

## Input congestion, technical inefficiency and output reduction in fuzzy data envelopment analysis

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### Abstract

During the last years, the concept of input congestion and technical inefficiency in data envelopment analysis (DEA), have been investigated by many researchers. The motivation of this paper is to present models which obtain the decreased output value due to input congestion and technical inefficiency. Moreover, we extend the models to estimate input congestion, technical inefficiency and output reduction in fuzzy data envelopment analysis, by using the concept of  $\alpha$  - cut sets.

**Keywords:** Fuzzy DEA, Congestion, Output reduction, Technical inefficiency.

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## 1 Introduction

The first model of data envelopment analysis (DEA) method, was introduced in (1978) by Charnes, Cooper and Rhodes (CCR) [3]. Banker, Charnes and Cooper (BCC) in (1984) introduced a variable return to scale version of the CCR model, namely BCC model [2]. Subsequently other models was exhibited. DEA method has a high ability in measuring relative efficiency of decision making units with multiple inputs and outputs and compare them. Hence, researchers investigate DEA in many different feature, such as input congestion.

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Congestion indicates an economic state where inputs are consumed excessively. This occurs whenever reducing some inputs can increase some outputs without worsening other inputs and outputs. First Färe et al. [12] introduced an implementable form for analyzing congestion. Cooper et al. [10] introduced an alternative DEA approach for congestion and then they studied [6] the management of congestion in the Chinese industry, where they showed how elimination inefficiencies of management could lead to output augmentation without decreasing labors in textiles and automobiles industries. Recently, Jahanshahloo and Khodabakhshi [14] provided a model to determine suitable combination of inputs for improving outputs in DEA with determining input congestion. In 2009, Khodabakhshi [17] proposed a one-model approach to determine input congestion.

Input congestion is one of reasons which cause decrement in output. The other reason of output reduction is managerial inefficiency. In this paper we determine the contribution of each reason in output reduction. Subsequently due to on stream inputs and outputs are imprecise data, we extend models to determine the input congestion and it's undesirable effects on output in scope of fuzzy data.

The rest of the paper is organized as follows: The congestion models is mentioned in section 2. Section 3 is devoted to the models for determining output reduction. The crisp numerical examples is presented in section 4. Section 5 develops the fuzzy congestion and fuzzy output reduction models. Section 6 contains fuzzy numerical example. Section 7 concludes the paper.

## 2 Input congestion models

Suppose that all input and output elements are non-negative deterministic number. Let  $DMU_j$ , ( $j=1,2,\dots,n$ ) be  $n$  decision making units (DMU) that convert  $m$  inputs  $x_{ij}$  ( $i=1,\dots,m$ ) into  $s$  outputs  $y_{rj}$  ( $r=1,\dots,s$ ) and  $DMU_o$  is a under evaluation DMU. One of the basic models used to evaluate efficiency is BCC [2]. This model evaluate  $DMU_o$  as follow:

$$\begin{aligned}
 & \text{Maximize} \quad \phi_o + \epsilon(\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+) \\
 & \text{s.t.} \quad x_{io} = \sum_{j=1}^n x_{ij}\lambda_j + s_i^-, \quad i = 1, \dots, m \\
 & \quad \quad 0 = \sum_{j=1}^n y_{rj}\lambda_j - y_{ro}\phi_o - s_r^+, \quad r = 1, \dots, s \\
 & \quad \quad 1 = \sum_{j=1}^n \lambda_j \\
 & \quad \quad \lambda_j, s_i^-, s_r^+ \geq 0
 \end{aligned} \tag{2.1}$$

Consider, in model (1)  $\epsilon$  is a non-Archimedean element, namely it is not a real number and defined to be smaller than any positive real number. To avoid the assigning a value to  $\epsilon$ , we can solve this model via a two-stage procedure. First, we maximize  $\phi_o$  while ignoring the slacks in the objective. Then, we replace  $\phi_o$  with  $\phi_o^*$  (optimal

value of first stage) and maximize the sum of the slacks. By using the above model, efficiency is defined as follow:

**Definition 1** (Efficiency).  $DMU_o$  is efficient if the following two conditions are satisfied:

- (i)  $\phi_o^* = 1$
- (ii)  $s_i^{-*} = s_r^{+*} = 0 \quad \forall i, r.$

Now we present some definition of needed concepts.

**Definition 2** (Input Congestion). Input congestion occurs whenever the increasing one or more inputs decreases some outputs without improving other inputs or outputs. Conversely, congestion occurs when decreasing some of the inputs increases some outputs without worsening other inputs or outputs.

Congestion is a particularly severe form of technical inefficiency, to clarify its relationship with technical inefficiency, we provide the definition of technical inefficiency as follows.

**Definition 3** (Technical Inefficiency). Technical inefficiency is present when it is possible to improve some inputs or outputs without worsening other inputs or outputs.

Following we present the procedure which has been proposed by Cooper et al. [10] to identify source and amount of input congestion. Inefficiency is a necessary condition for the presence of congestion. Therefore, if  $DMU_o$  found to be inefficient, utilize optimal solution of the model (1),  $(\phi_o^*, \lambda_j^*, s_i^{-*}, s_r^{+*})$ , in following model to determine amount of technical inefficiency:

$$\begin{aligned}
 & \text{Maximize} \quad \sum_{i=1}^m \delta_i^- \\
 & \text{s.t.} \quad x_{io} - s_i^{-*} = \sum_{j=1}^n x_{ij} \lambda_j - \delta_i^-, \quad i = 1, \dots, m \\
 & \quad \phi_o^* y_{ro} + s_r^{+*} = \sum_{j=1}^n y_{rj} \lambda_j, \quad r = 1, \dots, s \\
 & \quad 1 = \sum_{j=1}^n \lambda_j \\
 & \quad \delta_i^- \leq s_i^{-*}, \quad i = 1, \dots, m \\
 & \quad \delta_i^-, \lambda_j \geq 0
 \end{aligned} \tag{2.2}$$

and finally,  $(\delta_i^{+*})$  is used to determine congestion amounts as follows:

$$s_i^{-c*} = s_i^{-*} - \delta_i^{-*}, \quad i = 1, \dots, m \tag{2.3}$$

Cooper et al. [10] replaced the two pervious models by a following single model to determine input congestion:

$$\begin{aligned}
& \text{Maximize } \phi_o + \epsilon(\sum_{r=1}^s s_r^+ - \sum_{i=1}^m s_i^{-c}) \\
& \text{s.t. } x_{io} = \sum_{j=1}^n x_{ij}\lambda_j + s_i^{-c}, \quad i = 1, \dots, m \\
& 0 = \sum_{j=1}^n \lambda_j y_{rj} - \phi_o y_{ro} - s_r^+, \quad r = 1, \dots, s \\
& 1 = \sum_{j=1}^n \lambda_j \\
& \lambda_j, s_i^{-c}, s_r^+ \geq 0
\end{aligned} \tag{2.4}$$

In this approach we solve one model whereas in previous approach we must solve two models. But in the one-model approach the technical inefficiency is not estimated.

### 3 Estimating output reduction

Output reduction can be occurred because of two reasons, input congestion and technical inefficiency. In this section we determine the contributions of each reason in output decrement. Total output reduction due to both reasons can be achieved from model (1) as follow:

$$\Delta_r^* = (y_{ro}\Phi_o^* + s_r^*) - y_{ro}, \quad r = 1, \dots, s \tag{3.1}$$

Now we use the definition 3 to estimate the part of output which has been reduced due to technical inefficiency ( $\Delta_r^{T*}$ ). The model we propose for this purpose is formulated as follow:

$$\begin{aligned}
& \text{Maximize } \sum_{r=1}^s \Delta_r^T \\
& \text{s.t. } x_{io} = \sum_{j=1}^n x_{ij}\lambda_j, \quad i = 1, \dots, m \\
& y_{ro} = \sum_{j=1}^n y_{rj}\lambda_j - \Delta_r^T, \quad r = 1, \dots, s \\
& 1 = \sum_{j=1}^n \lambda_j \\
& \Delta_r^T, \lambda_j \geq 0
\end{aligned} \tag{3.2}$$

and finally output reduction amount due to congesting inputs can achieve by following equation:

$$\Delta_r^{C*} = \Delta_r^* - \Delta_r^{T*}, \quad r = 1, \dots, s \tag{3.3}$$

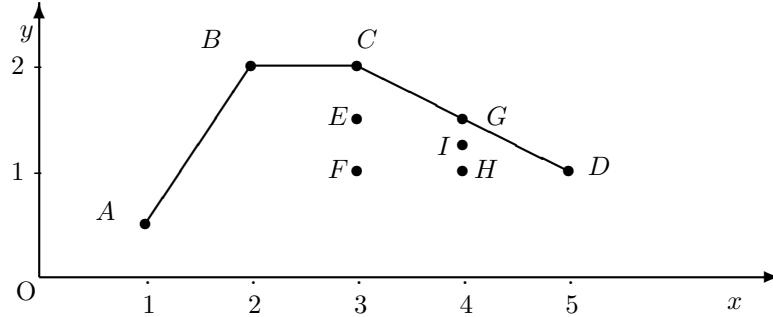


Figure 1: Example 1

Table 1: Numerical result of Example 1

DMU	Input	Output	$\Phi_o^*$	$s_r^{-**}$	$s_r^{+**}$	$\delta_i^{+**}$	$s_i^{-c**}$	$\Delta^*$	$\Delta^{T*}$	$\Delta^{C*}$
A	1	0.5	1	0	0	-	-	0	-	-
B	2	2	1	0	0	-	-	0	-	-
C	3	2	1	1	0	0	1	0	-	-
D	5	1	2	3	0	1	2	1	0	1
E	3	1.5	4/3	1	0	1	0	0.5	0.5	0
F	3	1	2	1	0	1	0	1	1	0
G	4	1.5	4/3	2	0	1	1	0.5	0	0.5
H	4	1	2	2	0	1	1	1	0.5	0.5
I	4	1.2	5/3	2	0	1	1	0.8	0.3	0.5

## 4 Crisp numerical examples

**Example 1.** Consider the following DMUs which depicted in Figure 1. Each DMU consume one input to produce one output that depicted in column 2 and 3 of table 1.

*DMUs* A and B are efficient and for them  $s_r^{+**} = 0$  and  $\Delta^* = 0$ , hence we did not calculate  $\delta_i^{+**}$ ,  $s_i^{-c**}$ ,  $\Delta^{T*}$  and  $\Delta^{C*}$  for these *DMUs*.

Product of *DMU<sub>C</sub>* is same as *DMU<sub>B</sub>* whereas it consumes 1 unit input more than *DMU<sub>B</sub>*. There is no output reduction associated with this excessive input, hence there is only technical inefficiency in C.

The G's output is 0.5 unit less than C's output whereas G consume 1 unit more input. Hence G has both input congestion and technical inefficiency.

H is evidently inefficient because G obtain 0.5 unit output more than G from utilized input by H. Comparing the outputs of G and H shows that G has 0.5 unit output reduction due to technical inefficiency, on the other side it has 0.5 unit output reduction because of input congestion similar to H. Hence H has 1 unit total output reduction due to both reasons. Also, other *DMU* have similar interpretation.

**Example 2.** Now we investigate the empirical example by using the data of Chinese

Table 2: Data of Chinese textile industry example.

Year	Labor	Capital	Output	$\phi_O^*$	$\delta_1^{+*}$	$s_1^{-c*}$	$\delta_2^{+*}$	$s_2^{-c*}$	$\Delta^{T*}$	$\Delta_1^{C*}$
1981	389.00	19.86	856.02	1.00	-	-	-	-	-	-
1982	412.30	21.16	866.85	1.50	-	-	0	0.72	0	430.74
1983	423.50	17.08	956.04	1.00	-	-	-	-	-	-
1984	417.30	18.10	1082.94	1.00	-	-	-	-	-	-
1985	570.00	12.61	1273.20	1.00	-	-	-	-	-	-
1986	600.50	13.45	1230.72	1.34	-	-	-	-	416.85	0
1987	641.10	15.91	1410.66	1.81	-	-	-	-	1147.12	0
1988	715.30	23.72	1728.16	2.77	0	65.39	-	-	1977.55	1080.08
1989	736.00	25.97	2109.57	2.35	0	45.00	0	0.52	0	2844.59
1990	745.00	18.24	2291.08	1.53	0	43.16	-	-	375.25	838.79
1991	756.00	14.40	2533.27	1.00	-	-	-	-	-	-
1992	743.00	17.50	2899.16	1.14	0	30.72	-	-	0	418.64
1993	684.00	25.08	3520.74	1.40	0	1.79	-	-	1386.63	3542.41
1994	691.00	25.45	4949.93	1.00	-	-	-	-	-	-
1995	673.00	29.35	4604.00	1.07	-	-	0	3.98	0	322.28
1996	634.00	23.05	4722.29	1.00	-	-	-	-	-	-
1997	595.00	25.02	4760.28	1.00	-	-	-	-	-	-

textile industry<sup>1</sup> which have been depicted in table 2. Data are labor and capital as inputs, and a single output. The unit of capital and output is a 1 million Ren Min Bi (Chinese monetary unit) and labor is expressed in unit of 1000 persons. Here, we consider every year in this industry as a one DMU as Cooper et al. [10]. Unlike Cooper et al. we use the downright economic approach for the problem and investigate output reduction due to congestion in both capital and labor.

As we see, in 1982 there exist 0.72 unit capital congestion which cause 430.74 unit decrement in its output. Also in 1992, 30.72 unit excessive labor, reduced 418.64 unit of its output. In 1989 there exist both capital and labor congestion which cause 2844.59 units output reduction. Also in 1987 there exist 1147.12 units output reduction due to technical inefficiency.

## 5 Extension the previous models to fuzzy DEA

In many problems data are imprecise and researchers try to express them as interval, ordinal, chancy or fuzzy numbers. The fuzzy sets theory was first introduced by Zadeh [20], and because of its ability to describe imprecise phenomena, was considered by many researchers in different fields. In this section we develop previous models and equations for fuzzy imprecise data. First we recall some concepts of fuzzy theory.

**Definition 4** (fuzzy set). If  $X$  is a collection of objects denoted generically by  $x$ , then a fuzzy set  $\tilde{x}$  in  $X$  is a set of ordered pairs:

$$\tilde{x} = \{(x, \mu_{\tilde{x}}(x)) : x \in X\}$$

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<sup>1</sup>Source of data: Cooper et al.[6]

that,  $\mu_{\tilde{x}}(x)$  is called the membership function which for each  $x \in X$  associates a number in  $[0,1]$ , indicating to what degree  $x$  is a member of  $X$ .

**Definition 5** (nonnegative triangular fuzzy number). A nonnegative triangular fuzzy number  $\tilde{x}$  that showed by triple  $(x^L, x^M, x^R)$  that  $0 \leq x^L \leq x^M \leq x^R$ , is a fuzzy set, such that its membership function is:

$$\mu_{\tilde{x}}(x) = \begin{cases} 0 & x < x^L \\ \frac{x-x^L}{x^M-x^L} & x^L \leq x < x^M \\ 1 & x = x^M \\ \frac{x-x^R}{x^M-x^R} & x^M < x \leq x^R \\ 0 & x > x^R \end{cases}$$

**Definition 6** ( $\alpha$  – cut of fuzzy set  $\tilde{x}$ ). A  $\alpha$ -cut of fuzzy set  $\tilde{x}$  is a crisp subset of  $X$  which, denoted by:

$$\tilde{x}_\alpha = \{x : \mu_{\tilde{x}}(x) \geq \alpha, x \in X\}$$

Possibility theory was formulated in terms of fuzzy set theory by Zadeh. He suggested that fuzzy sets can be used as a basis for the theory of possibility similar to the way that measure theory provides the basis for the theory of probability. He introduced the "fuzzy variable", which is associated with a possibility distribution in the same manner that a random variable is associated with a probability distribution.

**Definition 7** (possibility). If  $X$  be the universal set, a distribution of possibility is a function  $\pi$  from  $X$  to  $[0, 1]$  such that:

Axiom 1:  $\pi(\emptyset)=0$

Axiom 2:  $\pi(X)=1$

Axiom 3:  $\pi(U \cup V)=\max\{\pi(U) , \pi(V)\}$ , for any disjoint subsets  $U$  and  $V$ .

If  $\tilde{x}$  be the fuzzy set in  $X$  the distribution of possibility of  $x$  ( $\pi_x$ ) equals to  $\mu_{\tilde{x}}(x)$  from numerical aspect.

$$\pi_x(u) = \mu_{\tilde{x}}(u), \forall u \in X$$

We showed some of the previous concepts in following figure:

When  $\mu_{\tilde{x}}(x) = 1$  we sure that variable  $x$  is equal to crisp number  $x^M$  and it is impossible that  $x$  is equal to another number.  $\mu_{\tilde{x}}(x) = \alpha < 1$  indicates we do not confident  $x = x^M$  but it may occur, as  $x$  can be equal to any number in interval  $[\alpha X^M + (1 - \alpha)X^L , \alpha X^M + (1 - \alpha)X^R]$ . In this case the possibility of  $x = x^M$  equals to 1 whereas the possibility of,  $x$  is equal to one number of  $[\alpha X^M + (1 - \alpha)X^L , \alpha X^M + (1 - \alpha)X^R]$  lie in  $[\alpha , 1]$  and possibility of other cases is equal to 0.

In this section we develop previous models for fuzzy data using the  $\alpha$  - cut sets. Suppose that  $DMU_j$ , ( $j=1,2,\dots,n$ ) be  $n$  decision making units that convert  $m$  nonnegative fuzzy inputs  $\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^M, x_{ij}^R)$ , ( $i=1,\dots,m$ ) into  $s$  nonnegative fuzzy outputs  $\tilde{y}_{rj} = (y_{ij}^L, y_{ij}^M, y_{ij}^R)$ , ( $r=1,\dots,s$ ).

For each  $\alpha$ , inputs and outputs lie in the below intervals:

$$x_{ij} \in [\alpha x_{ij}^M + (1 - \alpha)x_{ij}^L, \alpha x_{ij}^M + (1 - \alpha)x_{ij}^R], \quad i = 1, 2, \dots, m, j = 1, 2, \dots, n \quad (5.1)$$

$$y_{ij} \in [\alpha y_{ij}^M + (1 - \alpha)y_{ij}^L, \alpha y_{ij}^M + (1 - \alpha)y_{ij}^R], \quad r = 1, 2, \dots, s, j = 1, 2, \dots, n \quad (5.2)$$

Hence we can not determine efficiency status of DMUs deterministically. Here we use the best-worst and worst-best state of DMUs to obtain minimum and maximum of  $\Phi_o$ ,  $\delta_i^+$ ,  $s_i^{-c}$ ,  $\Delta^T$  and  $\Delta^C$  and, then construct one interval for each of them. Best and worst status of input interval are minimum and maximum value of that interval respectively. Best and worst status of output interval are maximum and minimum value of that interval respectively.

In the best-worst state, inputs and outputs of  $DMU_o$  lie in the best status and inputs and outputs of other DMUs lie in the worst status of their intervals as follow:

$$x_{io} := \alpha x_{io}^M + (1 - \alpha)x_{io}^L, \quad y_{io} := \alpha y_{io}^M + (1 - \alpha)y_{io}^R \quad (5.3)$$

$$x_{ij} := \alpha x_{ij}^M + (1 - \alpha)x_{ij}^R, \quad y_{ij} := \alpha y_{ij}^M + (1 - \alpha)y_{ij}^L, \quad j = 1, \dots, n, j \neq o \quad (5.4)$$

In the worst-best state inputs and outputs of  $DMU_o$  lie in the worst status and inputs and outputs of other DMUs lie in the best status of their intervals as follow:

$$x_{io} := \alpha x_{io}^M + (1 - \alpha)x_{io}^R, \quad y_{io} := \alpha y_{io}^M + (1 - \alpha)y_{io}^L \quad (5.5)$$

$$x_{ij} := \alpha x_{ij}^M + (1 - \alpha)x_{ij}^L, \quad y_{ij} := \alpha y_{ij}^M + (1 - \alpha)y_{ij}^R, \quad j = 1, \dots, n, j \neq o \quad (5.6)$$

For each  $\alpha$  - cut of fuzzy data, we use the best-worst and worst-best state of data in model (1), to determine minimum and maximum of  $\Phi_o$ ,  $s_i^-$ ,  $s_r^+$ , as follow:

best-worst state:

$$\text{Maximize } \phi_o + \epsilon(\sum_{i=1}^m s_i^- - \sum_{r=1}^s s_r^+)$$

$$\text{s.t. } \alpha x_{io}^M + (1 - \alpha)x_{io}^L =$$

$$\sum_{j=1, j \neq o}^n (\alpha x_{ij}^M + (1 - \alpha)x_{ij}^R)\lambda_j + (x_{io}^M + (1 - \alpha)x_{io}^L)\lambda_o + s_i^-$$

$$0 = \sum_{j=1, j \neq o}^n (\alpha y_{ij}^M + (1 - \alpha)y_{ij}^L)\lambda_j + (y_{io}^M + (1 - \alpha)y_{io}^R)(\lambda_o - \phi_o) - s_r^+ \quad (5.7)$$

$$1 = \sum_{j=1}^n \lambda_j$$

$$\lambda_j, s_i^-, s_r^+ \geq 0, \quad i = 1, \dots, m, j = 1, 2, \dots, n, r = 1, 2, \dots, s.$$

worst - beststate :



$$\begin{aligned}
& \text{Maximize } \phi_o + \epsilon(\sum_{i=1}^m s_i^- - \sum_{r=1}^s s_r^+) \\
& \text{s.t. } \alpha x_{io}^M + (1 - \alpha)x_{io}^R = \\
& \sum_{j=1, j \neq o}^n (\alpha x_{ij}^M + (1 - \alpha)x_{ij}^L)\lambda_j + (x_{io}^M + (1 - \alpha)x_{io}^R)\lambda_o + s_i^- \\
& 0 = \sum_{j=1, j \neq o}^n (\alpha y_{ij}^M + (1 - \alpha)y_{ij}^R)\lambda_j + (y_{io}^M + (1 - \alpha)y_{io}^L)(\lambda_o - \phi_o) - s_r^+ \\
& 1 = \sum_{j=1}^n \lambda_j \\
& \lambda_j, s_i^-, s_r^+ \geq 0, \quad i = 1, \dots, m, j = 1, 2, \dots, n, r = 1, 2, \dots, s.
\end{aligned} \tag{5.8}$$

We use the result of these models as a minimum and maximum of intervals of  $\Phi_o$ ,  $s_i^-$  and  $s_r^+$ . Hence, while all fuzzy data lie in their  $\alpha$ -cuts, efficiency and slacks belong in the intervals  $(\phi_o^{* \min}, \phi_o^{* \max})$ ,  $(s_i^{-* \min}, s_i^{-* \max})$  and  $(s_r^{+* \min}, s_r^{+* \max})$ .

Note that, When  $\alpha = 1$  is used, we are optimist to the reliability of utilized data in our models. In this level used data are crisp values ( $x_{ij} = x_{ij}^M$  and  $y_{rj} = y_{rj}^M$ ) and (14) and (15) have same results. Whereas if  $\alpha = 0$ , we are pessimist to the trustworthiness of our data.

Now in the same way, we determine technical inefficiency and input congestion intervals in  $\alpha$ -level by (2) and (3) as follow:

best-worst state:

By using result of (14) we determine  $\delta_i^{-*}$  and  $s_i^{-c*}$  in the best-worst state as follow:

$$\begin{aligned}
& \text{Maximize } \sum_{i=1}^m \delta_i^- \\
& \text{s.t. } (\alpha x_{io}^M + (1 - \alpha)x_{io}^L) - s_i^{-*} = \\
& \sum_{j=1, j \neq o}^n (\alpha x_{ij}^M + (1 - \alpha)x_{ij}^R)\lambda_j + (x_{io}^M + (1 - \alpha)x_{io}^L)\lambda_o - \delta_i^- \\
& \phi_o^* y_{ro} + s_r^{+*} = \sum_{j=1, j \neq o}^n (\alpha y_{ij}^M + (1 - \alpha)y_{ij}^L)\lambda_j + (y_{io}^M + (1 - \alpha)y_{io}^R)\lambda_o \\
& 1 = \sum_{j=1}^n \lambda_j \\
& \delta_i^- \leq s_i^{-*} \\
& \lambda_j, \delta_i^- \geq 0, \quad i = 1, \dots, m, j = 1, 2, \dots, n, r = 1, 2, \dots, s.
\end{aligned} \tag{5.9}$$

$$s_i^{-c*} = s_i^{-*} - \delta_i^{-*} \tag{5.10}$$

worst-best state:

By using result of (15) we determine  $\delta_i^{-*}$  and  $s_i^{-c*}$  in the worst-best state as follow:

$$\begin{aligned}
& \text{Maximize } \sum_{i=1}^m \delta_i^- \\
& \text{s.t. } (\alpha x_{io}^M + (1 - \alpha)x_{io}^R) - s_i^{-*} = \\
& \sum_{j=1, j \neq o}^n (\alpha x_{ij}^M + (1 - \alpha)x_{ij}^L)\lambda_j + (x_{io}^M + (1 - \alpha)x_{io}^R)\lambda_o - \delta_i^- \\
& \phi_o^* y_{ro} + s_r^{+*} = \sum_{j=1, j \neq o}^n (\alpha y_{ij}^M + (1 - \alpha)y_{ij}^R)\lambda_j + (y_{io}^M + (1 - \alpha)y_{io}^L)\lambda_o \quad (5.11) \\
& 1 = \sum_{j=1}^n \lambda_j \\
& \delta_i^- \leq s_i^{-*} \\
& \lambda_j, s_i^-, s_r^+ \geq 0, \quad i = 1, \dots, m, j = 1, 2, \dots, n, r = 1, 2, \dots, s.
\end{aligned}$$

$$s_i^{-c*} = s_i^{-*} - \delta_i^{-*} \quad (5.12)$$

We make input congestion and technical inefficiency intervals,  $(s_i^{-c*min}, s_i^{-c*max})$  and  $(\delta_i^{-*min}, \delta_i^{-*max})$ , of  $DMU_o$  at  $\alpha$ -level by result of (16) to (19).

Now we use the result of (14) to (19) for defining efficiency, technical inefficiency and input congestion in fuzzy data envelopment analysis as follow:

**Definition 7** ( $\alpha$ -level efficiency).  $DMU_o$  in  $\alpha$ -level,

- (i) is full efficient, if  $\phi_o^* max = 1$  and  $s_i^{-c* max} = s_r^{+* max} = 0 \forall i, r$ .
- (ii) is full inefficient, if  $\phi_o^* min > 1$  or  $\exists i, s_i^{-c* min} > 0$  or  $\exists r, s_r^{+* min} > 0$ .
- (iii) is partial efficient, if (i) does not hold,  $\phi_o^* min = 1$  and  $s_i^{-c* min} = s_r^{+* min} = 0 \forall i, r$ .

**Definition 8** ( $\alpha$ -level input congestion).  $DMU_o$  in  $\alpha$ -level,

- (i) has full input congestion, if  $\exists i, s_i^{-c* min} > 0$ .
- (ii) has partial input congestion, if (i) does not hold and  $\exists i, s_i^{-c* max} > 0$ .

**Definition 9** ( $\alpha$ -level technical inefficiency).  $DMU_o$  in  $\alpha$ -level,

- (i) has full technical inefficiency in its input, if  $\exists i, \delta_i^{-* min} > 0$ .
- (ii) has partial technical inefficiency in its input if (i) does not hold and  $\exists i, \delta_i^{-* max} > 0$ .

Full efficiency in  $\alpha$ -level means that, in this level,  $DMU_o$  is efficient under any condition. In other words,  $DMU_o$  is efficient for any  $x_{ij}$  and  $y_{ij}$  of their  $\alpha$ -cuts intervals. Partial efficiency indicates  $DMU_o$  is not efficient for all numbers in  $\alpha$ -cuts of data, rather it is efficient for some  $x_{ij}$  and  $y_{ij}$  of  $\alpha$ -cut intervals.  $\alpha$ -level full

inefficiency shows that  $DMU_o$  is not efficient for any number in  $\alpha$  – cut intervals of data. Also other definitions have similar interpretation.

Now we extend the (5), (6) and (7) to determine source and amount of output reduction in fuzzy DEA.

best-worst state:

From (14) we can determine total output reduction in the best-worst status as follow:

$$\Delta_r^* = (y_{ro}\Phi_o^* + s_r^*) - y_{ro}, \quad r = 1, \dots, s \quad (5.13)$$

if  $\Delta_r^* > 0$  we determine output reduction due to technical inefficiency and input congestion in the best-worst status as follow:

$$\begin{aligned} & \text{Maximize} \quad \sum_{r=1}^s \Delta_r^T \\ & \text{s.t} \quad (\alpha x_{io}^M + (1 - \alpha)x_{io}^L) = \\ & \quad \text{sum}_{j=1, j \neq o}^n (\alpha x_{ij}^M + (1 - \alpha)x_{ij}^R)\lambda_j + (x_{io}^M + (1 - \alpha)x_{io}^L)\lambda_o \\ & \quad (\alpha y_{io}^M + (1 - \alpha)y_{io}^R) = \sum_{j=1, j \neq o}^n (\alpha y_{ij}^M + (1 - \alpha)y_{ij}^L)\lambda_j + (y_{io}^M + (1 - \alpha)y_{io}^R)\lambda_o - \Delta_r^T \\ & \quad 1 = \sum_{j=1}^n \lambda_j \\ & \quad \lambda_j, \Delta_r^T \geq 0, \quad i = 1, \dots, m, \quad j = 1, 2, \dots, n, \quad r = 1, 2, \dots, s. \end{aligned} \quad (5.14)$$

$$\Delta_r^{C*} = \Delta_r^* - \Delta_r^{T*}, \quad r = 1, \dots, s \quad (5.15)$$

worst-best state:

Similarly, From (15) we can determine total output reduction in the worst-best status as follow:

$$\Delta_r^* = (y_{ro}\Phi_o^* + s_r^*) - y_{ro}, \quad r = 1, \dots, s \quad (5.16)$$

if  $\Delta_r^* > 0$ , in the same way, we determine output reduction due to technical inefficiency

and input congestion in worst-best status as follow:

$$\begin{aligned}
& \text{Maximize } \sum_{r=1}^r \Delta_r^T \\
& \text{s.t. } (\alpha x_{io}^M + (1 - \alpha)x_{io}^R) = \\
& \sum_{j=1, j \neq o}^n (\alpha x_{ij}^M + (1 - \alpha)x_{ij}^L)\lambda_j + (x_{io}^M + (1 - \alpha)x_{io}^R)\lambda_o \\
& (\alpha y_{io}^M + (1 - \alpha)y_{io}^L) = \sum_{j=1, j \neq o}^n (\alpha y_{ij}^M + (1 - \alpha)y_{ij}^R)\lambda_j + (y_{io}^M + (1 - \alpha)y_{io}^L)\lambda_o - \Delta_r^T \\
& 1 = \sum_{j=1}^n \lambda_j \\
& \lambda_j, \Delta_r^T \geq 0, \quad i = 1, \dots, m, \quad j = 1, 2, \dots, n, \quad r = 1, 2, \dots, s.
\end{aligned} \tag{5.17}$$

$$\Delta_r^{C*} = \Delta_r^* - \Delta_r^{T*}, \quad r = 1, \dots, s \tag{5.18}$$

Hence intervals of output reduction due to technical inefficiency and input congestion in  $\alpha$ -level,  $(\Delta_r^{T*min}, \Delta_r^{T*max})$  and  $(\Delta_r^{C*min}, \Delta_r^{*max})$ , can be constructed by (21,24) and (22,25) respectively.

## 6 Fuzzy numerical example

**Example 3.** In this section we present numerical example with nonnegative triangular fuzzy data as shown in table 3.

Table 3: Data of fuzzy numerical example.

DMU	A	B	C	D	E
Input	(9,10,11)	(19,20,20)	(28,30,32)	(48,50,54)	(34,35,36)
output	(3,5,6)	(18,20,21)	(28,30,34)	(14,15,17)	(21,22,23)

We use (14) to (25) to determine intervals of efficiency, technical inefficiency, input congestion and output reduction for each DMU in five  $\alpha$ -cut of fuzzy data. In table 4 the left and right element of each parenthesis shows the minimum and maximum value of corresponding variable respectively.

In DMUs A and C in all levels,  $\phi_o^{*max} = 1$  and  $s_i^{-c*max} = s_r^{+*max} = 0$ , thus these DMUs are full efficient (F.E) in all levels.

$DMU_D$  in all levels, has  $\phi_D^{*min} > 1$  hence D is full inefficient (F.I) always.

$DMU_B$  in all levels has  $s_i^{-c*max} = s_r^{+*max} = 0$ , but only when, possibility of reliability of used data, is 1 or more than 0.75,  $\phi_B^{*max} = 1$ . Hence only in this two levels B is full efficient (F.E).  $\phi_B^{*max} > 1$  and  $\phi_B^{*min} = 1$  for B in levels  $\alpha = 0, 0.25$  and 0.5 hence in this levels it is a partial efficient (P.E). Note that in level  $\alpha = 0.5$ ,  $\phi_B^*$  changes between 1 and 1.04, namely in some times it is efficient and in others it

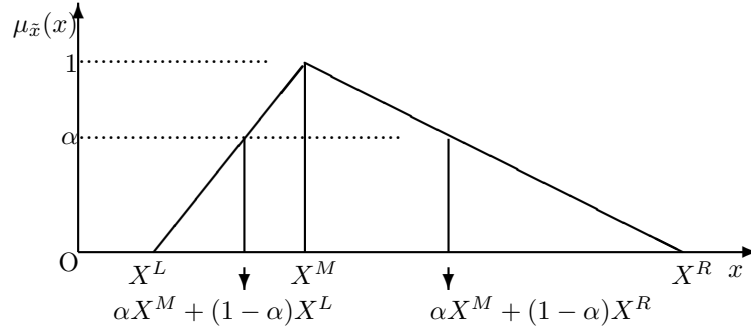


Figure 2: Nonnegative Triangular fuzzy number

Table 4: Efficiency, input congestion and output reduction in different levels.

DMU	$\alpha$	$\phi_\alpha^*$	$\delta_i^{-*}$	$s_i^{-c*}$	$\Delta^{*T}$	$\Delta^{*c}$	Status
A	0	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
	0.25	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
	0.5	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
	0.75	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
	1	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
B	0	(1, 1.23)	(0,0)	(0,0)	(0, 4.21)	(0,0)	P.E
	0.25	(1, 1.13)	(0,0)	(0,0)	(0, 2.41)	(0,0)	P.E
	0.5	(1, 1.04)	(0,0)	(0,0)	(0, 1.16)	(0,0)	P.E
	0.75	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
	1	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
C	0	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
	0.25	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
	0.5	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
	0.75	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
	1	(1, 1)	(0,0)	(0,0)	(0,0)	(0,0)	F.E
D	0	(1.65, 2.43)	(0,0)	(16.0, 26.0)	(0,0)	(11, 20.00)	F.I
	0.25	(1.73, 2.32)	(0,0)	(17.0, 24.5)	(0,0)	(12, 18.75)	F.I
	0.5	(1.81, 2.21)	(0,0)	(18.0, 23.0)	(0,0)	(13, 17.50)	F.I
	0.75	(1.90, 2.10)	(0,0)	(19.0, 21.5)	(0,0)	(14, 16.25)	F.I
	1	(2.00, 2.00)	(0,0)	(20.0, 20.0)	(0,0)	(15, 15.00)	F.I
E	0	(1.22, 1.62)	(0,0)	(2.00, 8.00)	(3.72, 7.38)	(1.34, 5.64)	F.I
	0.25	(1.25, 1.55)	(0,0)	(2.75, 7.25)	(3.93, 6.24)	(1.76, 5.45)	F.I
	0.5	(1.29, 1.49)	(0,0)	(3.50, 6.50)	(4.08, 5.30)	(2.45, 5.24)	F.I
	0.75	(1.33, 1.43)	(0,0)	(4.25, 5.75)	(4.19, 4.79)	(3.15, 4.56)	F.I
	1	(1.36, 1.36)	(0,0)	(5.00, 5.00)	(4.25, 4.25)	(3.67, 3.67)	F.I

is inefficient, therefore it is called partial efficient. Whereas in levels  $\alpha = 0.75$  and 1,  $\phi_B^*$  equals to constant value 1, hence in these levels, B is called full efficient.

For  $DMU_D$ , we find  $\Delta_r^{*min} > 0$  in all level, hence we determined source and amounts of output reduction of this  $DMU$ . It has full input congestion in all levels. This means that in each level and under any condition  $DMU_D$  has input congestion.

As we see in table, the range of intervals is extending when the reliability of used data ( $\alpha$ ) is decreasing. We showed this subject for  $s_i^{-c*}$ ,  $\Delta_r^{*T}$  and  $\phi_D^*$  of  $DMU_D$  by following figures. These figures show that the result of our models are fuzzy numbers. For example, Figure 4 shows that, it is a possible that  $\Delta^{c*}$  is any number of interval (11 20), but when possibility of reliability of used data is more than 0.5,  $\Delta^{c*}$  lie between 13 and 17.5. In other word the possibility of  $13 \leq \Delta^{c*} \leq 17.5$  is more than 0.5. Also possibility of  $\Delta^{c*} = 15$  equal to 1. Whereas  $\Delta^{c*} > 20$  or  $\Delta^{c*} < 11$  is impossible. Note that,  $s_i^{-c*}$  is triangular fuzzy number similar to  $\Delta^{c*}$ . Also  $\phi^*$  is a fuzzy number but has not linear diagram.

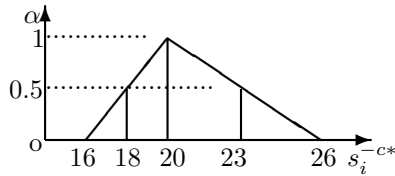


Figure 3: Input congestion

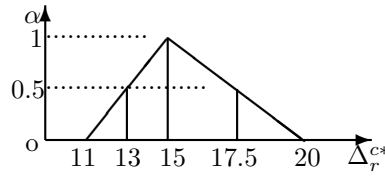
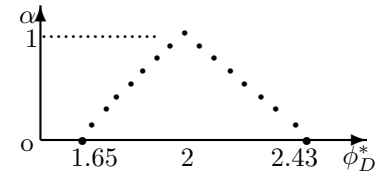
Figure 4:  $\Delta_r^{c*}$ 

Figure 5: Efficiency

## 7 Conclusion

In this paper we studied input congestion and technical inefficiency and their undesirable effects on outputs. First we investigated these phenomena in some DEA models with deterministic data. Then, considering, on stream inputs and outputs are imprecise, we extend models to determine source and amount of output reduction in fuzzy DEA. Decision makers by using achieved information of these models, can adopt better decision about congesting inputs and managerial inefficiencies and then exterminate them. The concept of output reduction and its reasons in fuzzy data envelopment analysis can be reinvestigated with other approaches, in future researches.

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