Mathematics Scientific Journal

Vol. 7, No. 2, (2012), 39-54

A numerical approach for solving a nonlinear inverse diffusion problem by Tikhonov regularization

H. Molhem [a,*], R. Pourgholi [b], M. Borghei [a]

[a]*Department of Physics , Faculty of Science, Islamic Azad University, Karaj Branch, Karaj, Iran.*

[b]*School of Mathematics and Computer Sciences, Damghan University, P.O.Box 36715-364, Damghan, Iran.*

Received 12 December 2010; accepted 23 October 2011

Abstract

In this paper, we propose an algorithm for numerical solving an inverse nonlinear diffusion problem. In additional, the least-squares method is adopted to find the solution. To regularize the resultant ill-conditioned linear system of equations, we apply the Tikhonov regularization method to obtain the stable numerical approximation to the solution. Some numerical experiments confirm the utility of this algorithm as the results are in good agreement with the exact data.

Key words: Inverse nonlinear diffusion problem, Laplace transform, Finite difference method, Least-squares method, Regularization method, SVD Method.

* Corresponding author: +989123507515
Email address: `molhem@kiau.ac.ir.` (H. Molhem).

1 Introduction

Quantitative understanding of the heat transfer processes occurring in industrial applications requires accurate knowledge of internal heat sources, the thermal properties of the material or surface conditions. In practical situations these unknown quantities are to be determined from transient temperature measurements or transient displacement measurements at one or more interior locations. These measurements can be fitted and then unknown quantities may be estimated. Such problems are called inverse problems which have become an attractive subject recently. In many situation it is difficult to analytically determine the heat transfer that enters or leaves a heat conducting material. Thermocouples and similar devices, however, allow accurate temperature measurements to be taken in most situations. Such temperature measurements provide the data necessary to determine the surface heat flux by employing an inverse technique.

Inverse heat conduction problems (IHCPs) appear in many important scientific and technological fields. Hence analysis, design implementation and testing of inverse algorithms are also are great scientific and technological interest. Mathematically, the inverse problems belong to the class of problems called the ill-posed problems. That is, their solution does not satisfy the general requirement of existence, uniqueness, and stability under small changes to the input data. To overcome such difficulties, a variety of techniques for solving IHCPs have been proposed.

Numerical solution of an inverse nonlinear diffusion problem requires to determine an unknown diffusion coefficient from an additional information. These new data are usually given by adding small random errors to the exact values from the solution to the direct problem. This paper presents the inverse determination of the diffusion coefficient of an unknown porous medium [1].

The outline of this paper is as follows. In the section 2, we formulate an inverse nonlinear parabolic problem. In the section 3, we linearize nonlinear terms by Taylor's series expansion, remove time-dependent terms by Laplace transform technique, discretize governing equations by finite difference method and used least squares method for correction unknown coefficients. Numerical experiments in section 4, confirm our theoretical results for an unknown porous medium.

2 Mathematical model of diffusion problem

The one dimensional diffusion equation is a partial differential equation which describes density fluctuations in a material undergoing diffusion.The equation is usually written as $\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}(a(u(x,t))\frac{\partial u(x,t)}{\partial x})$ where $u(x,t)$ is the density of the diffusing material at location x and time t and $a(u(x,t))$ is the collective diffusion coefficient for density u at location x.

The mathematical model of an inverse nonlinear diffusion problem with initial and boundary conditions is the following form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(a(u)\frac{\partial u}{\partial x}),\ 0 < x < 1, 0 < t < T, \tag{2.1}$$

$$u(x,0) = p(x), \quad 0 < x < 1, \tag{2.2}$$

$$-a(u(0,t))\frac{\partial u(0,t)}{\partial x} = g(t),\ 0 < t < T, \tag{2.3}$$

$$\frac{\partial u(1,t)}{\partial x} = q(t),\ 0 < t < T, \tag{2.4}$$

$$u(0,t) = f(t),\ 0 < t < T, \tag{2.5}$$

where T is a given positive constant, and $g(t)$, $p(x)$ and $q(t)$ are piecewise-continuous known functions, while $u(x,t)$ and diffusion coefficient $a(u(x,t)) > 0$, [2], are unknown which remain to be determined. For an unknown function $a(u)$ we must therefore provide additional information (5) to provide a unique solution $(u, a(u))$ to the inverse problem (2.1)-(2.5). Parabolic problems and nonlinear parabolic problems including equation (2.1) have been previously treated by many authors who considered certain special case of this type of problem [6-11]. In [6], Cannon and Duchateau defined an auxiliary inverse problem and sought a class of admissible coefficient

$a(u)$ which minimize an error functional. They have shown the existence of a solution to their auxiliary problem in a specified admissible class of functions. In this article, under certain conditions on $g(t)$, $p(x)$, $q(t)$ and $f(t)$, we shall identify both $u(x,t)$ and diffusion coefficient $a(u)$ at any time by using the overspecified condition (5), initial and boundary conditions (2)-(4).

3 Description of the numerical scheme

Consider the one-dimensional nonlinear problem described by the problem (1)-(5), where (1) and (3) are nonlinear. The application of the present numerical method to find the solution of problem (1)-(5),can be divided into the following steps.

3.1 Linearizing the nonlinear terms

Since the application of the Laplace transform technique is only restricted to the linear system, so that the nonlinear terms in equations (1) and (3) must be linearized. Therefore, we used Taylor's series expansion for linearized nonlinear terms and we obtain [9]

$$\frac{\partial}{\partial x}(a(u)\frac{\partial u}{\partial x}) = (\frac{\partial}{\partial u}K(u))_{u=\bar{u}}\frac{\partial^2 u}{\partial x^2} = a(\bar{u})\frac{\partial^2 u}{\partial x^2}, \tag{3.1}$$

where

$$K(u) = \int_0^u a(\rho)d\rho,$$

is a nonlinear function. Similarly

$$-a(u(0,t))\frac{\partial u(0,t)}{\partial x} = -a(\bar{u}(0,t))\frac{\partial u(0,t)}{\partial x}, \tag{3.2}$$

where $\bar{u} = \left(\bar{u}_0,\, \bar{u}_1,\, .\, .\, .,\, \bar{u}_N \right)$ denotes the previously iterated solution.

3.2 Remove time dependent terms

For remove time dependent terms from equations (2),(4),(6), and (7) the method of the Laplace transform is employed. The Laplace transform of a real function $\zeta(t)$ and its inversion formula are defined as

$$\tilde{\zeta}(s) = \pounds(\zeta(t)) = \int_0^\infty \exp(-st)\zeta(t)dt,$$

and

$$\zeta(t) = \pounds^{-1}(\tilde{\zeta}(s)) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \exp(st)\tilde{\zeta}(s)ds,$$

where $s = \nu+i\omega$, $\nu, \omega \in R$. The Laplace transform of equations (2),(4),(6), and (7) give

$$a(\bar{u})\frac{\partial^2\tilde{u}}{\partial x^2} = s\tilde{u} - p(x),\ \ 0 < x < 1 \tag{3.3}$$

$$-a(\bar{u})\frac{\partial\tilde{u}}{\partial x} = G(s),\ \ x = 0, \tag{3.4}$$

$$\frac{\partial\tilde{u}}{\partial x} = Q(s),\ \ x = 1, \tag{3.5}$$

where $\tilde{u}$, $\frac{\partial\tilde{u}}{\partial x}$, $\frac{\partial^2\tilde{u}}{\partial x^2}$, $Q(s)$ and $G(s)$ are Laplace transform of u, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $q(t)$ and $g(t)$ respectively.

3.3 Finite difference method for discrediting

In this step, we use central finite difference approximation for discrediting problem (8)-(10). Therefore

$$a(\bar{u}_\mu)\frac{\tilde{u}_{\mu+1} - 2\tilde{u}_\mu + \tilde{u}_{\mu-1}}{h^2} - s\tilde{u}_\mu = -p(\mu h), \tag{3.6}$$

$$-a(\bar{u}_0)\frac{\tilde{u}_1 - \tilde{u}_{-1}}{2h} = G(s),\ \ x = 0, \tag{3.7}$$

$$\frac{\tilde{u}_{N+1} - \tilde{u}_{N-1}}{2h} = Q(s),\ \ x = 1, \tag{3.8}$$

where $\mu = 0, 1, ..., N$. Problem (11)-(13) may be written in the following matrix form

$$A\tilde{U} = B, \tag{3.9}$$

where

$$A = \begin{pmatrix} -2a(\bar{u}_0) - sh^2 & 2a(\bar{u}_0) & 0 & 0 \\ a(\bar{u}_1) & -2a(\bar{u}_1) - sh^2 & a(\bar{u}_1) & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & a(\bar{u}_{N-1}) & -2a(\bar{u}_{N-1}) - sh^2 & a(\bar{u}_{N-1}) \\ 0 & 0 & 2a(\bar{u}_N) & -2a(\bar{u}_N) - sh^2 \end{pmatrix},$$

and

$$\tilde{U}^t = \left(\tilde{u}_0 \; \tilde{u}_1 \; . \; . \; . \; \tilde{u}_{N-1} \; \tilde{u}_N \right),$$

$$B^t = \left(b_0 \; b_1 \; ... \; b_{N-1} \; b_N \right),$$

where $b_0 = -h^2 p(0) - 2hG(s)$, $b_i = -h^2 p(ih), i = 1, ..., N-1$ and $b_N = -h^2 p(Nh) - 2ha(\bar{u}_N)Q(s)$. Note that equation (14) is a linear equation. The Cholesky Decomposition algorithm is used to solve $\tilde{U}$ and the numerical inversion of the Laplace transform technique ([12]-[13]) is applied to invert the transformed result to the physical quantity $U^t = (u_0 \, u_1 \, ... \, u_N)$. These updated values of U are used to calculate A and B for iteration. This computational procedure is performed repeatedly until desired convergence is achieved. The unknown function $a(u)$ is difficult to be approximated by a polynomial function for the whole time domain considered. Therefore the time domain $t_0 \leq t \leq T$ will be divided into some intervals where t_0 is the initial measurement time. Each of the intervals is assumed to be $t_{m-1} \leq t \leq t_m$ where $t_m = t_0 + m \triangle t$,

$m = 1, ..., N$ and $\triangle t = \frac{T-t_0}{N}$. . In this work the polynomial form proposed for the unknown $a(u)$ before performing the inverse calculation. Therefore $a(u)$ approximated as

$$a(u) = a_0 + a_1 u + a_2 u^2 + ... + a_q u^q, \tag{3.10}$$

where $\{a_0, a_1, ..., a_q\}$ are constants which remain to be determined simultaneously.

3.4 Least-squares minimization technique

To minimize the sum of the squares of the deviations between $u_0(t)$ (calculated) and $f(t)$, at the specific times $t = t_r$, we use least squares method. The error in the estimate is

$$E(a_0, a_1, ..., a_q) = \sum_{j=1}^{N} (u_0(t_j) - f(t_j))^2, \tag{3.11}$$

which remain to be minimized. The estimated values of a_i are determined until the value of $E(a_0, a_1, ..., a_q)$ is minimum. The computational procedure for estimating unknown coefficients a_i is well addressed in [9], therefore the correction linear system corresponding to the values of a_i can be expressed as

$$\Lambda\Theta = C, \tag{3.12}$$

where

$$\Lambda = \begin{pmatrix} \sum_{j=1}^{N}(\Upsilon_j^0)^2 & \sum_{j=1}^{N}\Upsilon_j^0\Upsilon_j^1 & ... & \sum_{j=1}^{N}\Upsilon_j^0\Upsilon_j^q \\ \sum_{j=1}^{N}\Upsilon_j^0\Upsilon_j^1 & \sum_{j=1}^{N}(\Upsilon_j^1)^2 & ... & \sum_{j=1}^{N}\Upsilon_j^1\Upsilon_j^q \\ & ... & & \\ \sum_{j=1}^{N}\Upsilon_j^0\Upsilon_j^q & \sum_{j=1}^{N}\Upsilon_j^1\Upsilon_j^q & ... & \sum_{j=1}^{N}(\Upsilon_j^q)^2 \end{pmatrix},$$

$$C = \begin{pmatrix} -\sum_{j=1}^{N} \Upsilon_j^0 e_j & \dots & -\sum_{j=1}^{N} \Upsilon_j^q e_j \end{pmatrix}^T,$$

$$\Theta = \begin{pmatrix} h_0 & h_1 & \dots & h_q \end{pmatrix}^T, \; e_j = u_0(t_j) - f(t_j),$$

$$\Upsilon_j^i = \frac{\partial u_0(t_j)}{\partial a_i}, \; i = 0, ..., q, \; j = 1, ..., N,$$

and h_i denotes the correction for initial values of a_i.

The Tikhonov regularized solution ([3]-[4]-[5]) to the system of linear algebraic equation

$$\Lambda\Theta = C,$$

is given by

$$\Theta_\alpha : \; \phi_\alpha(\Theta_\alpha) = \min_{\Theta} \phi_\alpha(\Theta),$$

where ϕ_α represents the zeroth order Tikhonov functional given by

$$\phi_\alpha(\Theta) = \parallel \Lambda\Theta - C \parallel^2 + \alpha^2 \parallel \Theta \parallel^2 .$$

Solving$\nabla\phi_\alpha(\Theta) = 0$ with respect to Θ, then we obtain, the Tikhonov regularized solution of the regularized equation

$$\Theta_\alpha = (\Lambda^T\Lambda + \alpha^2 I)^{-1}\Lambda^T C.$$

In our computation we use the L-curve scheme to determine a suitable value of α ([3]-[5]).

4 Numerical experiment

In this section, we are going to demonstrate some numerical results for unknown coefficient in the inverse problem (1)-(5).

It is noticeable that the accuracy of the scheme presented is evaluated by comparison with the SVD method [14].

Example 1. In this example, let us consider the following inverse non-linear parabolic problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}(a(u)\frac{\partial u}{\partial x}),\ \ 0 < x < 1, 0 < t < T, && (4.1)\\
u(x,0) &= x,\ \ 0 < x < 1, && (4.2)\\
-a(u(0,t))\frac{\partial u(0,t)}{\partial x} &= -1 - t,\ \ 0 < t < T, && (4.3)\\
\frac{\partial u(1,t)}{\partial x} &= 1, \quad 0 < t < T, && (4.4)\\
u(0,t) &= t,\ \ 0 < t < T, && (4.5)
\end{aligned}
$$

with unique exact solution

$$a(u) = 1 + u, \quad u(x,t) = x + t.$$

Tables 1 and 2 show the comparison between the exact solution and approximate solution result from our method by Tikhonov regularization and SVD regularization. To solve the problem 4.1-4.5 the unknown coefficient $a(u)$ defined as the following form

$$a(u) = a_0 + a_1 u.$$

For determine a_0 and a_1 we use

$$E(a_0, a_1) = \sum_{j=0}^{N} (u_0(t_j) - f(t_j))^2,$$

therefore the coefficients can be obtained. The estimated values of a_0, a_1 are $a_0 = 1.011972$ and $a_1 = 1.015392$.

j	Tikhonov $u(0, jk)$	SVD $u(0, jk)$	Exact $u(0, jk)$
1	0.104994	0.766667	0.1
2	0.197793	0.866667	0.2
3	0.299521	0.966667	0.3
4	0.397968	0.392234	0.4
5	0.505003	0.475689	0.5

Table 1. The comparison between exact and Tikhonov and SVD solutions for $u(0, jk)$ when $x = ih$, $t = jk$, $k = \frac{1}{10}$, $h = \frac{1}{6}$, $\tau_i = 0.05$.

j	Tikhonov $u(\frac{4}{6}, jk)$	SVD $u(\frac{4}{6}, jk)$	Exact $u(\frac{4}{6}, jk)$
1	0.762569	0.702231	0.766667
2	0.862768	0.883412	0.866667
3	0.959519	0.953321	0.966667
4	1.065250	0.985645	1.066667
5	1.166207	1.102333	1.166667

Table 2. The comparison between exact and Tikhonov and SVD solutions for $u(\frac{4}{6}, jk)$ when $x = ih$, $t = jk$, $k = \frac{1}{10}$, $h = \frac{1}{6}$, $\tau_i = 0.05$.

Example 2. In this example, let us consider the following inverse non-linear parabolic problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(a(u)\frac{\partial u}{\partial x}), \ 0 < x < 1, 0 < t < T, \tag{4.6}$$

$$u(x, 0) = (1 - x^2)^{1/2}, \ 0 < x < 1, \tag{4.7}$$

$$a(u(0, t))\frac{\partial u(0, t)}{\partial x} = 0, \ 0 < t < T, \tag{4.8}$$

$$\frac{\partial u(1, t)}{\partial x} = -\frac{1}{1 + 4t}(\frac{1}{(1 + 4t)^{1/2}} - \frac{1}{1 + 4t})^{-1/2}, \quad 0 < t < T, \tag{4.9}$$

$$u(0, t) = \frac{1}{(1 + 4t)^{1/4}}, \ 0 < t < T, \tag{4.10}$$

with unique exact solution

$$a(u) = u^2, \quad u(x, t) = (\frac{1}{(1 + 4t)^{1/2}} - \frac{x^2}{1 + 4t})^{1/2}.$$

Tables 3 and 4 show the comparison between the exact solution and approximate solution result from our method by Tikhonov regularization and SVD regularization.

To solve the problem (4.6)-(4.10) the unknown coefficient $a(u)$ defined as the following form

$$a(u) = a_0 + a_1 u^2.$$

For determine a_0 and a_1 we use

$$E(a_0, a_1) = \sum_{j=0}^{N} (u_0(t_j) - f(t_j))^2,$$

therefore the coefficients can be obtained. The estimated values of a_0, a_1 are $a_0 = 0.0223002$ and $a_1 = 1.37936$.

j	Tikhonov $u(0, jk)$	SVD $u(0, jk)$	Exact $u(0, jk)$
1	0.922343	0.908767	0.919325
2	0.865587	0.874436	0.863340
3	0.834357	0.812341	0.821097
4	0.790013	0.776992	0.787511
5	0.755643	0.735467	0.759836

Table 3. The comparison between exact and Tikhonov and SVD solutions for $u(0, jk)$ when $x = ih$, $t = jk$, $k = \frac{1}{10}$, $h = \frac{1}{3}$, $\tau_i = 0.05$.

j	Tikhonov $u(\frac{2}{3}, jk)$	SVD $u(\frac{2}{3}, jk)$	Exact $u(\frac{2}{3}, jk)$
1	0.723563	0.709987	0.726425
2	0.702234	0.723411	0.706005
3	0.677891	0.653452	0.687153
4	0.683349	0.678354	0.670249
5	0.654356	0.6775463	0.655135

Table 4. The comparison between exact and Tikhonov and SVD solutions for $u(\frac{2}{3}, jk)$ when $x = ih$, $t = jk$, $k = \frac{1}{10}$, $h = \frac{1}{3}$, $\tau_i = 0.05$.

Example 3. In this example, let us consider the following inverse non-linear parabolic problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(a(u)\frac{\partial u}{\partial x}),\ 0 < x < 1, 0 < t < T, \tag{4.11}$$
$$u(x,0) = e^x,\ 0 < x < 1, \tag{4.12}$$
$$a(u(0,t))\frac{\partial u(0,t)}{\partial x} = 1 - e^{-t},\ 0 < t < T, \tag{4.13}$$
$$\frac{\partial u(1,t)}{\partial x} = e^{1-t}, \quad 0 < t < T, \tag{4.14}$$
$$u(0,t) = e^{-t},\ 0 < t < T, \tag{4.15}$$

with unique exact solution

$$a(u) = e^{t-x} - 1, \quad u(x,t) = e^{x-t}.$$

Tables 5, 6, 7, and 8 show the comparison between the exact solution and approximate solution result from our method by Tikhonov regularization and SVD regularization.

j	Tikhonov $u(0, jk)$	SVD $u(0, jk)$	Exact $u(0, jk)$
1	0.902249	0.931245	0.904837
2	0.803214	0.795439	0.818731
3	0.731245	0.731242	0.740818
4	0.669875	0.654312	0.670320
5	0.606452	0.603214	0.606531

Table 5. The comparison between exact and Tikhonov and SVD solutions for $u(0, jk)$ when $x = ih$, $t = jk$, $k = \frac{1}{10}$, $h = \frac{1}{10}$, $\tau_i = 0.05$.

j	Tikhonov $u(\frac{6}{10}, jk)$	SVD $u(\frac{6}{10}, jk)$	Exact $u(\frac{6}{10}, jk)$
1	1.64543	1.63443	1.64872
2	1.49887	1.50543	1.49182
3	1.33987	1.31265	1.34986
4	1.223692	1.20341	1.22140
5	1.11234	1.14321	1.10517

Table 6. The comparison between exact and Tikhonov and SVD solutions for $u(\frac{6}{10}, jk)$ *when* $x = ih$, $t = jk$, $k = \frac{1}{10}$, $h = \frac{1}{10}$, $\tau_i = 0.05$.

j	Tikhonov $a(u(0, jk))$	SVD $a(u(0, jk))$	Exact $a(u(0, jk))$
1	0.105854	0.102134	0.105171
2	0.223213	0.213698	0.221403
3	0.347865	0.335987	0.349859
4	0.494652	0.473426	0.491825
5	0.646984	0.659768	0.648721

Table 7. The comparison between exact and Tikhonov and SVD solutions for $a(u(0, jk))$ *when* $x = ih$, $t = jk$, $k = \frac{1}{10}$, $h = \frac{1}{10}$, $\tau_i = 0.05$.

j	Tikhonov $a(u(\frac{6}{10}, jk))$	SVD $a(u(\frac{6}{10}, jk))$	Exact $a(u(\frac{6}{10}, jk))$
1	-0.395467	-0.412364	-0.393469
2	-0.329123	-0.376549	-0.329680
3	-0.258863	-0.234562	-0.259182
4	-0.181543	-0.154378	-0.181269
5	-0.095432	-0.090875	-0.095162

Table 8. The comparison between exact and Tikhonov and SVD solutions for $a(u(\frac{6}{10}, jk))$ when $x = ih$, $t = jk$, $k = \frac{1}{10}$, $h = \frac{1}{10}$, $\tau_i = 0.05$.

5 Conclusion

A numerical method to estimate unknown coefficient is proposed for an inverse nonlinear parabolic problem and the following results are obtained.

1. The present study, successfully applies the numerical method involving the Laplace transform technique and the finite difference method in conjunction with the least-squares scheme to an IHCP.

2. From the illustrated examples it can be seen that the proposed numerical method is efficient and accurate to estimate the thermal diffusivity in a one-dimensional nonlinear inverse diffusion problem.

3. Owing to the application of the Laplace transform, the present method is not a time-stepping procedure. Thus the unknown thermal diffusivity at any specific time can be predicted without any step-by-step computations from $t = t_0$. We also apply other different sets of the initial guesses, such as $\{a_0, a_1, ..., a_\iota\} = \{0.4, 0.4, ..., 0.4\}$, $\{0.7, 0.7, ..., 0.7\}$ and $\{1.1, 1.1, ..., 1.1\}$, results show that the effect of the initial guesses on the accuracy of the estimates is not significant for the present method.

4. Numerical results show that, thermal diffusivity evolutions estimated by the Tikhonov regularization is accurate that those obtained by the SVD regularization with noisy data.

5. we can used the result of this article in following subject: Thermal management of electronic devices and systems, Thermodynamics, Thermometer, Temperature of the universe (Heat death of the universe) , Thermal resistance in electronics, Heat pipe, improve Calibration thermometer, Atomic diffusion, Mass diffusivity, Phase transformations in solids and so on.

6 Acknowledgments

This work was supported by Islamic Azad University, Karaj Branch, Karaj, Iran.

References

[1] J. Bear , *Dynamics of Fluids in Porous Media, 2^{nd} edn. Elsevier*, New York, 1975.

[2] J. R. Cannon, *The One-Dimensional Heat Equation. Addison-Wesley*, Menlo Park, California, 1984.

[3] P. C. Hansen, Analysis of discrete ill-posed problems by means of the L-curve, *SIAM Rev.* **34** (1992), 561–80.

[4] A. N. Tikhonov, V. Y. Arsenin, *On the solution of ill-posed problems*, New York, Wiley, 1977.

[5] C. L. Lawson, R. J. Hanson, Solving least squares problems, Philadelphia, PA,*SIAM*, 1995, First published by Prentice-Hall, 1974.

[6] J. R. Cannon, P. Duchateau, An inverse problem for a nonlinear diffusion equation, *SIAM J. appl. Math.* **39** (1980), 272–289.

[7] O. A. Ladyzhenskaya, V. A. Sollonikov, N. N. Uralceva, Linear and Quasilinear Equations Parabolic Type,*Amer. Math. Soc.* Providence, RI (1967).

[8] H. T. Chen, S. M. Chang, Application of the hybrid method to inverse heat conduction problems, *Int. J. Heat Mass Transfer,* **33** (1990), 621–628.

[9] A. Shidfar, R. Pourgholi, M. Ebrahimi, A Numerical Method for Solving of a Nonlinear Inverse Diffusion Problem, *Comput. Math. Appl.* **52** (2006), 1021–1030.

[10] A. Shidfar, R. Pourgholi, Application of finite difference method to analysis an ill-posed problem, *Appl. Math. Comput.* **168** (2005), 1400–1408 .

[11] R. Pourgholi, N. Azizi, Y. S. Gasimov, F. Aliev, H. K. Khalafi, Removal of Numerical Instability in the Solution of an Inverse Heat Conduction Problem, *Communications in Nonlinear Science and Numerical Simulation*, (2008) In Press.

[12] F. Durbin, Numerical inversion of Laplace transforms: effcient improvement to Dubner and Abateś method, *Comp. J.* **17** (1973), 371–376.

[13] G. Honig and U. Hirdes, A method for the numerical inversion of Laplace transforms, *J. Comp. Appl. Math.* **9** (1984), 113-132.

[14] G. H. Golub, C. F. Van Loan, *Matrix computations*, John Hopkins university press, Baltimore, MD, 1983.