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Common fixed point theorems of contractive mappings sequence in partially ordered G-metric spaces

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Abstract

We consider the concept of Ω-distance on a complete partially ordered G-metric space and prove some common fixed point theorems.

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1 Introduction

The Banach fixed point theorem for contraction mapping has been generalized and extended in many direction [1-15]. Nieto and Rodriguez-

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Lopez [16], Ran and Reurings [17] and Petrusel and Rus [18] presented some new results for contractions in partially ordered metric spaces. The main idea in [12,16,17] involves combining the ideas of an iterative technique in the contraction mapping principle with those in the monotone technique. Also, Mustafa and sims [19] introduced the concept of G-metric. Some authors [20-24] have proved some fixed point theorems in these spaces. In [25] Gajić proved a common fixed point theorem for a sequence of mappings on this space. Recently, Saadati et al. [26], using the concept of G-metric, defined an Ω-distance on complete G-metric space and generalized the concept of ω-distance due to Kada et al. [27].
In [28,29] some fixed pointtheorems proved and generalized under this concept.
In this paper, we extend some fixed point theorems by using this concept in partially ordered G-metric spaces.
At first we recall some definitions and lemmas. For more information see [19,26].

Definition 1.1 *[19] Let* X *be a non-empty set. A function* $G : X\times X\times X \longrightarrow [0,\infty)$ *is called a G-metric if the following conditions are satisfied:*

(i) $G(x,y,z) = 0$ if $x = y = z$ (coincidence),
(ii) $G(x,x,y) > 0$ for all $x,y \in X$, where $x \neq y$,
(iii) $G(x,x,z) \leq G(x,y,z)$ for all $x,y,z \in X$, with $z \neq y$,
(iv) $G(x,y,z) = G(p\{x,y,z\})$, where p is a permutation of x,y,z (symmetry),
(v) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

A G-metric is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Definition 1.2 *[19] Let* $(X,\, G)$ *be a G-metric space,*

(1) a sequence $\{x_n\}$ in X is said to be G-Cauchy sequence if, for each
$\varepsilon > 0$, there exists a positive integer n_0 such that for all

$m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$.

(2) a sequence $\{x_n\}$ in X is said to be G-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$, $G(x_m, x_n, x) < \varepsilon$.

Definition 1.3 *[19] Let (X, G) be a G-metric space. Then a function*
$\Omega : X \times X \times X \longrightarrow [0, \infty)$ is called an Ω-distance on X if the following conditions are satisfied:

(a) $\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z)$ for all $x, y, z, a \in X$,
(b) for any $x, y \in X, \Omega(x, y, .), \Omega(x, ., y) : X \to [0, \infty)$ are lower semi-continuous,
(c) for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Omega(x, a, a) \leq \delta$ and
$\Omega(a, y, z) \leq \delta$ imply $G(x, y, z) \leq \varepsilon$.

Example 1.1 *[26] Let (X, d) be a metric space and $G : X^3 \longrightarrow [0, \infty)$ defined by*

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\},$$

for all $x, y, z \in X$. Then $\Omega = G$ is an Ω-distance on X.

Example 1.2 *[26] In $X = \mathbb{R}$ we consider the G-metric G defined by*

$$G(x, y, z) = \frac{1}{3}(\mid x - y \mid + \mid y - z \mid + \mid x - z \mid),$$

for all $x, y, z \in \mathbb{R}$. Then $\Omega : \mathbb{R}^3 \longrightarrow [0, \infty)$ defined by

$$\Omega(x, y, z) = \frac{1}{3}(\mid x - y \mid) + \mid x - z \mid,$$

for all $x, y, z \in \mathbb{R}$ is an Ω-distance on $\mathbb{R}$.

For more example see [26].

Lemma 1.1 *[26] Let X be a metric space with metric G and Ω be* *an Ω-distance on X. Let $\{x_n\}, \{y_n\}$ be sequences in X, $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then we have the following:*

(1) If $\Omega(y, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(y, y, z) < \varepsilon$ and hence $y = z$.
(2) If $\Omega(y_n, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y_m, z) \leq \beta_n$ for $m > n$, then $G(y_n, y_m, z) \to 0$ and hence $y_n \to z$.
(3) If $\Omega(x_n, x_m, x_l) \leq \alpha_n$ for any $l, m, n \in \mathbb{N}$ with $n \leq m \leq l$, then $\{x_n\}$ is a G-Cauchy sequence.
(4) If $\Omega(x_n, a, a) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a G-Cauchy sequence.

Definition 1.4 *[26] G-metric space X is said to be Ω-bounded if there is a constant $M > 0$ such that $\Omega(x, y, z) \leq M$ for all $x, y, z \in X$.*

2 Conclusion

In this section, we obtain common fixed point theorems for sequence of mappings satisfying contractiv and expansive conditions on partially ordered complete G-metric spaces.

Definition 2.1 *Suppose $(X, \leq)$ is a partially ordered space and $T : X \to X$ is a mapping of X into itself. We say that T is non-decreasing if for $x, y \in X$,*

$$x \leq y \Longrightarrow T(x) \leq T(y).$$

Theorem 2.1 *Let $(X, \leq)$ and $(Y, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X and Y such that (X, G) and (Y, G) are complete G-metric space and Ω_1 is an Ω-distance on X, Ω_2 is Ω-distance on Y such that X be Ω_1-bounded and Y be Ω_2-bounded. Let $T_n : X \longrightarrow Y$, $n \in \mathbb{N}$ and $S_n : Y \longrightarrow X$, $n \in \mathbb{N} \cup \{0\}$ be a non-decreasing and continuous sequence of mappings with following properties:*

(a) for all $x, y, z \in X$, $x', y', z' \in Y$ and $i, j, k \in \mathbb{N}$ such that $0 \le r < 1$,

$$\Omega_1(S_iT_ix, S_jT_jy, S_kT_kz) \le r \max\ \{\Omega_1(y, S_jT_jy, S_kT_kz), \Omega_1(x, y, z), \Omega_2(T_ix, T_jy, T_kz)\},$$

$$\Omega_2(T_iS_{i-1}x', T_jS_{j-1}y', T_kS_{k-1}z') \le r \max\{\Omega_2(y', T_jS_{j-1}y', T_kS_{k-1}z'), \Omega_2(x', y', z'), \Omega_1(S_{i-1}x', S_{j-1}y', S_{k-1}z')\};$$

(b) for every $x, y, z \in X$ with $y \neq S_nT_ny$, $n \in \mathbb{N}$,

$$\inf\{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \le z\} > 0;$$

(c) for every $x', y', z' \in Y$ with $y' \neq T_nS_{n-1}y'$, $n \in \mathbb{N}$,

$$\inf\{\Omega(x', y', x') + \Omega(x', y', z') + \Omega(x', z', y') : x' \le z'\} > 0;$$

(d) $\Omega_2(T_ix, T_iy, T_iz) = 0$ for each $x, y, z \in X$ and $\Omega_1(S_ix', S_iy', S_iz') = 0$ for each $x', y', z' \in Y$.

Then $\{S_nT_n\}$ has a unique common fixed point u in X and $\{T_nS_{n-1}\}$ has a unique common fixed point w in Y. Furthermore, $\lim_{n\to\infty} T_nu = w$ and $\lim_{n\to\infty} S_nw = u$.

Proof: Let $x_0 \in X$ such that $S_nT_n(x_{n-1}) = x_n$ and $T_n(x_{n-1}) = y_n$ and $x_n \le x_{n+1}$ for any $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ and $t \ge 0$,

$$\begin{aligned}\Omega_1(x_n, x_{n+1}, x_{n+t}) &= \Omega_1(S_nT_n(x_{n-1}), S_{n+1}T_{n+1}(x_n), S_{n+t}T_{n+t}(x_{n+t-1}))\\ &\le r \max\{\Omega_1(x_{n-1}, x_n, x_{n+t-1}), \Omega_1(x_n, x_{n+1}, x_{n+t}),\\ &\qquad \Omega_2(T_n(x_{n-1}), T_{n+1}(x_n), T_{n+t}(x_{n+t-1}))\}\\ &= r \max\{\Omega_1(x_{n-1}, x_n, x_{n+t-1}), \Omega_1(x_n, x_{n+1}, x_{n+t}),\\ &\quad \Omega_2(y_n, y_{n+1}, y_{n+t})\}.\end{aligned}$$

Then,

$$\Omega_1(x_n, x_{n+1}, x_{n+t}) \le r \max\{\Omega_1(x_{n-1}, x_n, x_{n+t-1}), \Omega_2(y_n, y_{n+1}, y_{n+t})\}.$$

Similarly,

$$\Omega_2(y_n, y_{n+1}, y_{n+t}) \leq r \max\{\Omega_2(y_{n-1}, y_n, y_{n+t-1}), \Omega_1(x_{n-1}, x_n, x_{n+t-1})\}.$$

So,

$$\Omega_1(x_n, x_{n+1}, x_{n+t}) \leq r^n \max\{\Omega_1(x_0, x_1, x_t), \Omega_2(y_1, y_2, y_{t+1})\},$$

and

$$\Omega_2(y_n, y_{n+1}, y_{n+t}) \leq r^n \max\{\Omega_1(x_0, x_1, x_t), \Omega_2(y_0, y_1, y_t)\}.$$

Now, for any $l > m > n$ with $m = n + k$ and $l = m + t$ $(k, t \in \mathbb{N})$, we have

$$\lim_{n,m,l\to\infty} \Omega_1(x_n, x_m, x_l) = 0.$$

Since X is Ω_1-bounded and,

$$\begin{aligned}
\Omega_1(x_n, x_m, x_l) &\leq \Omega_1(x_n, x_{n+1}, x_{n+1}) + \Omega_1(x_{n+1}, x_m, x_l) \\
&\leq \Omega_1(x_n, x_{n+1}, x_{n+1}) + \Omega_1(x_{n+1}, x_{n+2}, x_{n+2}) \\
&+ \cdots + \Omega_1(x_{m-1}, x_m, x_l) \\
&\leq r^n M + r^{n+1} M + \cdots + r^{m-1} M \\
&\leq \sum_{j=0}^{n-m+1} r^{n-j} M \\
&\leq \frac{r^n}{1-r} M.
\end{aligned}$$

So, by $0 \leq r < 1$ and Part (3) of Lemma (1.6), $\{x_n\}$ is a G-Cauchy sequence. Since X is G-complete, $\{x_n\}$ converges to a point $u \in X$. Similarly, $\{y_n\}$ is a G-Cauchy sequence such that has a limit w in Y. Fixed $n \in \mathbb{N}$ and by the lower semi-continuity of Ω, we have

$$\Omega_1(x_n, x_m, u) \leq \liminf_{p\to\infty} \Omega_1(x_n, x_m, x_p) \leq \frac{r^n}{1-r} M, \qquad m \geq n$$

$$\Omega_1(x_n, u, x_l) \leq \liminf_{p\to\infty} \Omega_1(x_n, x_p, x_l) \leq \frac{r^n}{1-r} M, \qquad l \geq n.$$

Assume that $u \neq S_n T_n u$. Since $x_n \leq x_{n+1}$, we have

$$
\begin{aligned}
0 &< \inf\{\Omega_1(x_n, u, x_n) + \Omega_1(x_n, u, x_{n+1}) + \Omega_1(x_n, x_{n+1}, u)\} \\
&\leq 3 \inf\{\frac{r^n}{1-r} M : n \in \mathbb{N}\} \\
&= 0,
\end{aligned}
$$

which is a contraction. Therefor, $u = S_nT_nu$ and consequently u is a common fixed point $\{S_nT_n\}$. Similarly, w is a common fixed point $\{T_nS_{n-1}\}$.
To prove the uniqueness, suppose $\{S_nT_n\}$ has another fixed point u'. Then,

$$
\begin{aligned}
\Omega_1(u, u', u') &= \Omega_1(S_nT_nu, S_nT_nu', S_nT_nu') \\
&\leq r \max\{\Omega_1(u, u', u'), \Omega_1(u', S_nT_nu', S_nT_nu'), \\
&\quad \Omega_2(T_nu, T_nu', T_nu')\} \\
&= r \max\{\Omega_1(u, u', u'), \Omega_1(u', u', u'), \\
&\quad \Omega_2(T_nu, T_nu', T_nu')\}.
\end{aligned}
$$

By (d) either $\Omega_1(u, u', u') = 0$ or $\Omega_1(u, u', u') \leq r\Omega_1(u', u', u')$. Since,

$$
\begin{aligned}
\Omega_1(u', u', u') &= \Omega_1(S_nT_nu', S_nT_nu', S_nT_nu') \\
&\leq r \max\{\Omega_1(u', u', u'), \Omega_1(u', S_nT_nu', S_nT_nu'), \\
&\quad \Omega_2(T_nu', T_nu', T_nu')\},
\end{aligned}
$$

then, $\Omega_1(u', u', u') = 0$ and consequently $\Omega_1(u, u', u') = 0$. By Part (c) of Definition (1.3) fixed point of $\{S_nT_n\}$ is unique. Similarly, w is a unique fixed point of $\{T_nS_{n-1}\}$. By continuity of $\{T_n\}$, we have

$$
\lim_{n\to\infty} T_nu = \lim_{n\to\infty} T_n(x_{n-1}) = \lim_{n\to\infty} y_n = w.
$$

Similarly, $\lim_{n\to\infty} S_nw = u$. □

Corollary 2.1 *Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X such that X is Ω-*

bounded. Let $T_n : X \longrightarrow X$, $n \in \mathbb{N}$ be a non-decreasing sequence of mappings with property that for any $i, j, k \in \mathbb{N}$, we have:

(a) for all $x, y, z \in X$ and $0 \leq r < 1$,

$$\Omega(T_i x, T_j y, T_k z) \leq r \max\{\Omega(x, y, z), \Omega(y, T_j y, T_k z)\};$$

(b) for every $x, y, z \in X$ with $y \neq T_n y, n \in \mathbb{N}$,

$$\inf\{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0.$$

Then $\{T_n\}$ has a unique common fixed point u in X and $\Omega(u, u, u) = 0$.

Proof: It is sufficient that put $\Omega = \Omega_1 = \Omega_2$, $X = Y$ and $S_n = I_n$ that I_n is identity mapping on X in Theorem (2.2). □

Theorem 2.2 *Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X such that X is Ω-bounded. Let $T_n : X \longrightarrow X$, $n \in \mathbb{N}$ be a non-decreasing sequence of mappings with property that for any $i, j, k \in \mathbb{N}$, we have:*

(a) for all $x, y, z \in X$ and $0 \leq r < 1$, $\Omega(T_i x, T_j y, T_k z) \leq r\Omega(x, y, z)$;

(b) for every $x, y, z \in X$ with $y \neq T_n y, n \in \mathbb{N}$,

$$\inf\{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0.$$

Then $\{T_n\}$ has a unique common fixed point u in X and $\Omega(u, u, u) = 0$.

Proof: Theorem is proved by similar proof of Theorem 2.1. □

Corollary 2.2 *Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X such that X is Ω-*

bounded. Let $T_n : X \longrightarrow X$, $n \in \mathbb{N}$ be a non-decreasing sequence of mappings with property that for some $m \in \mathbb{N}$ and each $i, j, k \in \mathbb{N}$, we have:

(a) for all $x, y, z \in X$ and $0 \leq r < 1$, $\Omega(T_i^m x, T_j^m y, T_k^m z) \leq r\Omega(x, y, z)$;

(b) for every $x, y, z \in X$ with $y \neq T_n y, n \in \mathbb{N}$,

$$\inf\{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0.$$

Then $\{T_n\}$ has a unique common fixed point u in X and $\Omega(u, u, u) = 0$.

Proof: By Theorem 2.2, the sequence $\{T_n^m\}$ has the unique common fixed point u. But,

$$T_n u = T_n(T_n^m u) = T_n^{m+1} u = T_n^m(T_n u).$$

So, $T_n u$ is the fixed point $\{T_n^m\}$. Now, by uniqueness of the fixed point, $T_n u = u$. □

Definition 2.2 *Let (X, G) be a G-metric space, Ω be an Ω-distance on X and T be a selfmapping on X. Then T is called expansive mapping with respect Ω if there exists a constant $a > 1$ such that for all $x, y, z \in X$, we have:*

$$\Omega(Tx, Ty, Tz) \geq a\Omega(x, y, z).$$

Theorem 2.3 *Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X such that X is Ω-bounded. Let $T_n : X \longrightarrow X$, $n \in \mathbb{N}$ be a non-decreasing sequence of surjective mappings and $S_n : X \longrightarrow X$, $n \in \mathbb{N}$ be a non-decreasing sequence of injective mappings with property that for any $i, j, k \in \mathbb{N}$, we have:*

(a) for all $x, y, z \in X$ and $a > 1$, $\Omega(T_i x, T_j y, T_k z) \geq a\Omega(S_i x, S_j y, S_k z)$;

(b) for all $n \in \mathbb{N}$, T_n *and* S_n *commute;*

(c) for every $x, y, z \in X$ *with* $y \neq T_n y, n \in \mathbb{N}$,

$$\inf\{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0$$

Then $\{T_n\}$ *and* $\{S_n\}$ *have a unique common fixed point* u *in* X *and* $\Omega(u, u, u) = 0$.

Proof: If $T_i x = T_i y$ for any $i \in \mathbb{N}$ and $x, y \in X$, then,

$$\Omega(T_i x, T_j y, T_j y) \geq a\Omega(S_i x, S_j y, S_j y);$$
$$\Omega(T_j y, T_i x, T_i y) \geq a\Omega(S_j y, S_i x, S_i y);$$

thus,

$$\Omega(S_i x, S_j y, S_j y) \leq \frac{1}{a}\Omega(T_i x, T_j y, T_j y);$$
$$\Omega(S_j y, S_i x, S_i y) \leq \frac{1}{a}\Omega(T_j y, T_i x, T_i y).$$

Now, since $a > 1$ and X is Ω-bounded then, for any $\varepsilon > 0$, we choose $\delta = \frac{1}{a}M$, which implies, $\Omega(S_i x, S_j y, S_j y) \leq \delta$ and $\Omega(S_j y, S_i x, S_i y) \leq \delta$. By Part (c) of Definition (1.3), $G(S_i x, S_i x, S_i y) \leq \varepsilon$. Since ε is arbitrary, hence $S_i x = S_i y$. Now, by injectivity S_i for every $i \in \mathbb{N}$, we imply that $x = y$. So, T_n is injective and consequently invertible. Let H_n be the inverse mapping of T_n for any $n \in \mathbb{N}$. Then,

$$\begin{aligned}\Omega(x, y, z) &= \Omega(T_i(H_i x), T_j(H_j y), T_k(H_k z)) \\ &\geq a\Omega(S_i(H_i x), S_j(H_j y), S_k(H_k z)).\end{aligned}$$

So, for each $x, y, z \in X$ and any $i, j, k \in \mathbb{N}$, we obtain

$$\Omega(S_i o H_i x, S_j o H_j y, S_k o H_k z) \leq r\Omega(x, y, z),$$

where $r = \frac{1}{a}$. Then $\Omega(G_i x, G_j y, G_k z) \leq r\Omega(x, y, z)$, where $G_n = S_n o H_n$. By Theorem 2.1, G_n or $S_n o H_n$ have a unique common fixed point u in X ,i.e. $G_n u = u = S_n o H_n u$. It follows that $T_n(S_n(H_n u) =$

$T_n u$,. Since, T_n and S_n commute, we obtain

$$S_n(T_n(H_n u) = T_n u \Longrightarrow S_n u = T_n u,$$

for any $n \in \mathbb{N}$. If we put $x = u$, $y = H_j u$ and $z = H_k u$, we have

$$\Omega(T_i u, T_j(H_j u), T_k(H_k u)) \geq a\Omega(S_i u, S_j(H_j u), S_k(H_k u)).$$

So,

$$\Omega(T_i u, u, u) \geq a\Omega(S_i u, u, u) = a\Omega(T_i u, u, u).$$

Since $a > 1$, then $\Omega(T_i u, u, u) = 0$. By putting $x = H_i u, y = H_j u, z = H_k u$ and similar proof $\Omega(u, u, u) = 0$. Now by Part (3) of Definition (1.3), $T_i u = u$. Hence $T_n u = S_n u = u$ and u is a unique common fixed point of T_n and S_n. □
The following corollary is a generalization of [18, theorem 2.1].

Corollary 2.3 *Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X such that X is Ω-bounded. Let $T_n : X \longrightarrow X$, $n \in \mathbb{N}$ be a non-decreasing sequence of surjective mappings with property that for any $i, j, k \in \mathbb{N}$, we have:*

(a) for all $x, y, z \in X$ and $a > 1$, $\Omega(T_i x, T_j y, T_k z) \geq a\Omega(x, y, z)$;

(b) for every $x, y, z \in X$ with $y \neq T_n y, n \in \mathbb{N}$,

$$\inf\{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0.$$

Then $\{T_n\}$ has a unique common fixed point u in X and $\Omega(u, u, u) = 0$.

Proof: Follows from Theorem 2.3, by taking $S_n = I_n$ for any $n \in \mathbb{N}$ such that I_n is identity mapping on X. □

Corollary 2.4 *Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete*

G-metric space and Ω *is an* Ω*-distance on* X *such that* X *is* Ω*-bounded. Let* $T_n : X \longrightarrow X$*,* $n \in \mathbb{N}$ *be a non-decreasing sequence of surjective mappings with property that for each* $i, j, k \in \mathbb{N}$*, we have:*

(a) for all $x, y, z \in X$ *and* $a > 1$ *,*

$$\Omega(T_i x, T_j y, T_k z) \geq a \max\{\Omega(x, y, y) + \Omega(y, y, z), \Omega(x, z, z) + \Omega(z, y, z)\},$$

(b) for every $x, y, z \in X$ *with* $y \neq T_n y, n \in \mathbb{N}$*,*

$$\inf\{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0.$$

Then $\{T_n\}$ *has a unique common fixed point* u *in* X *and* $\Omega(u, u, u) = 0$.

Proof: Since by Part (a) of Definition (1.3),

$a \max\{\Omega(x, y, y) + \Omega(y, y, z), \Omega(x, z, z) + \Omega(z, y, z)\} \geq a\Omega(x, y, z)$. So, Theorem 2.3 implies that $\{T_n\}$ has a unique common fixed point u in X and $\Omega(u, u, u) = 0$. □

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