Hammerstein type:

$$u(x) = \int_0^x K(x,t)\phi[t,u(t)]dt + g(x) \tag{2.15}$$

as before, we let

$$W(t) = \phi[t,u(t)] \tag{2.16}$$

by substituting (19) into (18) we obtain:

$$u(x) = \int_0^x K(x,t)W(t)dt + g(x) \tag{2.17}$$

substituting (20) into (19) leads to

$$W(t) = \phi[t, \int_0^t K(t,x)W(x)dx + g(t)]. \tag{2.18}$$

We approximate equation (20) by

$$u_k(x) = \int_0^x K(x,t)W_k(t)dt + g(x) \tag{2.19}$$

by substituting the approximations (14) and (15) into (21) we obtain:

$$\begin{aligned} \mathbf{w}^t\mathbf{B}(t) &= \phi[t, \int_0^t \mathbf{B}^t(t)\mathbf{K}\mathbf{B}(x)\mathbf{B}^t(x)\mathbf{w}dx + g(t)] \\ &= \phi[t, \mathbf{B}^t(t)\mathbf{K}\mathbf{F}(t)\mathbf{w} + g(t)] \end{aligned} \tag{2.20}$$

where, $\mathbf{F}(t) = \int_0^t \mathbf{B}(x)\mathbf{B}^t(x)dx$. In section 3, we consider evaluation of $\mathbf{F}(t)$ at the collocation points t_j using properties of Block-Pulse functions (BPfs).

3 Evaluation of $\mathbf{F}(t)$ at the collocation points t_j

For this purpose we use the materials mentioned in section 2. So we have

$$\begin{aligned}
\mathbf{F}(t_j) &= \int_0^{\frac{j-0.5}{k}} \mathbf{B}(x)\mathbf{B}^t(x)dx \\
&= \int_0^{\frac{1}{k}} \mathbf{B}(x)\mathbf{B}^t(x)dx + \int_{\frac{1}{k}}^{\frac{2}{k}} \mathbf{B}(x)\mathbf{B}^t(x)dx + \cdots + \int_{\frac{j-2}{k}}^{\frac{j-1}{k}} \mathbf{B}(x)\mathbf{B}^t(x)dx \\
&+ \int_{\frac{j-1}{k}}^{\frac{j-.5}{k}} \mathbf{B}(x)\mathbf{B}^t(x)dx \\
&= \begin{bmatrix} \frac{1}{k} & & \emptyset \\ & 0 & \\ & & \ddots \\ \emptyset & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & \emptyset \\ & \frac{1}{k} & & \\ & & 0 & \\ & & & \ddots \\ \emptyset & & & 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 & & & & \emptyset \\ & 0 & & & \\ & & \ddots & & \\ & & & \frac{1}{k} & \\ & & & & \ddots \\ \emptyset & & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & & & \emptyset \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \ddots & & \\ & & & & \frac{1}{2k} & \\ & & & & & \ddots \\ \emptyset & & & & & 0 \end{bmatrix} \\
&= \frac{1}{k} Diag[1, 1, ..., 1, \tfrac{1}{2}, 0, ..., 0] \\
&= \frac{1}{k}\mathbf{D}^j,
\end{aligned}$$

where, diagonal matrix $\mathbf{D}^j$ is a $k \times k$ matrix with the elements

$$\mathbf{D}^j_{mn} = \begin{cases} 1, & m = n = 1, 2, ..., j-1 \\ \frac{1}{2}, & m = n = j \\ 0 & m = n = j+1, ..., k. \end{cases}$$

Evaluating (23) at the collocation points t_j leads to

$$\begin{aligned}
\mathbf{w}^t\mathbf{B}(t_j) &= \phi[t_j, \mathbf{B}^t(t_j)\mathbf{K}\mathbf{F}(t_j)\mathbf{w} + g(t_j)] \\
&= \phi[t_j, \frac{1}{k}\mathbf{B}^t(t_j)\mathbf{K}\mathbf{D}^j\mathbf{w} + g(t_j)],
\end{aligned} \tag{3.1}$$

Solving nonlinear system of algebraic equations (25) gives column vector $\mathbf{w}$. Therefore from (11) we can approximate $W(t)$ by $W_k(t)$ and from (22) we get desired approximation $u_k(t)$ for $u(t)$.

4 Numerical Examples

Now for implementation the presented method in this paper, consider the numerical examples in the cases nonlinear Fredholm and Volterra integral equations.

4.0.0.1 Example 1:

$$u(x) + \int_0^1 e^{x-2t}[u(t)]^3 dt = e^{x+1}, \quad 0 \leq x < 1,$$

with exact solution $u(x) = e^x$.

4.0.0.2 Example 2:

$$u(x) - \int_0^1 [4tx + \pi x \sin(\pi t)] \frac{1}{u^2(t) + t^2 + 1} dt = \sin(\frac{\pi}{2}x) - 2x \ln 3, \quad 0 \leq x < 1,$$

with exact solution $u(x) = \sin(\frac{\pi}{2}x)$.

4.0.0.3 Example 3:

$$u(x) = 1 + \sin^2 x - \int_0^x 3 \sin(x-t)[u(t)]^2 dt, \quad 0 \leq x < 1,$$

with exact solution $u(x) = \cos x$.

4.0.0.4 Example 4:

$$u(x) = x + \cos x - 1 + \int_0^x \sin[u(t)]dt, \quad 0 \leq x < 1,$$

with exact solution $u(x) = x$.

Table 1 shows the computed error $\|e\| = \|u(x) - u_k(x)\|$ for the examples 1-4 with $k = 32$.

Table 1

t	Example 1	Example 2	Example 3	Example 4
0.1	2×10^{-4}	8×10^{-5}	4×10^{-5}	1×10^{-4}
0.2	9×10^{-3}	9×10^{-4}	3×10^{-4}	5×10^{-4}
0.3	1×10^{-3}	1×10^{-4}	1×10^{-4}	4×10^{-4}
0.4	1×10^{-3}	1×10^{-4}	3×10^{-4}	4×10^{-4}
0.5	1×10^{-3}	2×10^{-4}	8×10^{-4}	6×10^{-4}
0.6	1×10^{-3}	2×10^{-4}	1×10^{-3}	3×10^{-4}
0.7	1×10^{-3}	3×10^{-4}	2×10^{-3}	7×10^{-4}
0.8	1×10^{-3}	3×10^{-4}	2×10^{-3}	7×10^{-4}
0.9	1×10^{-3}	4×10^{-4}	1×10^{-3}	9×10^{-3}

5 Conclusion

In present paper, Block Pulse functions together with the collocation points are applied to solve the nonlinear Fredholm and Volterra integral equations of Hammerstein type. For nonlinear integral equation,Galerkin and collocation methods can be quite expensive to implement. Specially, in the case of collocation method, by substituting (5) into (8) and evaluating new equation at the collocation points $t_j \in [0, 1)$ we obtain

$$\sum_{n=1}^{k} u_n B_n(t_j) = \int_0^1 K(t_j, x)\phi[x, \sum_{n=1}^{k} u_n B_n(x)]dx + g(t_j),$$

for $j = 1, 2, \ldots, k$. In the iterative solution of this system, many integrals will need to be computed, which usually becomes quite expensive. In particular, the integral on the right side will need to re-evaluated with each new iterate. But by definition (9) and substituting (11) into (10), the collocation method for (10) is

$$\sum_{n=1}^{k} w_n B_n(t_j) = \phi[t_j, \sum_{n=1}^{k} w_n \int_0^1 K(t_j, x) B_n(x) dx + g(t_j)],$$

the integral of the right side of latter equation need be evaluated only once, since they are dependent only on the basis, not on the unknowns $\{u_n\}$. Many fewer integrals need be calculated to solve this system. Also example 1 and example 2 are solved in [8] using Petrov-Galerkin method(PGm). Comparing the results shows PGm is more accurate then BPfs method. But, it seems the number of calculations of BPfs method is lower. Also, the benefits of this method are low cost of setting up the equations due to properties of BPfs mentioned in section 1. In addition, the nonlinear system of algebraic equations is sparse. Finally, this method can be easily extended and applied to nonlinear Volterra-Fredholm integral equations. Numerical examples show the accuracy of the presented method. Approximations may be more accurate by using larger k.

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