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A Numerical Approach for Solving Forth Order Fuzzy Differential Equations Under Generalized Differentiability

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Abstract

In this paper a numerical method for solving forth order fuzzy differential equations under generalized differentiability is proposed. This method is based on the interpolating a solution by piecewise polynomial of degree 8 in the range of solution . We investigate the existence and uniqueness of solutions. Finally a numerical example is presented to illustrate the accuracy of the new technique.

Key words: Fuzzy differential equations, Numerical Method, Generalized differentiability

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1 Introduction

Fuzzy differential equations are a suitable tool to model problem in science and engineering. There are many idea to define a fuzzy derivative and in consequence, to study fuzzy differential equations. The first and most popular approach is using the Hukuhara differentiability for fuzzy valued function. Hukuhara differentiability has the drawback that the solution of fuzzy differential equations need to have increasing length of its support, so in order to overcome this weakness, Bede and Gal [12], introduced the strongly generalized differentiability of fuzzy valued function. This concept allows us to solve the above-mentioned shortcoming, also the strongly generalized derivative is defined for a larger class of fuzzy valued functions than the Hukuhara derivatives.

Higher-order fuzzy differential equations under generalized differentiability is presented by Khastan in [23]. Khastan proposed a analytic method to solve higher-order fuzzy differential equations based on the selection different type of derivatives, they obtained several solution to fuzzy initial value problem. In this paper a numerical method for forth order fuzzy differential equations is presented. The idea of this method is based on interpolating the solution by polynomial of degree 8 in the range of solution, the step size used is of length $H = 4h$. Also existence and uniqueness of the solutions are proved.

The paper is organized as follows: In section 2, some basic definitions are brought. A new method for solving forth order fuzzy differential equations, also the existence and uniqueness are introduced in section 3 and 4 . A numerical example is presented in section 5 and finally conclusion is drawn.

2 Notation and definitions

First notations which shall be used in this paper are introduced.
We denote by $\mathbb{R}_{\mathcal{F}}$, the set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which

are defined over the real line.
For $0 < r \le 1$, set $[u]^r = \left\{ t \in \mathbb{R} \middle| u(t) \ge r \right\}$, and $[u]^0 = cl\left\{ t \in \mathbb{R} \middle| u(t) > 0 \right\}$.
We represent $[u]^r = [u^-(r), u^+(r)]$, so if $u \in \mathbb{R}_{\mathcal{F}}$, the r-level set $[u]^r$ is a closed interval for all $r \in [0,\ 1]$. For arbitrary $u,\ v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$, the addition and scalar multiplication are defined by $[u+v]^r = [u]^r + [v]^r$, $[ku]^r = k[u]^r$ respectively.

A triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_{\mathcal{F}}$, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \le b \le c$ such that $u^-(r) = a + (b-a)r$ and $u^+(r) = c - (c-b)r$ are the endpoints of r-level sets for all $r \in [0, 1]$.

Definition 2.1 *([19]) The Hausdorff distance between fuzzy numbers is given by* $D : \mathbb{R}_{\mathcal{F}} \times, \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$,

$$D(u,\ v) = \sup_{r \in [0,\ 1]} \max \left\{ |u^-(r) - v^-(r)|,\ |u^+(r) - v^+(r)| \right\}. \tag{2.1}$$

Consider $u, v, w, z \in \mathbb{R}_{\mathcal{F}}$ *and* $\lambda \in \mathbb{R}$*, then the following properties are well-known for metric* D*,*

1. $D(u \oplus w,\ v \oplus w) = D(u,\ v)$, *for all* $u, v, w \in \mathbb{R}_{\mathcal{F}}$,
2. $D(\lambda u,\ \lambda v) = |\lambda| D(u,\ v)$, *for all* $u, v \in \mathbb{R}_{\mathcal{F}}$, $\lambda \in \mathbb{R}$
3. $D(u \oplus v,\ w \oplus z) \le D(u,\ w) + D(v,\ z)$, *for all* $u, v, w, z \in \mathbb{R}_{\mathcal{F}}$,
4. $D(u \ominus v,\ w \ominus z) \le D(u,\ w) + D(v,\ z)$, *as long as* $u \ominus v$ *and* $w \ominus z$ *exist, where* $u,\ v,\ w,\ z \in \mathbb{R}_{\mathcal{F}}$.

where, $\ominus$ *is the Hukuhara difference(H-difference), it means that* $w \ominus v = u$ *if and only if* $u \oplus v = w$.

Definition 2.2 *([12]) Let* $u, v \in \mathbb{R}_{\mathcal{F}}$*. If there exists* $w \in \mathbb{R}_{\mathcal{F}}$ *such that*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i) \quad u = v + w, \\ \qquad or \\ (ii) \quad v = u + (-1)w, \end{cases}$$

Then w is called the generalized Hukuhara difference of u and v.

Remark 2.1 *Throughout the rest of this paper, we assume that $u \ominus_{gH} v \in \mathbb{R}_{\mathcal{F}}$.*

Note that a function $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ is called fuzzy-valued function. The r-level representation of this function is given by $f(t;r) = [f^-(t;r)\ ,\ f^+(t;r)]$, for all $t \in [a,b]$ and $r \in [0,1]$.

Definition 2.3 *([15]) A fuzzy valued function $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is said to be continuous at $t_0 \in [a,b]$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $D(f(t), f(t_0)) < \epsilon$, whenever $t \in [a,b]$ and $|t - t_0| < \delta$. We say that f is fuzzy continuous on $[a,b]$ if f is continuous at each $t_0 \in [a,\ b]$.*

Definition 2.4 *([15]) The generalized Hukuhara derivative of the fuzzy-valued function $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ at $t_0 \in (a,b)$ is defined as*

$$f'_{gH}(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h}. \tag{2.2}$$

If $f'_{gH}(t_0) \in \mathbb{R}_{\mathcal{F}}$ satisfying (2.2) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at t_0.

Definition 2.5 *([15]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ and $t_0 \in (a,b)$, with $f^-(t;r)$ and $f^+(t;r)$ both differentiable at t_0 for all $r \in [0,1]$. We say that*

- *f is $[(i) - gH]$-differentiable at t_0 if*

$$f'_{i.gH}(t_0;r) = [(f^-)'(t_0;r)\ ,\ (f^+)'(t_0;r)], \tag{2.3}$$

- *f is $[(ii) - gH]$-differentiable at t_0 if*

$$f'_{ii.gH}(t_0;r) = [(f^+)'(t_0;r)\ ,\ (f^-)'(t_0;r)]. \tag{2.4}$$

Definition 2.6 *([15]) We say that a point $t_0 \in (a,b)$, is a switching point for the differentiability of f, if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$ such that*

type(I) *at t_1 (2.3) holds while (2.4) does not hold and at t_2 (2.4) holds and (2.3) does not hold, or*

type(II) *at t_1 (2.4) holds while (2.3) does not hold and at t_2 (2.3) holds and (2.4) does not hold.*

Theorem 2.1 *[9] Let $T = [a, a+\beta] \subset \mathbb{R}$, with $\beta > 0$ and $f \in \mathcal{C}^n_{gH}([a,b], \mathbb{R}_{\mathcal{F}})$. For $s \in T$*

(i) *If $f^{(i)}, \ i = 0, 1, \ldots, n-1$ are $[(i) - gH]$-differentiable, provided that type of gH-differentiability has no change. Then*

$$f(s) = f(a) \oplus f'_{i.gH}(a) \odot (s-a) \oplus f''_{i.gH}(a) \odot \frac{(s-a)^2}{2!} \oplus \ldots \oplus f^{(n-1)}_{i.gH}(a) \odot \frac{(s-a)^{n-1}}{(n-1)!} \oplus R_n(a,s),$$

where

$$R_n(a,s) := \int_a^s \left(\int_a^{s_1} \ldots \left(\int_a^{s_{n-1}} f^{(n)}_{i.gH}(s_n) ds_n \right) ds_{n-1} \ldots \right) ds_1.$$

(ii) *If $f^{(i)}, \ i = 0, 1, \ldots, n-1$ is $[(ii) - gH]$-differentiable, provided that type of gH-differentiability has no change. Then*

$$f(s) = f(a) \ominus (-1) f'_{ii.gH}(a) \odot (s-a) \ominus (-1) f''_{ii.gH}(a) \odot \frac{(a-s)^2}{2!} \ominus (-1) \ldots \ominus (-1) f^{(n-1)}_{ii.gH}(a) \odot \frac{(a-s)^{n-1}}{(n-1)!} \ominus (-1) R_n(a,s),$$

where

$$R_n(a,s) := \int_a^s \left(\int_a^{s_1} \ldots \left(\int_a^{s_{n-1}} f^{(n)}_{ii.gH}(s_n) ds_n \right) ds_{n-1} \ldots \right) ds_1.$$

(iii) *If $f^{(i)}$ are $[(i) - gH]$-differentiable for $i = 2k-1, \ k \in \mathbb{N}$, and $f^{(i)}$ are $[(ii) - gH]$-differentiable for $i = 2k, \ k \in \mathbb{N} \cup \{0\}$. Then*

$$
\begin{aligned}
f(s) &= f(a) \ominus (-1) f'_{ii.gH}(a) \odot (s-a) \oplus f''_{i.gH}(a) \\
&\odot \frac{(a-s)^2}{2!} \ominus (-1) \ldots \ominus (-1) f^{(\frac{i-1}{2})}_{ii.gH}(a) \odot \frac{(a-s)^{\frac{i}{2}-1}}{(\frac{i}{2}-1)!} \\
&\oplus f^{(\frac{i}{2})}_{i.gH}(a) \odot \frac{(a-s)^{\frac{i}{2}}}{(\frac{i}{2})!} \ominus (-1) \ldots \ominus (-1) R_n(a,s),
\end{aligned}
$$

where

$$
\begin{aligned}
&R_n(a,s) := \\
&\quad \int_a^s \left(\int_a^{s_1} \ldots \left(\int_a^{s_{n-1}} f^{(n)}_{i.gH}(s_n) ds_n \right) ds_{n-1} \ldots \right) ds_1.
\end{aligned}
$$

(iv) *Suppose that* $f \in \mathcal{C}^n_{gH}([a,b],\ \mathbb{R}_{\mathcal{F}})$ *,* $n \geq 3$.

Furthermore let f *in* $[a,\xi]$ *is* $[(i)-gH]$*-differentiable and in* $[\xi,b]$ *is* $[(ii)-gH]$*-differentiable, in fact* ξ *is switching point type I for first order derivative of* f *and* $t_0 \in [a,\xi]$ *in a neighborhood of* ξ*. Moreover suppose that second order derivative of* f *in* ζ_1 *of* $[t_0,\xi]$ *have switching point type* II*. Moreover type of differentiability for* $f^{(i)}$*,* $i \leq n$ *on* $[\xi,\ b]$ *don't change. So*

$$
\begin{aligned}
f(s) &= f(t_0) \oplus f'_{i.gH}(t_0) \odot (\xi - t_0) \ominus f''_{ii.gH}(t_0) \\
&\odot (t_0 - \zeta_1) \odot (\xi - t_0) \oplus f''_{i.gH}(\zeta_1) \\
&\odot \left(\frac{(\xi-\zeta_1)^2}{2} - \frac{(t_0-\zeta_1)^2}{2} \right) \ominus (-1) f'_{ii.gH}(\xi) \\
&\odot (s-\xi) \ominus (-1) f''_{ii.gH}(\xi) \odot \frac{(s-\xi)^2}{2!} \\
&\ominus (-1) \int_{t_0}^{\xi} \left(\int_{t_0}^{\zeta_1} \left(\int_{t_0}^{s_2} f'''_{ii.gH}(s_4) ds_4 \right) ds_2 \right) ds_1 \\
&\oplus \int_{t_0}^{\xi} \left(\int_{\zeta_1}^{s_1} \left(\int_{\zeta_1}^{s_3} f'''_{i.gH}(s_5) ds_5 \right) ds_3 \right) ds_1 \\
&\ominus (-1) \int_{\xi}^{s} \left(\int_{\xi}^{t_1} \left(\int_{t_0}^{t_2} f'''_{ii.gH}(t_3) dt_3 \right) dt_2 \right) dt_1.
\end{aligned}
$$

3 Proposed Method

Consider the following forth order fuzzy differential equation

$$\begin{cases} y^{(4)}(t) = f(t, y(t)), & t \in I = [0, T], \\ y(0) = y_0, y'(0) = y'_0, y''(0) = y''_0, y^{(3)}(0) = y_0^{(3)}, \end{cases} \tag{3.1}$$

where the derivative $y^{(i)}$, $i = 1, 2, 3, 4$, is considered in the sense of gH-differentiability. The interval I may be $[0, T]$ for some $T > 0$ or $I = [0, \infty)$. We assume that $f : I \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is sufficiently smooth function, and there exists $k > 0$ such that

$$D(f(t, x), f(t, z)) \leq kD(x, z) \;\forall t \in I,\; x, z \in \mathbb{R}_{\mathcal{F}}. \tag{3.2}$$

Our construction of the fuzzy approximate solution $s(t)$ is as follows: let $y(t)$ be the fuzzy solution of (3.1), we divided the range of solution $[0, T]$ into sub-intervals of equal length $H = 4h = \frac{T}{n}$, and let $I_k = [kH, (k + 1)H]$, where $k = 0, \cdots, n - 1$. In this paper we approximate fuzzy solution of (3.1) by fuzzy piecewise polynomial of order 8. Piecewise approximation solution $s(t)$ on $I_k = [kH, (k + 1)H]$, is construct step by step as follows:

Step One: We define the first component of $s(t)$ by $s_0(t)$, in two cases:

Case(i): Let us suppose that the unique solution of problem (3.1), $y^{(i)}(t)$ are $[(i)-gH]$-differentiable, therefore $s_0(t)$, where in this cases is called $s_{0,1}(t)$ for $0 \leq t \leq H$ is as following

$$s_{0,1}(t) = \sum_{i=0}^{4} y_{i.gH}^{(i)}(0) \odot \frac{t^i}{i!} \oplus \sum_{i=5}^{8} \alpha_{i,0} \odot \frac{t^i}{i!}, \quad 0 \leq t \leq H, \tag{3.3}$$

Case(ii): Now, consider $y^{(i)}(t)$ are $[(ii)-gH]$-differentiable, then $s_0(t)$ that is called $s_{0,2}(t)$ obtained for $0 \leq t \leq H$ as follows:

$$s_{0,2}(t) = y(0) \ominus (-1) \sum_{i=1}^{4} y_{ii.gH}^{(i)}(0) \odot \frac{t^i}{i!} \oplus \sum_{i=5}^{8} \alpha_{i,0} \odot \frac{t^i}{i!}, \tag{3.4}$$

Case(iii): Now, consider $y^{(i)}(t)$ are $[(ii)-gH]$-differentiable for $i=0,2$, and $y^{(i)}(t)$ are $[(i)-gH]$-differentiable for $i=1,3$, then in this case $s_0(t)$, that is called $s_{0,3}(t)$ is obtained for $0 \leq t \leq H$ as follows:

$$\begin{aligned} s_{0,3}(t) = {} & y(0) \ominus (-1)y'_{ii.gH}(0) \odot t \oplus y''_{i.gH}(0) \odot \frac{t^2}{2} \\ & \ominus (-1)y^{(3)}_{ii.gH}(0) \odot \frac{t^3}{3!} \oplus y^{(4)}_{i.gH}(0) \odot \frac{t^4}{4!} \\ & \oplus \sum_{i=5}^{8} \alpha_{i,0} \odot \frac{t^i}{i!}, \end{aligned} \tag{3.5}$$

In Eqs (3.3),(3.4) and (3.5), the coefficients $\alpha_{i,0}$ for $i=5,6,7,8$ as yet undetermined and to be obtained where $s_0(t)$ satisfy the relations:

$$s_0^{(4)}(jh) = f(jh, s_0(jh)), \tag{3.6}$$

for $j=1,2,3,4$. By using Hausdorff distance(2.1), for $j=1,2,3$, we obtain:

$$\begin{aligned} (s_0^-)^{(4)}(jh,r) &= f^-(jh, s_0(jh,r)), \\ (s_0^+)^{(4)}(jh,r) &= f^+(jh, s_0(jh,r)) \end{aligned} \tag{3.7}$$

by solving system(3.7), the value of $\alpha_{i,0}$ for $i=5,6,7,8$ are obtained and $s_0(t)$ is constructed.

Step Two: The approximate solution $s(t)$ in interval $[kH,(k+1)H]$ for $k=1,\cdots,n-1$ is obtained as follows:

$$s(t) = \sum_{i=0}^{4} s_{4k}^{(i)}(t) \odot \frac{(t-4kh)^i}{i!} \oplus \sum_{i=5}^{8} \alpha_{i,k} \odot \frac{(t-4kh)^i}{i!}, \tag{3.8}$$

where $s_0(t)$ is obtained by step 1. The value of $\alpha_{i,k}$ are to be determined so that $s(t)$ satisfy the relations:

$$s^{(4)}(jh) = f(jh, s(jh)). \tag{3.9}$$

This means for $j=4k+1, 4k+2, 4k+3, 4k+4$; $k=1,\cdots,n-1$,

$$(s^-)^{(4)}(jh,r) = f^-(jh, s(jh,r)), \qquad (3.10)$$
$$(s^+)^{(4)}(jh,r) = f^+(jh, s(jh,r)),$$

by solving system (3.10), the values of $\alpha_{i,k}$ are obtained.

Therefore the approximate solution is obtained as follows

$$s(t) = \begin{cases} s_{0,1}(t) & 0 \leq t \leq H, \\ \sum_{i=0}^{4} s_{4k}^{(i)}(t) \odot \frac{(t-4kh)^i}{i!} \oplus \sum_{i=5}^{8} \alpha_{i,k} \odot \frac{(t-4kh)^i}{i!}, & kH \leq t \leq (k+1)H \end{cases} \qquad (3.11)$$

if $y(t)$ is $[(i) - gH]$-differentiable and

$$s(t) = \begin{cases} s_{0,2}(t) & 0 \leq t \leq H, \\ \sum_{i=0}^{4} s_{4k}^{(i)}(t) \odot \frac{(t-4kh)^i}{i!} \oplus \sum_{i=5}^{8} \alpha_{i,k} \odot \frac{(t-4kh)^i}{i!}, & kH \leq t \leq (k+1)H \end{cases} \qquad (3.12)$$

if $y(t)$ is $[(ii) - gH]$-differentiable, and

$$s(t) = \begin{cases} s_{0,3}(t) & 0 \leq t \leq H, \\ \sum_{i=0}^{4} s_{4k}^{(i)}(t) \odot \frac{(t-4kh)^i}{i!} \oplus \sum_{i=5}^{8} \alpha_{i,k} \odot \frac{(t-4kh)^i}{i!}, & kH \leq t \leq (k+1)H \end{cases} \qquad (3.13)$$

if $y^{(i)}(t)$ are $[(ii)-gH]$-differentiable for $i = 0, 2$, and $y^{(i)}(t)$ are $[(i)-gH]$-differentiable for $i = 1, 3$.

4 Existence and uniqueness

In this section we prove that there exist a unique piecewise approximation solution $s(t)$ where approximating the solution of forth order fuzzy differential equation (3.1), provided that the size of the subinterval h satisfies some constraints.

Theorem 4.1 *If $h = \min\{h_1, h_2, h_3, h_4\}$, where*

$$h_1 < \sqrt[4]{\frac{5}{L}}, \quad h_2 < \sqrt[4]{\frac{0.756}{L}}, \quad h_3 < \sqrt[4]{\frac{0.467}{L}}, \quad h_4 < \sqrt[4]{\frac{0.988}{L}} \tag{4.1}$$

then the approximate solution defined by (3.11) or (3.12), exists and unique.

Proof: Let $t = jh$ and $j = 3k + \eta$ for $\eta = 1, 2, 3$, therefore

$$s^{(4)}((4k+\eta)h) = s^{(4)}_{4k+\eta} = s^{(4)}_{4k} + \sum_{i=5}^{8} \alpha_{i,k} \frac{(\eta h)^{i-4}}{(i-4)!} \tag{4.2}$$

where $\eta = 1, \cdots, 4$. By solving system (4.2) we obtain:

$$\begin{aligned} \alpha^{+}_{5,k} = & \frac{1}{12h}(48(s^{+}_{4k+1})^{(4)} - 36(s^{+}_{4k+2})^{(4)} \\ & + 16(s^{+}_{4k+3})^{(4)} - 3(s^{+}_{4k+4})^{(4)} - 25(s^{+}_{4k})^{(4)}), \end{aligned} \tag{4.3}$$

$$\begin{aligned} \alpha^{-}_{5,k} = & \frac{1}{12h}(48(s^{-}_{4k+1})^{(4)} - 36(s^{-}_{4k+2})^{(4)} \\ & + 16(s^{-}_{4k+3})^{(4)} - 3(s^{-}_{4k+4})^{(4)} - 25(s^{-}_{4k})^{(4)}), \end{aligned} \tag{4.4}$$

$$\begin{aligned} \alpha^{+}_{6,k} = & \frac{-1}{12h^2}(104(s^{+}_{4k+1})^{(4)} - 114(s^{+}_{4k+2})^{(4)} \\ & + 56(s^{+}_{4k+3})^{(4)} - 11(s^{+}_{4k+4})^{(4)} - 35(s^{+}_{4k})^{(4)}), \end{aligned} \tag{4.5}$$

$$\begin{aligned} \alpha^{-}_{6,k} = & \frac{-1}{12h^2}(104(s^{-}_{4k+1})^{(4)} - 114(s^{-}_{4k+2})^{(4)} \\ & + 56(s^{-}_{4k+3})^{(4)} - 11(s^{-}_{4k+4})^{(4)} - 35(s^{-}_{4k})^{(4)}), \end{aligned} \tag{4.6}$$

$$\alpha^{+}_{7,k} = \frac{1}{2h^3}(18(s^{+}_{4k+1})^{(4)} - 24(s^{+}_{4k+2})^{(4)} + 14(s^{+}_{4k+3})^{(4)} - 3(s^{+}_{4k+4})^{(4)} - 5(s^{+}_{4k})^{(4)}), \tag{4.7}$$

$$\alpha^{-}_{7,k} = \frac{1}{2h^3}(18(s^{-}_{4k+1})^{(4)} - 24(s^{-}_{4k+2})^{(4)} + 14(s^{-}_{4k+3})^{(4)} - 3(s^{-}_{4k+4})^{(4)} - 5(s^{-}_{4k})^{(4)}), \tag{4.8}$$

$$\alpha^{+}_{8,k} = \frac{-1}{h^4}(4(s^{+}_{4k+1})^{(4)} - 6(s^{+}_{4k+2})^{(4)} + 4(s^{+}_{4k+3})^{(4)} - (s^{+}_{4k+4})^{(4)} - (s^{+}_{4k})^{(4)}), \tag{4.9}$$

$$\alpha^{-}_{8,k} = \frac{-1}{h^4}(4(s^{-}_{4k+1})^{(4)} - 6(s^{-}_{4k+2})^{(4)} + 4(s^{-}_{4k+3})^{(4)} - (s^{-}_{4k+4})^{(4)} - (s^{-}_{4k})^{(4)}), \tag{4.10}$$

To prove the existence and uniqueness of $s(t)$, let us define the operator $G_k : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by $\alpha_{j,k} = G_v(\alpha_{j,k})$ for $j = 5,6,7,8$ and $v = 1,2,3,4$. According to condition (3.2) and equations (4.3),(4.5),(4.7), (4.9) and (4.4),(4.6),(4.8), (4.10) we conclude that

$$D(G_1(\alpha_{5,k})\ ,\ G_1(\alpha^{*}_{5,k})) \leq L\frac{h^4}{12.5!}|48 - 36(2^5) + 16(3^5) - 3(4^5)|D(\alpha_{5,k}, \alpha^{*}_{5,k}), \tag{4.11}$$

$$D(G_2(\alpha_{6,k})\ ,\ G_2(\alpha^{*}_{6,k})) \leq L\frac{h^4}{12.6!}|104 - 114(2^6) + 56(3^6) - 11(4^6)|D(\alpha_{6,k}, \alpha^{*}_{6,k}), \tag{4.12}$$

$$D(G_3(\alpha_{7,k})\ ,\ G_3(\alpha^*_{7,k})) \tag{4.13}$$
$$\leq L\frac{h^4}{2.7!}|18-24(2^7)+14(3^7)-3(4^7)|D(\alpha_{7,k},\alpha^*_{7,k}),$$

$$D(G_4(\alpha_{8,k})\ ,\ G_4(\alpha^*_{8,k})) \tag{4.14}$$
$$\leq L\frac{h^8}{8!}|4-6(2^8)+4(3^8)-4^8|D(\alpha_{8,k},\alpha^*_{8,k}),$$

From Equations (4.11), (4.12),(4.13), (4.14), and

$$h_1 < \sqrt[4]{\frac{5}{L}}, \quad h_2 < \sqrt[4]{\frac{0.756}{L}}, \quad h_3 < \sqrt[4]{\frac{0.467}{L}}, \quad h_4 < \sqrt[4]{\frac{0.988}{L}}$$

it follows that $G_v, \quad v=1,2,3,4$ are contraction operators. This implies the existence and uniqueness of approximate method under the stated conditions of theorem.

5 Numerical Example

Example 5.1 *Consider the fuzzy initial value problem*

$$y^{(4)}(t) = y(t), \quad t \in [0,1],$$
$$y(0) = y'(0) = y'''(0) = y^{(3)}(0) = (r-1, 1-r),$$

$y(t)$ is $[(i)-gH]$-differentiable and the real solution is:

$$y^-(t,r) = (r-1)e^t,$$
$$y^+(t,r) = (1-r)e^t.$$

We consider $I_k = [kH, (k+1)H]$, for $k=0,1,\ H=4h$ and $h=0.125$. $s_0(t)$, $s_4(t)$ are obtained as follows:

$$
\begin{aligned}
s_0^-(t) = {} & (r-1) + t(r-1) + \frac{t^2}{2}(r-1) + \frac{t^3}{3!}(r-1) + \frac{t^4}{4!}(r-1) \\
& + \frac{t^5}{5!}(-.9999397099 + .9999397099r) + \frac{t^6}{6!}(-1.001990135 + 1.001990135r) \\
& + \frac{t^7}{7!}(-.9672052893 + .9672052893r) + \frac{t^8}{8!}(-1.287374418 + 1.287374418r),
\end{aligned}
$$

$$
\begin{aligned}
s_0^+(t) = {} & (1-r) + t(1-r) + \frac{t^2}{2}(1-r) + \frac{t^3}{3!}(1-r) + \frac{t^4}{4!}(1-r) \\
& + \frac{t^5}{5!}(.9999397099 - .9999397099r) + \frac{t^6}{6!}(1.001990135 - 1.001990135r)) \\
& + \frac{t^7}{7!}(.9672052893 - .9672052893r) + \frac{t^8}{8!}(1.287374418 - 1.287374418r),
\end{aligned}
$$

$$
\begin{aligned}
s_4^-(t) = {} & 1.648721270r - 1.648721270 \\
& + (k-0.5)(1.648721270r - 1.648721270) \\
& + \frac{(k-0.5)^2}{2}(1.648721270r - 1.648721270) \\
& + \frac{(k-0.5)^3}{3!}(1.648721270r - 1.648721270) \\
& + \frac{(k-0.5)^4}{4!}(1.648721270r - 1.648721270)) \\
& + \frac{(k-0.5)^5}{5!}(-1.648621871 + 1.648621871r) \\
& + \frac{(k-0.5)^6}{6!}(-1.652002422 + 1.652002422r) \\
& + \frac{(k-0.5)^7}{7!}(-1.594651978 + 1.594651978r) \\
& + \frac{(k-0.5)^8}{8!}(-2.122522161 + 2.122522161r),
\end{aligned}
$$

$$
\begin{aligned}
s_4^+(t) = & 1.648721270 - 1.648721270r \\
& + (k-0.5)(1.648721270 - 1.648721270r) \\
& + \frac{(k-0.5)^2}{2}(1.648721270 - 1.648721270r) \\
& + \frac{(k-0.5)^3}{3!}(1.648721270 - 1.648721270r) \\
& + \frac{(k-0.5)^4}{4!}(1.648721270 - 1.648721270r) \\
& + \frac{(k-0.5)^5}{5!}(1.648621871 - 1.648621871r) \\
& + \frac{(k-0.5)^6}{6!}(1.652002422 - 1.652002422r) \\
& + \frac{(k-0.5)^7}{7!}(1.594651978 - 1.594651978r) \\
& + \frac{(k-0.5)^8}{8!}(2.122522161 - 2.122522161r),
\end{aligned}
$$

Table 1
Error of proposed method by Hausdorff distance in example 5.1

t	Error of Proposed method
0	0
0.1	0
0.2	0
0.3	0
0.4	0.1×10^{-8}
0.5	0.1×10^{-8}
0.6	0.1×10^{-8}
0.7	0.1×10^{-8}
0.8	0.1×10^{-8}
0.9	0.3×10^{-8}

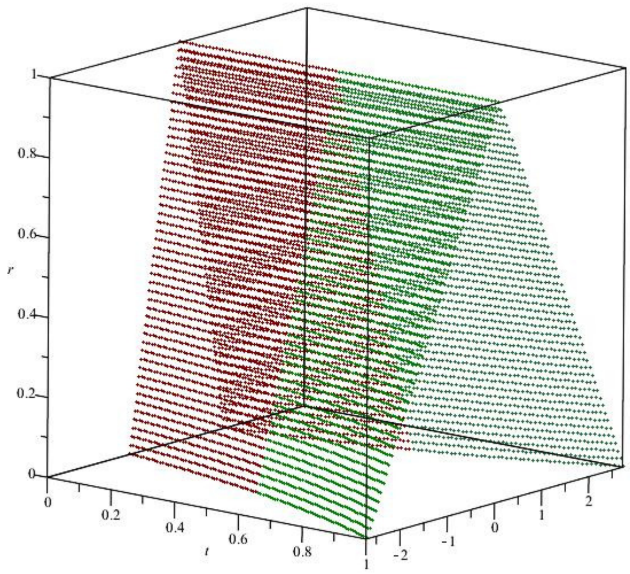

Fig. 1. Approximate solution for example 5.1. Red points: $s_0(t)$; green points:$s_4(t)$.

The approximated solution $s(t)$, for $i = 0, 1$, is plotted in Fig 1.

6 Conclusion

In this paper a new numerical method for solving forth order fuzzy differential equations under generalized differentiability was proposed. We used piecewise fuzzy polynomial of degree 8 based on the taylor expansion for approximating solutions of forth order fuzzy differential equations. Also, we can extend this method for $N-$th order fuzzy differential equations under generalized differentiability.

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