Theory of Approximation and Applications

Vol.11 , No.2, (2017), 15-35

Viscosity Approximation Methods for W-mappings in Hilbert space

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Received 7 June 2016; accepted 20 December 2016

Abstract

We suggest a explicit viscosity iterative algorithm for finding a common element of the set of common fixed points for W-mappings which solves some variational inequality. Also, we prove a strong convergence theorem with some control conditions. Finally, we apply our results to solve the equilibrium problems. Finally, examples and numerical results are also given.

Key words: Nonexpansive mapping, equilibrium problems, strongly positive linear bounded operator, fixed point, Hilbert space, W-mapping.

2010 AMS Mathematics Subject Classification : 47H09, 47H10, 47J20

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1 Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. A mapping T of C into itself is nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for each $x, y \in C$, we denote F(T) the set of fix points of T. The strong(weak) convergence of $\{x_n\}$ to x is written by $x_n \to x$ ($x_n \rightharpoonup x$) as $n \to \infty$.
For any $x \in H$, there exists a unique nearest pointin C, denoted it by $P_C(x)$ such that

$$\|x - P_C x\| \leq \|x - y\|, \text{ for all } y \in C,$$

such that a mapping P_C from H onto C is called the *metric projection*.
Recall that H satisfies the Opial's condition [6] if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H$ with $x \neq y$. A self mapping $f : C \to C$ is a contraction if there exists $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for each $x, y \in C$.
An operator A is said to be a strongly positive linear bounded operator on H, if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \text{ for all } x \in H.$$

Let F be a bifunction of $C \times C$ into R. The equilibrium problems for $C \times C \to C$, is to find $x \in C$ such that

$$F(x, y) \geq 0, \text{ for all } y \in C. \tag{1.1}$$

The set of solution of Eq.(1.1) is denoted by $EP(F)$. Several problems in physics, optimization, and economics reduce to find a solution of Eq.(1.1)

[1], [4]. We consider the following iteration [10]

$$
\begin{aligned}
U_{n,n+1} &:= I, \\
U_{n,n} &:= \lambda_n S_n U_{n,n+1} + (1-\lambda_n)I, \\
U_{n,n-1} &:= \lambda_{n-1} S_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\
&\ \vdots \\
U_{n,k} &:= \lambda_k S_k U_{n,k+1} + (1-\lambda_k)I, \\
&\ \vdots \\
U_{n,2} &:= \lambda_2 S_2 U_{n,3} + (1-\lambda_2)I, \\
W_n = U_{n,1} &:= \lambda_1 S_1 U_{n,2} + (1-\lambda_1)I,
\end{aligned}
\tag{1.2}
$$

where $\lambda_1, \lambda_2, \dots$ are real numbers such that $0 \leq \lambda_n \leq 1$, and $S_1, S_2, \dots$ be an infinite nonexpansive mappings. It is clear that nonexpansiviety of each S_n ensure the nonexapnsivity of W_n. Such a mapping W_n is called $W-$ mapping generated by $S_n, S_{n-1}, \dots, S_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$.
In this paper, by intuition from [7], a new iterative scheme is introduced. This scheme find a common solution of the equilibrium problem (EP) and fixed point problem for an infinite family of nonexpansive mappings. Also, we prove a strong convergence theorem.
The following lemmas will be useful for proving the main results of this aticle:

Lemma 1.1 *[8] Let C be a nonempty closed convex subset of a Banach space E and $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0,1)$. For any $n \geq 1$, let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, \dots, S_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. Then W_n is asymptotically regular and nonexpansive. Further, if E is strictly convex, then $F(W_n) = \bigcap_{i=1}^{n} F(S_i)$.*

Lemma 1.2 *[8] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0,1)$. Then, for every $x \in C$*

and $k \geq 1$ $\lim\limits_{n\to\infty} U_{n,k}x$ *exists.*

Lemma 1.3 *[8] Let C be a nonempty closed convex subset of stricly convex Banach E .Let $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then W is a nonexpansive mapping and $F(W) = \bigcap_{n=1}^{\infty} F(S_n)$.*

Lemma 1.4 *[2] Let C be a nonempty closed convex subset of a Hilbert space H and $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. If $K$is any bounded subset of C, then* $\limsup\limits_{n\to\infty} \|Wx - W_nx\| = 0$.

Lemma 1.5 *[5] Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| \leq I - \rho\overline{\gamma}$.*

Lemma 1.6 *[9] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n+\beta_nx_n$ for all integers $n \geq 1$ and $\limsup\limits_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim\limits_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 1.7 *[2] Let H be a real Hilbert space. Then the following holds:*

(a) $\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y\rangle$ *for all* $x, y \in H$,
(b) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$,
(c) $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$.

Lemma 1.8 *[1] Let K be a nonempty closed convex subset of H and F be a bi-function of $K \times K$ into R satisfying the following conditions:*

(A1) $F(x, x) = 0$ *for all* $x \in K$,
(A2) F *is monotone, that is,* $F(x, y) + F(y, x) \leq 0$ *for all* $x, y \in K$,
(A3) for each $x, y, z \in K$

$$\lim_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y),$$

(A4) *for each* $x \in K$, $y \mapsto F(x,y)$ *is convex and lower semi-continuous.Let* $r > 0$ *and* $x \in H$.
Then, there exists $z \in K$ *such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K.$$

Lemma 1.9 *[3] Let* K *be a nonempty closed convex subset of* H *and let* F *be a bifunction of* $K \times K$ *into* R *satisfying* $(A1) - (A4)$. *For* $r > 0$ *and* $x \in H$ *, define a mapping* $T_r : H \to K$ *as follows:*

$$T_r(x) = \{z \in K : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K\},$$

for all $x \in H$. *Then the following hold*

(i) T_r *is single valued map,*
(ii) T_r *is firmly nonexpansive, that is , for any* $x, y \in H$

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle,$$

(iii) $F(T_r) = EP(F)$,
(iv) $EP(F)$ *is closed and convex.*

Lemma 1.10 *[11] Assume* $\{a_n\}$ *be a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n,$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in real number such that*

(i) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$,

then $\lim_{n\to\infty} a_n = 0$.

2 Explicit viscosity iterative algorithm

In this section, a new iterative scheme for finding a common element of the set of solutions for a equilibrium problems and the set of common fixed point for an infinite family of mappings in Hilbert space, is introduced.

Theorem 2.1 *Let*

- *C be a nonempty closed convex subset of a real Hilbert space H,*
- *f be a $\rho-$contractive map on C,*
- *$J = \{1, 2, \ldots, k\}$ be a finite index set,*
- *For each $i \in J$, let G_i be a bifunction from $C \times C$ into R satisfying $(A1) - (A4)$,*
- *A be a strongly positive linear bounded operator on H with coefficient $\varpi > 0$,*
- *$\{S_n\} : H \to H$ be a family of infinite nonexpansive mappings,*
- *$\bigcap_{i=1}^{k} F(W) \bigcap EP(G_i) \neq \emptyset$ where $F(W) = \bigcap_{j=1}^{n} F(S_j)$,*
- *$\{x_n\}$ be the sequence generated as following :*

$$\begin{cases} G_1(u_{n,1}, y) + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n\rangle \geq 0, \\ G_2(u_{n,2}, y) + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n\rangle \geq 0, \\ \vdots \\ G_k(u_{n,k}, y) + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n\rangle \geq 0, \\ \theta_n = \frac{1}{k}\sum_{i=1}^{k} u_{n,i}, \\ y_n = \beta_n \gamma f(\theta_n) + (I - \beta_n A)\theta_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) W_n y_n, \end{cases}$$

where $\{W_n\}$ is a sequence defined by Eq.(1.2). Also, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, $r_n \subset (0, \infty)$ and $0 < \gamma < \frac{\varpi}{\rho}$.

Suppose

(C1): $\lim_{n\to\infty}\beta_n = 0,\ \sum_{n=1}^{\infty}\beta_n = \infty,$
(C2): $\liminf_{n\to\infty}\ r_n > 0,\quad \lim_{n\to\infty}(r_{n+1} - r_n) = 0;$
(C3): $0 < \liminf_{n\to\infty}\ \alpha_n \le \limsup_{n\to\infty}\ \alpha_n < 1,$
(C4): *for each* $i = 1, 2, \dots, k\ \ 0 < \lambda_i \le c < 1.$

Then

(i) *the sequence* $\{x_n\}$ *is bounded.*
(ii) $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$
(iii) $\lim_{n\to\infty}\|W_n y_n - y_n\| = 0.$

Proof. From (C1), we may assume that $\beta_n \le \|A\|^{-1}$ for all $n \ge 1$. By Lemma 1.5, we obtain $\|I - \beta_n A\| \le 1 - \beta_n\varpi$. It is clear that $P_{\bigcap_{i=1}^{k} F(W)\bigcap EP(G_i)}(I - A - \gamma f)$ is a contraction of C into itself. Indeed, for all $x, y \in C$

$$
\begin{aligned}
\|P_{\bigcap_{i=1}^{k} F(W)\bigcap EP(G_i)}(x) - & P_{\bigcap_{i=1}^{k} F(W)\bigcap EP(G_i)}(y)\| \\
&\le \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\
&\le \|I - A\|\|x - y\| + \gamma\|f(x) - f(y)\| \\
&\le (1 - \varpi)\|x - y\| + \gamma\rho\|x - y\| \\
&= (1 - (\varpi - \gamma\rho))\|x - y\|.
\end{aligned}
$$

(i): Let $x^* \in \bigcap_{i=1}^{k} F(W)\bigcap EP(G_i)$. Since $u_{n,i} = T_{r_n,i}x_n$ and $x^* = T_{r_n,i}x^*$, we see for any $n \ge N$

$$\|u_{n,i} - x^*\| = \|T_{r_n,i}x_n - T_{r_n,i}x^* \le \|x_n - x^*\|\|,$$

thus

$$\|\theta_n - x^*\| \le \|x_n - x^*\|. \tag{2.1}$$

Since f is $\rho-$contraction, we have

$$\begin{aligned}\|y_n - x^*\| &= \|\beta_n\gamma f(\theta_n) + (I - \beta_n A)\theta_n - x^*\| \\ &= \|\beta_n(\gamma f(\theta_n) - Ax^*) + (I - \beta_n A)(\theta_n - x^*)\| \\ &\leq \beta_n\|\gamma f(\theta_n) - Ax^*\| + \|I - \beta_n A\|\|\theta_n - x^*\| \\ &\leq \beta_n\gamma\|f(\theta_n) - f(x^*)\| \\ &+ \beta_n\|\gamma f(x^*) - Ax^*\| + (1 - \beta_n)\varpi\|x_n - x^*\|.\end{aligned}$$

From which it follows that

$$\|y_n - x^*\| \leq (1 - \beta_n(\varpi - \gamma\rho))\|x_n - x^*\| + \beta_n\|\gamma f(x^*) - Ax^*\|. \quad (2.2)$$

In viwe of Eq. (2.1) and Eq.(2.2), we obtain that

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n x_n + (1 - \alpha_n)W_n y_n - x^*\| \\ &= \|\alpha_n(x_n - x^*) + (1 - \alpha_n)(W_n y_n - x^*)\| \\ &\leq \alpha_n\|x_n - x^*\| + (1 - \alpha_n)\|y_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + (1 - \alpha_n)\{(1 - \beta_n(\varpi - \gamma\rho))\|x_n - x^*\| \\ &\quad +\beta_n\|\gamma f(x^*) - Ax^*\|\} \\ &= (1 - \alpha_n)(1 - \beta_n(\varpi - \gamma\rho))\|x_n - x^*\| \\ &\quad +\beta_n(\varpi - \gamma\rho)(1 - \alpha_n)\frac{\|f(x^*) - Ax^*\|}{\varpi - \gamma\rho}.\end{aligned}$$

It follows by induction that

$$\|x_{n+1} - x^*\| \leq max\ \{\|x_1 - x^*\|, \frac{\|f(x^*) - Ax^*\|}{\varpi - \gamma\rho}\}.$$

Therefore, the sequence $\{x_n\}$ is bounded and also $\{y_n\}, \{\theta_n\}$ are bounded.
(ii): Notic that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|(I - \beta_{n+1}A)(\theta_{n+1} - \theta_n) + (\beta_n - \beta_{n+1})A\theta_n \\ &\quad +\gamma\{\beta_{n+1}(f(\theta_{n+1}) - f(\theta_n)) + f(\theta_n)(\beta_{n+1} - \beta_n)\}\| \\ &\leq (1 - \beta_{n+1}\varpi)\|\theta_{n+1} - \theta_n\| + |\beta_n - \beta_{n+1}|\|A\theta_n\| \\ &\quad +\gamma\beta_{n+1}\rho\|\theta_{n+1} - \theta_n\| + \gamma|\beta_{n+1} - \beta_n|\|f(\theta_n)\|.\end{aligned}$$

It follows that

$$\|y_{n+1} - y_n\| \leq (1 - \beta_{n+1}(\varpi - \gamma\rho))\|\theta_{n+1} - \theta_n\| + |\beta_{n+1} - \beta_n|M. \quad (2.3)$$

where $M = Sup_{\ n\geq 1}\ \{\|A\theta_n\| + \|\gamma(\theta_n)\|\}$.

Moreover, we have

$$G_i(u_{n+1,i}, u_{n,i}) + \frac{1}{r_{n+1}}\langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1}\rangle \geq 0, \tag{2.4}$$

$$textbf1 \geq i \geq k. \tag{2.5}$$

and

$$G_i(u_{n,i}, u_{n+1,i}) + \frac{1}{r_n}\langle u_{n+1,i} - u_{n,i}, u_{n,i} - x_n\rangle \geq 0. \tag{2.6}$$

Combining Eq.(2.5) and Eq.(2.6), we obtain

$$\begin{aligned} 0 &\leq r_{n+1}\{G_i(u_{n+1,i}, u_{n,i}) + G_i(u_{n,i}, u_{n+1,i})\} \\ &\quad + \langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} - \frac{r_{n+1}}{r_n}(u_{n,i} - x_n)\rangle \\ &\leq \langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} - \frac{r_{n+1}}{r_n}(u_{n,i} - x_n)\rangle, \end{aligned}$$

from which it follows that

$$\langle u_{n,i} - u_{n+1,i}, u_{n,i} - u_{n+1,i} + x_{n+1} - x_n + x_n - u_{n,i} + \frac{r_{n+1}}{r_n}(u_{n,i} - x_n)\rangle \leq 0 \tag{2.7}$$

which implies that

$$\|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_n}\|x_n - u_{n,i}\|. \tag{2.8}$$

Using the condition (C2) and noting that there exists $b > 0$ such that $r_n > b > 0$, we obtain

$$\|\theta_{n+1} - \theta_n\| \leq \frac{1}{k}\sum_{i=1}^{k}\|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_n}\acute{M} \tag{2.9}$$

and

$$\|\theta_{n+1}-\theta_n\| \leq \|x_{n+1}-x_n\| + \frac{\acute{M}}{b}|r_{n+1}-r_n|, \tag{2.10}$$

where $\acute{M} := \frac{1}{k}\sum_{i=1}^{k}\|x_n - u_{n,i}\| < \infty$.
Moreover, we note that

$$\begin{aligned}
\|W_{n+1}y_n - W_ny_n\| &= \|\lambda_1 S_1 U_{n+1,2}y_n + (1-\lambda_1)y_n \\
&\quad -(\lambda_1 S_1 U_{n,2}y_n + (1-\lambda_1)y_n)\| \\
&\leq \lambda_1\|U_{n+1,2}y_n - U_{n,2}y_n\| \\
&\leq \lambda_1\|\lambda_2 S_2 U_{n+1,3}y_n + (1-\lambda_2 y_n) \\
&\quad -(\lambda_2 S_2 U_{n,3}y_n + (1-\lambda_2)y_n)\| \\
&\leq \lambda_1\lambda_2\|U_{n+1,3}y_n - U_{n,3}y_n\| \\
&\quad \vdots \\
&\leq (\prod_{m=1}^{n}\lambda_m)\|U_{n+1,n+1}y_n - U_{n,n+1}y_n\| \\
&= (\prod_{m=1}^{n}\lambda_m)\|\lambda_{n+1}S_{n+1}U_{n+1,n+2}y_n \\
&\quad +(1-\lambda_{n+1}y_n - y_n\| \\
&= (\prod_{m=1}^{n}\lambda_m)\|\lambda_{n+1}S_{n+1}y_n - \lambda_{n+1})y_n\|
\end{aligned}$$

$$= (\prod_{m=1}^{n+1}\lambda_m)\|S_{n+1}y_n - y_n\| \leq \hat{M}(\prod_{m=1}^{n+1}\lambda_m) \tag{2.11}$$

where $\hat{M} := Sup_{\ n\geq 1}\ \{\|S_{n+1}y_n - y_n\|\}$.
Combining Eq.(2.3), Eq.(2.10) and Eq.(2.11), we obtain

$$\begin{aligned}\|W_{n+1}y_{n+1} - W_n y_n\| &= \|W_{n+1}y_{n+1} - W_{n+1}y_n + W_{n+1}y_n - W_n y_n\| \\ &\leq \|y_{n+1} - y_n\| + \|W_{n+1}y_n - W_n y_n\| \\ &\leq \|\theta_{n+1}\theta_n\| + M|\beta_{n+1} - \beta_n| + \hat{M}(\prod_{m=1}^{n+1} \lambda_m) \\ &\leq \|x_{n+1} - x_n\| + \frac{\acute{M}}{b}|r_{n+1} - r_n| \\ &\quad + M|\beta_{n+1} - \beta_n| + \hat{M}(\prod_{m=1}^{n+1} \lambda_m).\end{aligned}$$

We have

$$\limsup_{n\to\infty}(\|W_{n+1}y_{n+1} - W_n y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

From Lemma 1.6, we see that

$$\|W_n y_n - x_n\| \to 0 \quad , as \quad n \to \infty. \tag{2.12}$$

which implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \alpha_n)\|W_n y_n - x_n\| = 0$$

(iii): We shall prove that $\lim_{n\to\infty}\|x_n - z_n\| = 0$.
Notic that

$$\begin{aligned}\|u_{n,i} - x^*\|^2 &\leq \langle T_{r_{n,i}}x_n - T_{r_{n,i}}x^*, x_n - x^*\rangle \\ &= \tfrac{1}{2}\{\|u_{n,i} - x^*\|^2 + \|x_n - x^*\|^2 - \|u_{n,i} - x_n\|^2\}\end{aligned}$$

thus

$$\|u_{n,i} - x^*\|^2 \leq \|x_n - x^*\|^2 - \|u_{n,i} - x_n\|^2. \tag{2.13}$$

From Eq.(2.13), we get

$$\begin{aligned} \|\theta_n - x^*\| &= \|\sum_{i=1}^{k}\frac{1}{k}(u_{n,i} - x_n)\|^2 \\ &\leq \|x_n - x^*\|^2 - \tfrac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x_n\|^2. \end{aligned} \tag{2.14}$$

It follows from Eq.(2.14) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(W_n y_n - W_n x^*)\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\{\|(I - \beta_n A)(\theta_n - x^*) \\ &\quad +\beta_n(\gamma f(\theta_n) - Ax^*)\|^2\} \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\{(1-\beta_n\varpi)\|\theta_n - x^*\|^2 \\ &\quad +\beta_n\|\gamma f(\theta_n) - Ax^*)\|^2\} \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|\theta_n - x^*\|^2 \\ &\quad +\beta_n\|\gamma f(\theta_n) - Ax^*)\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\{\|x_n - x^*\|^2 \\ &\quad -\frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x_n\|^2\} \\ &\quad +\beta_n\|\gamma f(\theta_n) - Ax^*)\|^2 \\ &\leq \|x_n - x^*\|^2 - (1-\alpha_n)\frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x_n\|^2 \\ &\quad +\beta_n\|\gamma f(\theta_n) - Ax^*)\|^2. \end{aligned}$$

Thanks to the conditions of (C1)- (C4) and Eq.(2.13), we conclude that

$$\begin{aligned} (1-\alpha_n)\frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad +\beta_n\|\gamma f(\theta_n) - Ax^*)\|^2 \\ &\leq \|x_{n+1} - x_n\|(\|x_{n+1} - x^*\| + \|x_n - x^*\|) \\ &\quad +\beta_n\|\gamma f(\theta_n) - Ax^*)\|^2 \end{aligned}$$

$$\lim_{n\to\infty}\|u_{n,i} - x_n\| = 0, \ \textit{for each} \ \ i = 1, 2, \ldots, k$$

also

$$\lim_{n\to\infty}\|\theta_n - x_n\| = \lim_{n\to\infty}\|u_{n,i} - x_n\| = 0 \tag{2.15}$$

$$\|y_n - \theta_n\| = \beta_n\|\gamma f(\theta_n) - A\theta_n\| \to 0 \quad as \quad n \to \infty. \tag{2.16}$$

Moreover, we know that

$$\|y_n - x_n\| \leq \|x_n - \theta_n\| + \|\theta_n - y_n\|$$

$$\|W_n y_n - y_n\| \leq \|W_n y_n - x_n\| + \|x_n - \theta_n\| + \|\theta_n - y_n\|.$$

In viwe of Eq.(2.12), Eq.(2.15) and Eq.(2.16) , we can obtain

$$\lim_{n\to\infty}\|y_n - x_n\| = 0 \tag{2.17}$$

$$\lim_{n\to\infty}\|W_n y_n - y_n\| = 0. \tag{2.18}$$

□

Theorem 2.2 *Suppose all assumptions of Theorem 2.1 are holds. Then the sequence $\{x_n\}$ converge strongly to $\tilde{x}$, which solves the variational inequality*

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - x_n\rangle \leq 0, \quad \tilde{x} \in \bigcap_{i=1}^{k} F(W) \bigcap EP(G_i).$$

Equivalently, $P_{\bigcap_{i=1}^{k} F(W)\bigcap EP(G_i)}(I - A - \gamma f)(\tilde{x}) = \tilde{x}$.

Proof. We shall prove that

$$\limsup_{n\to\infty}\langle (A - \gamma f)x^*, y_n - x^*\rangle \leq 0,$$

where $x^* = P_{\bigcap_{i=1}^k F(W) \bigcap EP(G_i)} f(x^*)$.
We choose a subsequence $\{y_{n_p}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle (A - \gamma f)x^*, y_n - x^*\rangle = \lim_{p\to\infty} \langle (A - \gamma f)x^*, y_{n_p} - x^*\rangle, \quad (2.19)$$

since $\{y_{n_p}\}$ is bounded, there exits a subsequence of $\{y_{n_p}\}$, we denote it by $\{y_{n_p}\}$ such that $y_{n_p} \rightharpoonup q$, $q \in C$.

We shall show that $q \in \bigcap_{i=1}^k F(W) \bigcap EP(G_i)$. On the contrary, suppose that $q \notin F(W)$. By Opial's condition

$$\begin{aligned}
\liminf_{p\to\infty} \|y_{n_p} - q\| &< \liminf_{p\to\infty} \|y_{n_p} - Wq\| \\
&\leq \liminf_{p\to\infty} \{\|y_{n_p} - Wy_{n_p}\| + \|Wy_{n_p} - Wq\|\} \\
&\leq \liminf_{p\to\infty} \{\|y_{n_p} - Wy_{n_p}\| + \|y_{n_p} - q\|\}.
\end{aligned}$$

By virtue of Lemma 1.4 and noticing Eq.(2.18)

$$\begin{aligned}
\lim_{p\to\infty} \|Wy_{n_p} - y_{n_p}\| &\leq \lim_{p\to\infty} \{\|Wy_{n_p} - W_{n_p} y_{n_p}\| + \|W_{n_p} y_{n_p} - y_{n_p}\|\} \\
&\leq \lim_{p\to\infty} \{Sup_{\ x\in C} \ \ \|Wx - W_{n_p} x\|\} \\
&\quad + \lim_{p\to\infty} \|W_{n_p} y_{n_p} - y_{n_p}\| = 0.
\end{aligned}$$

It follows that

$$\liminf_{p\to\infty} \|y_{n_p} - q\| < \liminf_{p\to\infty} \|y_{n_p} - q\|.$$

This is a contradiction. Therefore, we have $q \in F(W)$. Also, we prove $q \in \bigcap_{i=1}^k EP(G_i)$.
For each $i \in J = \{1, 2, \ldots, k\}$, since $G_i(u_{n_p}, y) + \frac{1}{r_{n_p}} \langle y, u_{n_p} - x_{n_p}\rangle \geq 0$,

from (A2), we see that

$$\begin{aligned}\tfrac{1}{r_{n_p}}\langle y-u_{n_p},u_{n_p}-x_{n_p}\rangle &\geq G_i(u_{n_p},y)+G_i(y,u_{n_p})\\ &\quad+\tfrac{1}{r_{n_p}}\langle y-u_{n_p},u_{n_p}-x_{n_p}\rangle\\ &\geq G_i(y,u_{n_p}),\end{aligned}$$

hence

$$\langle y-u_{n_p},\frac{u_{n_p}-x_{n_p}}{r_{n_p}}\rangle \geq G_i(y,u_{n_p}),\ \text{for all } y\in C.$$

Since $\frac{|u_{n_p}-x_{n_p}|}{r_{n_p}}\to 0$, $u_{n,i}\rightharpoonup q$, in viwe of (A4), we conclude

$$G_i(y,q)\leq 0, textbfforally\in C.$$

Let $0<t\leq 1$, $y\in C$ and $y_t=ty+(1-t)q$. It is clear that $G_i(y_t,q)\leq 0$.
From (A1)-(A4), we obtai

$$\begin{aligned}0=G_i(y_t,y_t)\leq tG_i(y_t,y)+(1-t)G_i(y_t,q)\leq tG_i(y_t,y),G_i(y,q)\geq 0,\\ textbfforally\in C.\end{aligned}$$

Thus $q\in\bigcap_{i=1}^k EP(G_i)$.
From Eq.(2.19), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(A-\gamma f)x^*,y_n-x^*\rangle &=\lim_{p\to\infty}\langle(A-\gamma f)x^*,y_{n_p}-x^*\rangle\\ &=\langle(A-\gamma f)x^*,x^*-q\rangle\leq 0.\end{aligned}$$

It follows from Eq.(2.17) and Eq.(2.19) that

$$\begin{aligned}\limsup_{n\to\infty}\langle(A-\gamma f)x^*,x^*-x_n\rangle &\leq \limsup_{n\to\infty}\langle(A-\gamma f)x^*,y_n-x_n\rangle\\ &\quad+\limsup_{n\to\infty}\langle(A-\gamma f)x^*,x^*-y_n\rangle\leq 0.\end{aligned}$$

Finally, we prove that $x_n\to q$ where $x^*=P_{\bigcap_{i=1}^k F(W)\bigcap EP(G_i)}f(x^*)$.
By virtue of Lemma 1.7

$$
\begin{aligned}
\|y_n - x^*\|^2 &= \|(I - \beta_n A)(\theta_n - x^*) + \beta_n(\gamma f(\theta_n) - Ax^*)\|^2 \\
&\le \|(I - \beta_n A)(\theta_n - x^*)\|^2 + 2\beta_n\langle \gamma f(\theta_n) - Ax^*, y_n - x^*\rangle \\
&\le \|(I - \beta_n A)(\theta_n - x^*)\|^2 \\
&\quad +2\beta_n\gamma\rho\|x_n - x^*\|\|y_n - x^*\| + 2\beta_n\langle \gamma f(\theta_n) - Ax^*, y_n - x^*\rangle \\
&\le (1 - \beta_n\varpi)^2\|x_n - x^*\|^2 + \beta_n\gamma\rho(\|x_n - x^*\|^2 + \|y_n - x^*\|^2) \\
&\quad +2\beta_n\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle.
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\|y_n - x^*\|^2 &\le \frac{(1 - \beta_n\varpi)^2 + \beta_n\gamma\rho}{1 - \beta_n\gamma\rho}\|x_n - x^*\|^2 \\
&\quad +\frac{2\beta_n}{1 - \beta_n\gamma\rho}\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle \\
&\le \{1 - \frac{2\beta_n(\varpi - \gamma\rho)}{1 - \beta_n\rho\gamma}\}\|x_n - x^*\|^2 \\
&\quad +\frac{2\beta_n(\varpi - \gamma\rho)}{1 - \beta_n\gamma\rho}\{\frac{1}{\varpi - \gamma\rho}\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle \\
&\quad +\frac{\beta_n\varpi^2}{2(\varpi - \gamma\rho)}L\},
\end{aligned}
$$

where $L = Sup\{\|x_n - x^*\|\}$.
Also

$$
\|x_{n+1} - x^*\|^2 \le \alpha_n\|x_n - x^*\|^2 + (1 - \alpha_n)\|y_n - x^*\|^2 \tag{2.20}
$$

it follows from Eq.(2.20) that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \{1 - (1 - \alpha_n)\frac{2\beta_n(\varpi - \gamma\rho)}{1 - \beta_n\rho\gamma}\}\|x_n - x^*\|^2 \\
&\quad +(1 - \alpha_n)\frac{2\beta_n(\varpi - \gamma\rho)}{1 - \beta_n\rho\gamma}\}\{\frac{1}{\varpi - \gamma\rho}\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle \\
&\quad +\frac{\beta_n\varpi^2}{2(\varpi - \gamma\rho)}L\}.
\end{aligned}
$$

Let $\xi_n := (1-\alpha_n)\frac{2\beta_n(\varpi-\gamma\rho)}{1-\beta_n\rho\gamma}$ and

$$\varepsilon_n := \frac{1}{\varpi-\gamma\rho}\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle + \frac{\beta_n\varpi^2}{2(\varpi-\gamma\rho)}L.$$

Therefore,

$$\|x_{n+1} - x^*\|^2 \le (1-\xi_n)\|x_n - x^*\|^2 + \xi_n\varepsilon_n. \tag{2.21}$$

Thanks to the condition (C1) and Eq.(2.21), we conclude that

$$\lim_{n\to\infty}\xi_n = 0, \sum_{n=1}^{\infty}\xi_n = \infty.$$

From Lemma 1.10 we can obtain $x_n \to x^*$. $\square$

3 Numerical Example

In this section, we get one example is presented to guarantee the Theorem (2.2).

Example 3.1 *Let $H = \mathbb{R}, C = [-1,1]$ and $G_1(x,y) = -3x^2 + xy + 2y^2, G_2(x,y) = -4x^2 + xy + 3y^2$ and $G_3(x,y) = -9x^2 + xy + 8y^2$. Also, we consider $S_n = I, f(x) = \frac{x}{5}$ and $A = I$ be a strongly positive linear bounded operator with coefficient $\gamma = 1$. It is easy to check that A and f satisfy all conditions in Theorem 2.2. For each $r > 0$ and $x \in C$, there exists $z \in C$ such that, for any $y \in C$,*

$$\begin{aligned}
&G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \\
\Leftrightarrow\ & -3z^2 + zy + 2y^2 + \frac{1}{r}(y-z)(z-x) \ge 0 \\
\Leftrightarrow\ & 2ry^2 + ((r+1)z - x)y - 3rz^2 - z^2 + zx \ge 0
\end{aligned}$$

Set $G(y) = 2ry^2 + ((r+1)z - x)y - 3rz^2 - z^2 + zx$. *Then* $G(y)$ *is a quadratic function of* y *with coefficients* $a = 2r, b = (r+1)z - x$ *and* $c = -3rz^2 - z^2 + zx$. *So*

$$\begin{aligned}\Delta &= [(r+1)z - x]^2 - 8r(zx - z^2 - 3rz^2)\\ &= (r+1)^2z^2 - 2(r+1)xz + x^2 + 24r^2z^2 + 8rz^2 - 8rzx\\ &= x^2 - 2(5rz + z)x + (25r^2z^2 + 10rz^2 + z^2)\\ &= [(x - (5rz + z))]^2.\end{aligned}$$

Since $G(y) \geq 0$ *for all* $y \in C$, *if and only if* $\Delta = [(x - (5rz + z))]^2 \leq 0$. *Therefore,* $z = \frac{x}{5r+1}$, *which yields* $T_{r_n,1} = u_n^{(1)} = \frac{x_n}{5r_n+1}$.
By the same argument, for G_2 *and* G_3, *one can conclude* $T_{r_n,2} = u_n^{(2)} \frac{x_n}{7r_n+1}$ *and* $T_{r_n,3} = u_n^{(3)} = \frac{x_n}{17r_n+1}$. *Let* $r_n = \frac{n}{n+1}$. *Hence*

$$\theta_n = \frac{u_n^{(1)} + u_n^{(2)} + u_n^{(3)}}{3} = \frac{1}{3}\frac{280n^3 + 344n^2 + 67n + 3}{864n^3 + 300n^2 + 32n + 1}x_n.$$

Suppose that $\alpha_n = \frac{2n-1}{10n-9}, \beta_n = \frac{1}{n}$ *and* $\lambda_n = \varepsilon$, *we have*

$$\begin{aligned}W_1 &= U_{11} = \lambda_1 S_1 U_{12} + (1 - \lambda_1)I,\\ W_2 &= U_{21} = \lambda_1 S_1 U_{22} + (1 - \lambda_1)I\\ &= \lambda_1 S_1\{\lambda_2 S_2 U_{23} + (1 - \lambda_2)I\} + (1 - \lambda_1)I,\\ &= \lambda_1\lambda_2 S_1 S_2 + \lambda_1(1 - \lambda_2)S_1 + (1 - \lambda_1)I,\\ W_3 &= U_{31} = \lambda_1 S_1 U_{32} + (1 - \lambda_1)I\\ &= \lambda_1 S_1\{\lambda_2 S_2 U_{33} + (1 - \lambda_2)I\} + (1 - \lambda_1)I,\\ &= \lambda_1\lambda_2 S_1 S_2 U_{33} + \lambda_1(1 - \lambda_2)S_1 + (1 - \lambda_1)I,\\ &= \lambda_1\lambda_2 S_1 S_2\{\lambda_{33} S_3 U_{34} + (1 - \lambda_3)I\} + \lambda_1(1 - \lambda_2)S_1 + (1 - \lambda_1)I,\\ &= \lambda_1\lambda_2\lambda_3 S_1 S_2 S_3 + \lambda_1\lambda_2(1 - \lambda_3)S_1 S_2 + \lambda_1(1 - \lambda_2)S_1 + (1 - \lambda_1)I.\end{aligned}$$

By iteration this manner, we have

$$\begin{aligned}W_n = U_{n1} &= \lambda_1\lambda_2\cdots\lambda_n S_1 S_2\cdots S_n + \lambda_1\lambda_2\cdots\lambda_{n-1}(1 - \lambda_n)S_1 S_2\cdots S_{n-1}\\ &\quad + \lambda_1\lambda_2\cdots\lambda_{n-2}(1 - \lambda_{n-1})S_1 S_2\cdots S_{n-2} + \cdots + \lambda_1(1 - \lambda_2)S_1 + (1 - \lambda_1 I.\end{aligned}$$

Let $T_n = I, \lambda_n = \varepsilon$, *we obtain*

$$W_n = [\varepsilon^n + \varepsilon^{n-1}(1-\varepsilon) + \cdots + \varepsilon(1-\varepsilon) + (1-\varepsilon)]I = I.$$

Hence

$$y_n = \left(\frac{280n^3 + 344n^2 + 67n + 3}{864n^3 + 300n^2 + 32n + 1}\right)\left(\frac{15n - 14}{15n}\right)x_n.$$

We have the following algorithm for the sequence $\{x_n\}$

$$x_{n+1} = \frac{2n-1}{10n-9}x_n + \frac{8n-8}{10n-9}y_n.$$

Choose $x_1 = 1$. By using MATLAB software, we obtain the following table and figure of the result.

n	x_n	n	x_n	n	x_n
1	1.0	11	0.003385691332	21	0.000002071711754
2	1.270565302	12	0.001637456001	22	0.0000009778618741
3	0.7706281483	13	0.0007885980277	23	0.0000004609899888
4	0.4251825949	14	0.0003784282172	24	0.0000002170787103
5	0.2242603285	15	0.0001810390789	25	0.0000001021162838
6	0.1151384546	16	0.00008637699279	26	0.00000004799105366
7	0.05805496956	17	0.00004111536174	27	0.00000002253432275
8	0.02889456779	18	0.00001953029496	28	0.00000001057245007
9	0.01424086637	19	0.000009260000453	29	0.000000004956540717
10	0.006965124578	20	0.000004383229046	30	0.000000002322073954

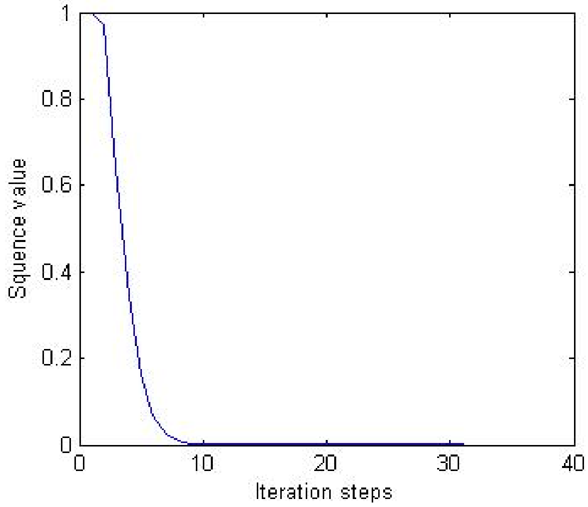

Fig. 1. The graph of $\{x_n\}$ with initial value $x_1 = 1$.

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