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Some fixed points for J-type multi-valued maps in $CAT(0)$ spaces

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Abstract

In this paper, we prove the existence of fixed point for J-type multi-valued map T in $CAT(0)$ spaces and also we prove the strong convergence theorems for Ishikawa iteration scheme without using the fixed point of involving map.

Key words: Ishikawa iteration scheme; $CAT(0)$ spaces; J-type mapping.

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1 Introduction

Let D be a nonempty subset of metric space $X := (X, d)$. The set D is called proximinal if for each $x \in X$, there exists an element $y \in D$ such that $d(x, y) = d(x, D)$, where $d(x, D) = \inf\{d(x, z) : z \in D\}$. Let $CB(D), P(D)$ denote the family of nonempty closed bounded and nonempty proximinal bounded subsets of D, respectively. The Hausdorff metric on $CB(D)$ is defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

for $A, B \in CB(D)$. A single-valued map $T : D \to D$ is called nonexpansive if $d(T(x), T(y)) \leq d(x, y)$ for $x, y \in D$. A multi-valued map $T : D \to CB(D)$ is said to be nonexpansive, if $H(T(x), T(y)) \leq d(x, y)$ for $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \to D$ (respectively, $T : D \to CB(D)$) if $p = T(p)$ (respectively, $p \in T(p)$). The set of fixed points of T is represented by $F(T)$. The mapping $T : D \to CB(D)$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(T(x), T(p)) \leq d(x, p)$ for all $x \in D$ and all $p \in F(T)$. It is clear that every nonexpansive multi-valued map T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive (see [8]) The following definition is from [5]:

Definition 1.1 *Let (X, d) be a metric space. $y_0 \in X$ is called a center for the mapping $T : D \to X$ if, for each $x \in D$,*

$$d(y_0, T(x)) \leq d(y_0, x).$$

$T : D \to X$ is called a J-type mapping, whenever it is continuous and it has some center $y_0 \in X$. In this case, $Z(T)$, denote the set of all centers of the mapping T.

Of course, if a mapping $T : D \to X$ has a center $y_0 \in D$, then trivially $T(y_0) = y_0$. Thus, fixed point results for J-type mappings are only nontrivial provided they have a center $y_0 \notin D$. It turns out that this class contains all contractions defined in closed sets of Banach spaces and even all the so called quasi-nonexpansive mappings (i.e. those for which every fixed point is a center).

Definition 1.2 *Let D be a bounded closed convex subset of a metric space (X,d). We say that $y_0 \in X$ is a center for a mapping $T : D \to CB(X)$ if, for each $x \in D$,*

$$H(T(x), \{y_0\}) \leq d(y_0, x),$$

we will say that $T : D \to CB(X)$ is a J-type mapping whenever it is upper semicontinuous and has some center $y_0 \in X$.

Of course, if a mapping $T : D \to CB(X)$ has a center $y_0 \in D$, then trivially $H(T(y_0), \{y_0\}) = 0$, that is, $T(y_0) = \{y_0\}$, which means that y_0 is a stationary point for T.

$T : D \to CB(D)$ is called hemicompact if, for any $\{x_n\}$ in D such that $d(x_n, T(x_n)) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in D$.

$T : D \to CB(D)$ is said to satisfy condition (I) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x, T(x)) \geq f(d(x, F(T)))$$

for all $x \in D$.

$T : D \to CB(X)$ is said to satisfy condition (II) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x, T(x)) \geq f(d(x, Z(T)))$$

for all $x \in D$.

(A)Let $T : D \to D$ be a single - valued mapping. The Ishikawa iteration scheme, starting from $x_0 \in D$, is the sequence $\{x_n\}$ defined by

$$y_n = \beta_n T(x_n) + (1-\beta_n)x_n, \quad \beta_n \in [0,1], \quad n \geq 0$$

$$x_{n+1} = \alpha_n T(y_n) + (1-\alpha_n)x_n, \quad \alpha_n \in [0,1], \quad n \geq 0$$

(B)Let $T : D \to P(D)$ be a multi-valued mapping and fix $p \in F(T)$. Sastry and Babu [6] defined the Ishikawa iteration scheme for this multi-valued mapping as below by $x_0 \in D$

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \quad \beta_n \in [0,1], \quad n \geq 0$$

where $z_n \in T(x_n)$, and

$$x_{n+1} = \alpha_n \acute{z}_n + (1-\alpha_n)x_n, \quad \alpha_n \in [0,1], \quad n \geq 0$$

where $\acute{z}_n \in T(y_n)$ such that $d(\acute{z}_n, p) = d(p, T(y_n))$.
They proved the Ishikawa iteration scheme for a multi-valued map T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. More precisely, they proved the following results for nonexpansive multi-valued map with compact domain.

Theorem 1.1 *Let E be a Hilbert space, K a nonempty compact convex subset of E, and $T : K \longrightarrow P(K)$ a nonexpansive map with a fixed point p. Assume (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n \beta_n = \infty$.Then the Ishkawa iterates $\{x_n\}$ defined by (B) convergence to a fixed point of T.*

Song and Wang[9] following Ishikawa iteration scheme:
(C)Let K be a nonempty convex subset of $X, \alpha_n, \beta_n \in [0,1]$ and $\gamma_n \in (0,\infty)$ such that $lim_{n\to\infty}\gamma_n = 0$. Choose $x_0 \in K$.Then

$$y_n = \beta_n z_n + (1-\beta_n)x_n,$$

$$x_{n+1} = \alpha_n \acute{z}_n + (1-\alpha_n)x_n,$$

where $||z_n - \acute{z}_n|| \leq H(T(x_n), T(y_n)) + \gamma_n$ and

$$||z_{n+1} - \acute{z}_n|| \leq H(T(x_{n+1}), T(y_n)) + \gamma_n$$

for $z_n \in T(x_n)$ and $\acute{z}_n \in T(y_n)$. Song and Wang[9] proved the following results. In the results, the domain of T is still compact.

Theorem 1.2 *Let E be a uniformly convex Banach space, K a nonempty compact convex subset of E, and $T : K \longrightarrow CB(K)$ a nonexpansive map with $F(T) \neq \emptyset$. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n \beta_n = \infty$.Then the Ishkawa iterates $\{x_n\}$ defined by (C) convergence to a fixed point of T.*

Recently Shahzad and Zegeye [8] introduced the modified Ishikawa iteration scheme as follows: (D)Let K be a nonempty convex subset of Banach spaces X.The Ishikawa iteration scheme is defined by $x_0 \in K$

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \quad \beta_n \in [0,1], \quad n \geq 0$$

where $z_n \in T(x_n)$, and

$$x_{n+1} = \alpha_n \acute{z}_n + (1-\alpha_n)x_n, \quad \alpha_n \in [0,1], \quad n \geq 0$$

where $\acute{z}_n \in T(y_n)$.
(E)Let $T : K \to P(K)$ be a multi-valued map, $P_T(x) = \{y \in T(x) : ||x-y|| = d(x, T(x))\}$. The Ishikawa iteration scheme for this multi-valued mapping as below by $x_0 \in K$

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \quad \beta_n \in [0,1], \quad n \geq 0$$

where $z_n \in P_T(x_n)$, and

$$x_{n+1} = \alpha_n \acute{z}_n + (1-\alpha_n)x_n, \quad \alpha_n \in [0,1], \quad n \geq 0$$

where $\acute{z}_n \in P_T(y_n)$. They also proved some results on the strong convergence of the sequence defined by (D) and(E).

2 $CAT(0)$ spaces

Recently, $CAT(0)$ spaces has been rapidly developed, and many papers have appeared (see [1], [7]). In this section introduce some definition and properties of $CAT(0)$ spaces.
Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(\acute{t}) = |t - \acute{t}|$ for all $t, \acute{t} \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$.

The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x to y for each $x, y \in X$. A subset $Y \subseteq X$ is said to convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of$\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane E^2 such that $d_{E^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space is said to be a $CAT(0)$ space [2] if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$: $d(x, y) \leq d_{E^2}(\bar{x}, \bar{y})$.

It is known that in a $CAT(0)$ space, the distance function is convex [2]. Complete $CAT(0)$ spaces are often called Hadamard spaces. Finally we observe that if x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies

$$d(x, \frac{y_1 \oplus y_2}{2})^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2 \qquad (2.1)$$

because equality holds in the Euclidean metric. In fact (see [2], p. 163), a geodesic metric space is a $CAT(0)$ space if and only if it satisfies inequality 2.1 (which is known as the CN inequality of Bruhat and Tits [3]).
The following lemma is a generalization of (CN) inequality which can be found in [4].

Lemma 2.1 *Let (X, d) be a $CAT(0)$ space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2$$

for all $t \in [0, 1]$ and $x, y, z \in X$.

It is worth mentioning that the results in $CAT(0)$ spaces can be applied to any $CAT(\kappa)$ space with $\kappa \leq 0$, since any $CAT(\kappa)$ space is a $CAT(\acute{\kappa})$ space for every $\acute{\kappa} \geq \kappa$(see [2] p165).

3 Ishikawa iteration schemes

We use the following iteration scheme.
(F) Let D be a nonempty convex subset of a $CAT(0)$ space X and $\alpha_n, \beta_n \in [0,1]$. The sequence of Ishikawa iterates is defined by $x_0 \in D$,

$$y_n = \beta_n z_n \oplus (1-\beta_n)x_n, \quad n \geq 0,$$

where $z_n \in T(x_n)$, and

$$x_{n+1} = \alpha_n \acute{z}_n \oplus (1-\alpha_n)x_n, \quad n \geq 0,$$

where $\acute{z}_n \in T(y_n)$.

Lemma 3.1 *Let X be a Complete $CAT(0)$ space, D a nonempty closed convex subset of X and $T : D \to CB(X)$ be J-type multi-valued map. Let $\{x_n\}$ be the Ishkawa iterates (F). Then $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in Z(T)$.*

Proof. Let $p \in Z(T)$. Then, using (H), we have

$$\begin{aligned}
d(y_n, p) &= d(\beta_n z_n \oplus (1-\beta_n)x_n, p) \\
&\leq \beta_n d(z_n, p) + (1-\beta_n)d(x_n, p) \\
&\leq \beta_n d(z_n, \{p\}) + (1-\beta_n)d(x_n, p) \\
&\leq \beta_n H(T(x_n), \{p\}) + (1-\beta_n)d(x_n, p) \\
&\leq \beta_n d(x_n, p) + (1-\beta_n)d(x_n, p) \\
&\leq d(x_n, p)
\end{aligned}$$

and

$$
\begin{aligned}
d(x_{n+1},p) &= d(\alpha_n \acute{z}_n + (1-\alpha_n)x_n, p)\\
&\leq \alpha_n d(\acute{z}_n, p) \oplus (1-\alpha_n)d(x_n,p)\\
&\leq \alpha_n d(\acute{z}_n, \{p\}) + (1-\alpha_n)d(x_n,p)\\
&\leq \alpha_n H(T(y_n), \{p\}) + (1-\alpha_n)d(x_n,p)\\
&\leq \alpha_n d(y_n,p) + (1-\alpha_n)d(x_n,p)\\
&\leq \alpha_n d(x_n,p) + (1-\alpha_n)d(x_n,p)\\
&\leq d(x_n,p).
\end{aligned}
$$

Consequently, the sequence $\{d(x_n,p)\}$ is decreasing and bounded below and thus
$\lim_{n\to\infty} d(x_n,p)$ exists for any $p \in Z(T)$. Also $\{x_n\}$ is bounded.

Theorem 3.1 *Let X be a Complete $CAT(0)$ space, D a nonempty closed convex subset of X and $T : D \to CB(X)$ be J-type multi-valued map. Let $\{x_n\}$ be the Ishikawa iterates (F). Assume that T satisfies condition (II) and $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Then T has a fixed point and also $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Let $p \in Z(T)$. Then, as in the proof of lemma 3.1, $\{x_n\}$ is bounded and so $\{y_n\}$ is bounded, therefore, there exists $R > 0$ such that $x_n - p, y_n - p \in \mathbf{B}_R(0)$ for all $n \geq 0$. Applying Lemma 2.1, we have

$$
\begin{aligned}
d(x_{n+1},p)^2 &= d(\alpha_n \acute{z}_n \oplus (1-\alpha_n)x_n, p)^2\\
&\leq \alpha_n d(\acute{z}_n, p)^2 + (1-\alpha_n)d(x_n,p)^2 - \alpha_n(1-\alpha_n)d(x_n,\acute{z}_n)\\
&\leq \alpha_n H(T(y_n), \{p\})^2 + (1-\alpha_n)d(x_n,p)^2 - \alpha_n(1-\alpha_n)d(x_n,\acute{z}_n)\\
&\leq \alpha_n d(y_n,p)^2 + (1-\alpha_n)d(x_n,p)^2.
\end{aligned}
$$

it follows that

$$
\begin{aligned}
d(y_n, p)^2 &= d(\beta_n z_n \oplus (1-\beta_n)x_n, p)^2 \\
&\le \beta_n d(z_n, p)^2 + (1-\beta_n)d(x_n, p)^2 - \beta_n(1-\beta_n)d(x_n, z_n) \\
&\le \beta_n H(T(x_n), \{p\})^2 + (1-\beta_n)d(x_n, p)^2 - \beta_n(1-\beta_n)d(x_n, z_n) \\
&\le \beta_n d(x_n, p)^2 + (1-\beta_n)d(x_n, p)^2 - \beta_n(1-\beta_n)d(x_n, z_n) \\
&\le d(x_n, p)^2 - \beta_n(1-\beta_n)g(d(x_n, z_n))\beta_n)d(x_n, z_n).
\end{aligned}
$$

So

$$
d(x_{n+1}, p)^2 \le \alpha_n d(x_n, p)^2 \oplus (1-\alpha_n)d(x_n, p)^2 - \alpha_n\beta_n(1-\beta_n)d(x_n, z_n).
$$

This implies that

$$
a^2(1-b)d(x_n, z_n) \le \alpha_n\beta_n(1-\beta_n)d(x_n, z_n) \le d(x_n, p)^2 - d(x_{n+1}, p)^2
$$

and so

$$
\sum_{n=1}^{\infty} a^2(1-b)d(x_n, z_n) < \infty.
$$

Thus, $\lim_{n\to\infty} d(x_n, z_n) = 0$. Also $d(x_n, T(x_n)) \le d(x_n, z_n) \to 0$ as $n \to \infty$. Since T satisfies condition (II) , we have $\lim_{n\to\infty} d(x_n, Z(T)) = 0$. Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, p_k) < \frac{1}{2^k}$ for some $\{p_k\} \subset Z(T)$ and all k. Note that by Lemma 3.1 we obtain

$$
d(x_{n_{k+1}}, p_k) \le (x_{n_k}, p_k) < \frac{1}{2^k}.
$$

We now show that $\{p_k\}$ is a Cauchy sequence in D. Notice that

$$
\begin{aligned}
d(p_{k+1}, p_k) &\le d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\
&< \tfrac{1}{2^{k+1}} + \tfrac{1}{2^k} \\
&< \tfrac{1}{2^{k-1}}.
\end{aligned}
$$

This shows that $\{p_k\}$ is a Cauchy sequence in D (because of closedness of D) and converges to $q \in D$. Since

$$\begin{aligned} d(p_k, T(q)) &\le H(T(q), \{p_k\}) \\ &\le d(q, p_k) \end{aligned}$$

and $p_k \to q$ as $k \to \infty$, it follows that $d(q, T(q)) = 0$ and thus $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Since $Z(T)$ is closed, $q \in Z(T)$ and $\lim_{n\to\infty} d(x_n, q)$ exists, it follows that $\{x_n\}$ converges strongly to q. As we see in [8] the author proved Theorem 2.5 with replacing condition (I) by the hemicompactness of T, so one can replace condition (II) in Theorem 3.1 by the hemicompactness of T in the same way and show that $\{x_n\}$ converges to a fixed point of T.

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