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The combined Sinc-Taylor expansion method to solve Abel's integral equation

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Abstract

In this paper, numerical solution of Abel's integral equation by using the Taylor expansion of the unknown function via collocation method based on sinc is considered. The method converts Abel's integral equation to a system of linear equations for the unknown function. A desired solution can be determined by solving the resulting system. Numerical examples show the accuracy and efficiency of the method.

Key words: Singular integral equation, Abel's integral equation, Sinc collocation method, Single exponential transformation.
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1 Introduction

The Abel's integral equation of the second kind is given by [1]

$$u(x) = f(x) + \int_0^x \frac{u(t)}{(x-t)^\alpha} dt, \quad 0 < \alpha < 1, \quad 0 < t \le x < 1, \qquad (1.1)$$

where α is a constant and $f(x)$ is a given function, $u(t)$ is unknown function and must be determined. Abel's integral equation appears in many different problems of basic and engineering sciences such as physics, chemistry, biology, electronics and mechanics [2,3,4,5]. So far, many approaches have been proposed for determining the numerical solution to Abel's integral equation such as [6,7,8,9,10].

Sinc method is a powerful numerical tool for finding fast and accurate solution in various areas of problems. In [11,12] a full overview of sinc function and appropriate conditions and theorems have been discussed. Recently [13] used sinc method to singular integral equations and take useful and interesting results. In this work, we consider Abel's integral equation of the second kind. The proposed method consists of reducing the Abel's integral equation into a system of algebraic equations, by expanding the unknown functions, as a series in terms of sinc functions with unknown coefficients.

The layout of the paper is as follows: in section 2, we give some basic definitions, assumptions and preliminaries of the sinc approximations and related topics. In section 3, the proposed method to solve Abel's integral equation is applied. Finally, section 4, contains the details of the proposed method and numerical implementation and some experimental results.

2 Basic definitions and preliminaries

Let f be a function defined on $\mathbb{R}$ and $h > 0$ is step size then the Whittaker cardinal defined by the series

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x), \tag{2.1}$$

whenever this series convergence, and

$$S(j,h)(x) = \frac{\sin[\pi(x-jh)/h]}{\pi(x-jh)/h}, j = 0, \pm 1, \pm 2, ..., \tag{2.2}$$

where $S(j,h)(x)$ is known as $j-th$ Sinc function evaluated at x.
To construct sinc approximation on the interval (a,b), which apply in this paper, the eye-shaped :

$$D_E = \{z = x + iy : |arg(\frac{z-a}{b-z})| < d \leq \frac{\pi}{2}\},$$

is mapped onto infinite strip $D_d = \{z = x + iy \quad |y| < d\}$ by $\phi(z) = \ln(\frac{z-a}{b-z})$, which is known as single exponential transformation. So, the basis sinc function on (a,b) is given by

$$S(j,h)o\phi(x) = Sinc(\frac{\phi(x)-jh}{h}),$$

where

$$Sinc(x) = \begin{cases} 1 \text{ if } x = 0 \\ \frac{\sin(\pi x)}{\pi x} \text{ if } x \neq 0. \end{cases}$$

Definition 1. [11] Let $B(D_E)$ be the class of functions F which are analytical in D_E, satisfy

$$\int_{\psi(s+L)} |F(z)dz| \to 0, \quad s \to \pm\infty,$$

where $L = iv : |v| < d \leq \frac{\pi}{2}$, and

$$N(F) = \int_{\partial D_E} |F(z)dz| < \infty.$$

.

Theorem 1. [11,14] If $|F(x)| \leq Ce^{-\alpha|\phi(x)|}$, $x \in \Gamma$, for positive constants C, α be selecting $h = \sqrt{\frac{\pi d}{\alpha N}}$, we have the following interpolation relation:

$$|F(x) - \sum_{j=-N}^{N} F(x_j) S(j,h) o\phi(x)| \leq C\sqrt{N} e^{-\sqrt{\pi d \alpha N}}, \quad x \in \Gamma.$$

3 Main Idea

Consider Eq. (1.1) and using Taylor expansion of $u(t)$ we have

$$u(t) = \sum_{k=0}^{n} \frac{(t-x)^k}{k!} u^{(k)}(x) + E_n, \tag{3.1}$$

where

$$E_n = \frac{(t-x)^{k+1}}{(k+1)!} u^{(k+1)}(\zeta_{t,x}), \quad \zeta_{t,x} \in (x,t). \tag{3.2}$$

Substituting (3.1) in (1.1) results

$$\begin{aligned} u(x) &= f(x) + \int_0^x \frac{1}{(x-t)^\alpha} \sum_{k=0}^{n} \frac{(t-x)^k}{k!} u^{(k)}(x) dt + IE_n \\ &= f(x) + \sum_{k=0}^{n} (-1)^k \frac{u^{(k)}(x)}{k!} \int_0^x (x-t)^{k-\alpha} dt + IE_n, \end{aligned} \tag{3.3}$$

where

$$IE_n = \int_0^x E_n \; dt. \tag{3.4}$$

Let

$$I_{\alpha,k} = \int_0^x (x-t)^{k-\alpha} dt, \quad k = 0, 1, 2, ..., n. \tag{3.5}$$

by easy computations it follows

$$I_{\alpha,0} = \int_0^x \frac{dt}{(x-t)^\alpha} = \frac{x^{1-\alpha}}{1-\alpha}. \tag{3.6}$$

and

$$I_{\alpha,k} = \int_0^x (x-t)^{k-\alpha} dt = \frac{1}{(k-\alpha)+1} x^{(k-\alpha)+1}, \quad k = 1, 2, ..., n. \tag{3.7}$$

Set

$$w_k(x) = \begin{cases} I_{\alpha,0} & \text{if } k = 0 \\ (-1)^k I_{\alpha,k} & \text{if } k \neq 0. \end{cases}$$

So we have

$$u(x) = f(x) + \sum_{k=0}^{n} \frac{u^{(k)}(x)}{k!} w_k(x). \tag{3.8}$$

To approximate $u^{(k)}(x)$ in (3.8) based on sinc functions we consider the following numerical expansion

$$u_m(x) = \sum_{j=-M}^{M} u_j S(j,h) o\phi(x), \quad m = 2M+1. \tag{3.9}$$

By applying (3.9) in (3.8) and ignoring IE_n term we get

$$u_m(x) = f(x) + \sum_{k=0}^{n} \frac{u_m^{(k)}(x)}{k!} w_k(x). \tag{3.10}$$

or equivalently

$$\sum_{j=-M}^{M} u_j S(j,h) o\phi(x) = f(x) + \sum_{k=0}^{n} \frac{w_k(x)}{k!} \{ \sum_{j=-M}^{M} u_j [S(j,h) o\phi(x)]^{(k)} \}. \tag{3.11}$$

Also

$$[S(j,h) o\phi(x)]^{(k)} = \frac{d^k}{d\phi^k} S(j,h) o\phi(x) \ \frac{d^k \phi}{dx^k}. \tag{3.12}$$

To find unknown $u_j, j = -M, ..., M$, we can apply the sinc collocation points $x_i = \phi^{-1}(ih), i = -M, ..., M$.

Define

$$\delta_{jk}^{(r)} = \frac{d^r}{d\phi^r} S(j,h) o\phi(x)|_{x_k = \phi^{-1}(kh)}, \quad r = 0, 1, 2, ..., n. \tag{3.13}$$

by easy computation it comes [11]

$$\delta_{jk}^{(2r)} = \frac{1}{h^{2r}} \begin{cases} \frac{(-1)^r \pi^{2r}}{2r+1} & \text{if } j = k \\ \frac{(-1)^{j-k}}{(j-k)^{2r}} \sum_{s=0}^{r-1} \frac{(-1)^{s+1}(2r)!}{(2s+1)!} \pi^{2s}(j-k)^{2s} & \text{if } j \neq k, \end{cases}$$

and

$$\delta_{jk}^{(2r+1)} = \frac{1}{h^{2r+1}} \begin{cases} 0 & \text{if } j = k \\ \frac{(-1)^{j-k}}{(j-k)^{2r+1}} \sum_{s=0}^{r} \frac{(-1)^{s}(2r+1)!}{(2s+1)!} \pi^{2s}(j-k)^{2s} & \text{if } j \neq k. \end{cases}$$

For values $k = 0, 1, 2, 3$ we have

$$\begin{aligned} &S(j,h)o\phi(x)|_{x_i} = \delta_{ji}^{(0)}, \\ &\frac{d}{dx} S(j,h)o\phi(x)|_{x_i} = \delta_{ji}^{(1)} \phi'(x_i), \\ &\frac{d^2}{dx^2} S(j,h)o\phi(x)|_{x_i} = \delta_{ji}^{(1)} \phi''(x_i) + (\phi'(x_i))^2 \delta_{ji}^{(2)}, \\ &\frac{d^3}{dx^3} S(j,h)o\phi(x)|_{x_i} = \delta_{ji}^{(1)} \phi'''(x_i) + 3\phi'(x_i)\phi''(x_i)\delta_{ji}^{(2)} + (\phi'(x_i))^3 \delta_{ji}^{(3)}. \end{aligned} \tag{3.14}$$

By applying (3.14) in (3.11) it follows that

$$\sum_{j=-M}^{M} u_j \delta_{ji}^{(0)} = f(x_i) + \sum_{k=0}^{n} \frac{w_k(x_i)}{k!} \{ \sum_{j=-M}^{M} u_j [S(j,h)o\phi(x)]_{x_i}^{(k)} \}. \tag{3.15}$$

By solving the linear system of equations (3.15), we obtain u_j which approximate $u(x)$ at sinc point x_j i.e. $u(x_j)$. The system of linear equations (3.15) has $(2M+1)$ equations and $(2M+1)$ variables which can be expressed in a matrix form

$$\Re(I - A)\tilde{\mathbf{u}} = \tilde{\mathbf{f}}, \tag{3.16}$$

where $\Re$ means real part and

$$A = [\sum_{k=0}^{n} \frac{w_k(x_i)}{k!}\{[S(j,h)o\phi(x)]_{x_i}^{(k)}\}]$$
$$\tilde{\mathbf{u}} = (u_{-M}, ..., u_M)^t, \quad \tilde{\mathbf{f}} = (f(x_{-M}), ..., f(x_M))^t. \tag{3.17}$$

4 Numerical Experiments

In this section, we apply the proposed method to solve the Abel's integral equation. All programs have been provided by Maple 13 with $d = \frac{\pi}{2}$ and $\alpha = 1$. Also, in order to show the error and the accuracy of approximation, we apply the following criteria:

1)Absolute error between the exact and approximated solution (L_∞error norm)is defined for $2N + 1$ by

$$\|.\|_\infty = Max_{i=-N..N}|u(x_i) - u_i|. \tag{4.1}$$

2)Run time of program which is showed by T(s),(s means second).

Example 1. Consider the following Abel's integral equation [1,9,**?**]:

$$u(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \qquad 0 < t \leq x < 1. \tag{4.2}$$

with the exact solution: $u(x) = x^2$.
To obtain results, we take different values of M as $M = 5, 10, 15, 18$ with $n = 3$. As seen in table 1, infinity norm (column $\|.\|_\infty$)decreases as M increases. The results of example 1 are showed in table 1.

M	$T(s)$	$\|.\|_\infty$
5	44.51	8.36E-004
10	96.37	7.42E-005
15	135.12	3.25E-007
18	141.68	9.41E-008

Table 1. Results of example 1 by sinc method.

In table 2, errors for different points are reported, in comparison with the results of [**?**], infinity norm in each row is smaller than [**?**]. Also, run time of program is remarkable. For example, in the sinc method for $M = 18$ we have a 37×37 system of linear equations. Figure 1 shows the exact and approximate solution of example 1. Figure 2, shows convergence behavior of the method in terms of infinity norm versus reciprocal of number of collocation points M. As shown in figure 2, infinity norm decreases by increasing the number of collocation points.

[15]

x_i	Ref	$Proposed\ Method$
0.0	1.8E-007	6.36E-010
0.2	1.9E-007	7.42E-009
0.4	3.3E-007	9.25E-008
0.6	2.8E-006	3.41E-008
0.8	1.8E-006	2.41E-007
1.0	4.8E-008	1.41E-008

Table 2. Results of example 1 by proposed method.

Fig. 1. The exact and the approximate solutions of example 1 with $M = 5$.

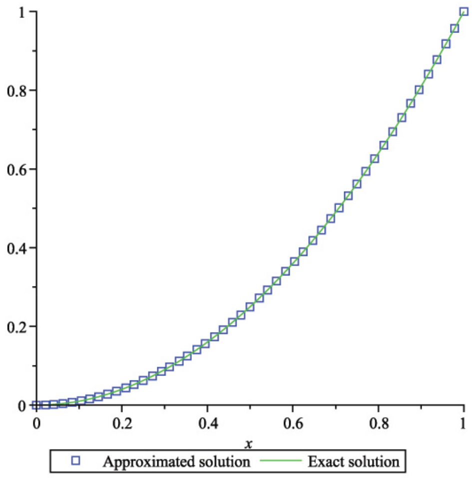
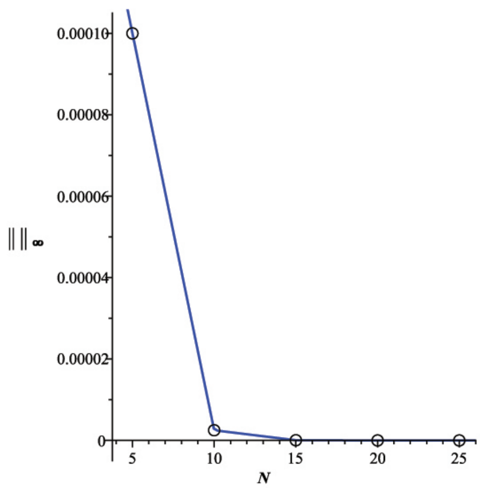


Fig. 2. The plot of error (with infinity norm) versus M in example 1.



Example 2. Consider the integral equation [1]:

$$u(x) = x^2(1 - x^2) - \frac{729}{15400}x^{\frac{14}{3}} + \frac{243}{2200}x^{\frac{11}{3}} - \frac{27}{400}x^{\frac{8}{3}} + \frac{1}{10}\int_0^x \frac{u(t)}{(x-t)^{\frac{1}{3}}}dt. \tag{4.3}$$

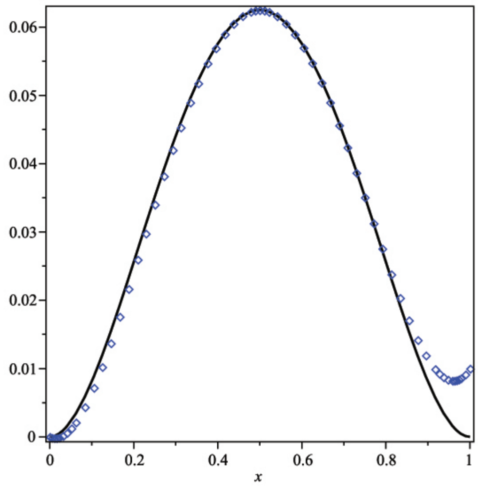
with the exact solution: $u(x) = x^4 - 2x^3 + x^2$.

Results in table 3, show good approximation based on sinc collocation method. Although number of derivatives is $n = 3$, but infinity norm and run time is suitable. Figure 3, shows difference between the exact and approximate solution of example 2. Also, figure 4, shows convergence behavior of the proposed method.

M	$T(s)$	$\|.\|_\infty$
5	23.24	2.35E-005
10	58.36	6.82E-007
15	89.54	3.25E-009
18	121.35	8.47E-010

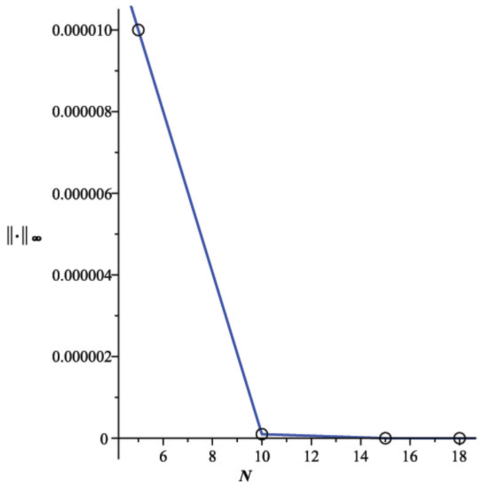
Table 3. Results of example 2 by proposed method.

Fig. 3. The exact and approximate solution of example 2.



However, the results show that the proposed method is practically reliable. Based on results, and other works [16,17] sinc collocation method

Fig. 4. The plot of error (with infinity norm) versus M in example 2.



gives better accuracy at the computational cost, also the implementing and coding are very easy.

5 Conclusion

In this paper, the combined Sinc-Taylor expansion method was applied to solve Abel's integral equation. Examples were presented to illustrate effectiveness of the method. Results show the high accuracy of method and useful properties of sinc method such as storing in time and memory. In addition this method is portable to other area of problems and easy to programming [18].

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