Theory of Approximation and Applications

Vol. 12, No.1, (2018), 1-9

Study on the new graph constructed by a commutative ring

Zahra Yarahmadi [a,*]

[a] *Department of Mathematics, Faculty of Science, Khorramabad Branch, Islamic Azad University, Khorramabad, I. R. Iran*

Received 12 July 2017; accepted 3 January 2018

Abstract

Let R be a commutative ring and $G(R)$ be a graph with vertices as proper and non-trivial ideals of R. Two distinct vertices I and J are said to be adjacent if and only if $I + J = R$. In this paper we study a graph constructed from a subgraph $G(R)\backslash\Delta(R)$ of $G(R)$ which consists of all ideals I of R such that $I \ntrianglelefteq J(R)$, where $J(R)$ denotes the Jacobson radical of R. In this paper we study about the relation between the number of maximal ideal of R and the number of partite of graph $G(R)\backslash\triangle(R)$. Also we study on the structure of ring R by some properties of vertices of subgraph $G(R)\backslash\triangle(R)$. In another section, it is shown that under some conditions on the $G(R)$, the ring R is Noetherian or Artinian. Finally we characterize clean rings and then study on diameter of this constructed graph.

Key words: Connected graph, diameter, n-partite graph, Commutative ring, Noetherian and Artinian ring,

2010 AMS Mathematics Subject Classification : 05C75, 13A15.

* Corresponding author's e-mail: z.yarahmadi@gmail.com, z.yarahmadi@khoiau.ac.ir

1 Introduction

Throughout this paper we consider only commutative ring not necessary unital. We recall some definitions that will be used in the paper. Let G be a graph and L be a set. A *coloring* of G by L is a function $c : V(G) \longrightarrow L$ with this property: if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. The *chromatic number* of G is the minimum number of colors which is needed for a proper coloring of G, which is denoted by $\chi(G)$. Recall that a graph is said to be *connected* if for each pair of distinct vertices v and w, there is a finite sequence of distinct vertices $v = v_1, v_2, ...v_n = w$ such that each $v_i v_{i+1}$ is an edge. For two vertices u and v in a graph G, the distance between u and v, is denoted by $d(u, v)$, is the length of the shortest path between u and v, if such a path exists; otherwise we define $d(u, v) = \infty$. The *diameter* of a graph G is defined

$$diam(G) = sup\{d(u, v)|u, v \in V(G)\}.$$

The diameter is 0 if the graph consist of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete. A *k-partite* graph is one whose vertex set can be partitioned into k subsets such that no edge has no both ends in any one subsets. A *complete k-partite* graph is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite* graph with part sizes m and n is denoted by $K_{m,n}$.The *clique* of the graph is its maximal complete subgraph. We denote the size of the largest clique of G by $\omega(G)$. Obviously for every graph G, $\chi(G) \geq \omega(G)$.

In [2], Beck considered $\Gamma(R)$ as a graph with vertices as elements of R, where two different vertices a and b are adjacent if and only if $ab = 0$. He showed that $\chi(\Gamma(R)) = \omega(\Gamma(R))$ for certain class of rings.

In [6], Sharama and Bhatwadekar defined another graph on R, $\Gamma(R)$, with vertex set $V(\Gamma(R))$ and edge set $E(\Gamma(R))$ as follows:

$$V(\Gamma(R)) = \{a \mid a \in R\},$$

$$E(\Gamma(R)) = \{ab \mid Ra + Rb = R\}.$$

They showed that $\chi(\Gamma(R)) < \infty$ if and only if R is a finite ring. In this case $\chi(\Gamma(R)) = \omega(\Gamma(R)) = t+l$, where t and l are the number of maximal ideals of R and the number of units of R, respectively.

Maimani et al. in [4] study further the graph defined by Sharama and Bhatwadekar. They study on connectivity and diameter of this graph. In addition, they completely characterize the diameter of comaximal graph of commutative rings. In [5] comaximal ideal graph was proposed on R, where R be a commutative ring not necessary unital. Comaximal ideal graph $G(R)$ with vertex set $V(G(R))$ and edge set $E(G(R))$ as follows:

$$V(G(R)) = \{I \mid I \neq \{0\},\ I \lhd R\},$$

$$E(G(R)) = \{IJ \mid I + J = R\}.$$

A ring R is quasi local if it has a unique maximal ideal. A quasi local ring R with unique maximal m is denoted by (R, m). Obviously R is quasi local ring if and only if $E(G(R)) = \emptyset$. In the graph $G(R)$, the induced subgraph $Max(R)$ is complete. In this case we have $\omega(G(R)) = |Max(R)|$. Moreover $\mathsf{m} \in V(G(R))$ is a maximal ideal of R not contained in nonezero ideals of R. So m is adjacent with all vertices of $G(R)$ and in this case $J(R) = \{0\}$. The notation we use is mostly standard and taken from standard graph theory textbooks, such as [3], [8] and [7].

2 Main Results

Throughout this section R is a commutative ring not necessary unital. Let $G_1(R) = \langle Max(R) \rangle$, $G_2(R) = G(R) \setminus G_1(R)$ and $\Delta(R) = \langle \{I | I \lhd R, I \subseteq J(R)\} \rangle$ be subgraphs of $G(R)$, where $J(R)$ is Jacobson radical of R.

Lemma 1 *The following statements are hold:*
i. $G_1(R)$ *is a complete graph.*
ii. $I \in \Delta(R)$ *if and only if* $deg_{G(R)} I = 0$.

Proof. (i) For each $\mathsf{m}, \mathsf{m}' \in Max(R)$, it is clear that $\mathsf{m} + \mathsf{m}' = R$. Then all of the maximal ideals are adjacent.

(ii) If $I \in \Delta(R)$ then for any $J \in V(G(R))$ there exists a maximal ideal m such that $J \subseteq \mathsf{m}$.Therefore $I + J \subseteq \mathsf{m} \subset R$ and there are no vertices that adjacent to I. So $deg_{G(R)} I = 0$.

Conversely suppose that $deg_{G(R)} I = 0$. If $I \ntrianglelefteq J(R)$, then there exists $\mathsf{m} \in Max(R)$ such that $I \nsubseteq \mathsf{m}$. So, $I + \mathsf{m} = R$, and it is contradiction.

We know that each vertex of $\Delta(R)$ is an isolated vertex of $G(R)$. Thus the main part of the graph $G(R)$ is the subgraph $G(R)\backslash\Delta(R)$. For this reason the main aim of this paper is to study the structure of this subgraph.

Theorem 2 *the following statement are equivalent:*
i. $G(R)\backslash\Delta(R)$ *is a complete bipartite graph.*
ii. $|Max(R)| = 2$

Proof. $(i \Rightarrow ii)$ If $|Max(R)| = 1$, obviously the graph $G(R)\backslash\Delta(R)$ is a single vertex and it isn't bipartite graph. Now assume that $|Max(R)| \geq 3$. Let m_1, m_2 and m_3 be maximal ideals of R. one can see that the graph $G(R)\backslash\Delta(R)$ is contained odd cycle $\mathsf{m}_1 - \mathsf{m}_2 - \mathsf{m}_3 - \mathsf{m}_1$ and it's contradiction.

$(ii \Rightarrow i)$ If $|Max(R)| = 2$ and m_1 and m_2 be maximal ideals of R. Then we consider the following sets of ideals:

$$A = \{I| \ \ I \subseteq \mathsf{m}_1, I \nsubseteq \mathsf{m}_2\},$$

$$B = \{J| \ \ J \subseteq \mathsf{m}_2, J \nsubseteq \mathsf{m}_1\}.$$

It is easy to see that:

$$V\Big(G(R)\backslash\Delta(R)\Big) = \{\mathsf{m}_1 \ , \mathsf{m}_2\} \cup A \cup B.$$

We partition the vertices of $G(R)\backslash\Delta(R)$ in two parts. Put the vertices of A and m_1 in first part and the vertices of B and m_2 in another part of the $G(R)\backslash\Delta(R)$. Now we can prove that $G(R)\backslash\Delta(R)$ is complete bipartite graph. Obviously the is no edges between the vertices of the first part and also between the vertices of second partite and it is easily to seen that $\mathsf{m}_1 + \mathsf{m}_2 = R$. Also for each $I \in A, I + \mathsf{m}_2 = R$ and for each $J \in B, J + \mathsf{m}_1 = R$. Now it's enough to show that for each $I \in A$ and $J \in B$, $I + J = R$. If $I + J \neq R$, then there is an ideal maximal m, such that $I+J \subseteq \mathsf{m}$. Without losing generality, we assume that $\mathsf{m}=\mathsf{m}_1$, $I+J \subseteq \mathsf{m}_1$. Then $J \subseteq \mathsf{m}_2$ and it is contradiction.

Theorem 3 *Let $n>1$, If the graph $G(R)\backslash\Delta(R)$ is a complete $n-$partite graph then $n \leq 2$.*

Proof. Assume the contrary $n>2$. Let $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3 \in Max(R)$. If $\mathsf{m}_1 \cap \mathsf{m}_2 \nsubseteq J(R)$, then $\mathsf{m}_1 \cap \mathsf{m}_2$ and m_1 are in the same partite and similarly for $\mathsf{m}_1 \cap \mathsf{m}_2$ and m_2. Therefore $\mathsf{m}_1 \cap \mathsf{m}_2 \subseteq J(R)$. Since $\mathsf{m}_1 \cap \mathsf{m}_2 \subseteq \mathsf{m}_3$, $\mathsf{m}_1 \subseteq \mathsf{m}_3$ or $\mathsf{m}_2 \subseteq \mathsf{m}_3$ and by maximality $\mathsf{m}_1=\mathsf{m}_3$ or $\mathsf{m}_2=\mathsf{m}_3$. This is a contradiction and so $n \leq 2$.

Theorem 4 *Let R be a ring with identity and $|Max(R)| \geq 2$. If there exists a vertex m of $G(R)\backslash\Delta(R)$ which is adjacent to every other vertex then the following statements are hold:*
i. m is a maximal ideal of R.
ii. $|Max(R)|=2$.
iii. $R \cong S \times F$, where F is a field. If $J(R)=0$ then S is a field, otherwise S is a quasi local ring.

Proof. Since (i) is clear, we just prove (ii) and (iii).
(ii) Suppose m is the vertex of $G(R)\backslash\Delta(R)$ which is adjacent to every other vertex. Then by (i) m is the maximal ideal of R. Assume m' is another maximal ideal of R. It is easy to see that $J(R) \subseteq \mathsf{m}\mathsf{m}'$. There are two cases:
Case 1. If $J(R) \subset \mathsf{m}\mathsf{m}'$ and $\mathsf{m}\mathsf{m}' \subset \mathsf{m}$, then $\mathsf{m}\mathsf{m}'$ and m are adjacent, a contradiction. If $\mathsf{m}\mathsf{m}'=\mathsf{m}$ then $\mathsf{m}=\mathsf{m}\mathsf{m}' \subseteq \mathsf{m}'$ and $\mathsf{m}=\mathsf{m}'$ or $\mathsf{m} \subset \mathsf{m}'$, lead again to a contradiction.
Case 2. $J(R)=\mathsf{m}\mathsf{m}'$. If m'' is another maximal ideal of R then $J(R)=\mathsf{m}\mathsf{m}' \subseteq \mathsf{m}''$. By maximality of m'', $\mathsf{m} \subseteq \mathsf{m}''$ or $\mathsf{m}' \subseteq \mathsf{m}''$ and this is contradiction.
(iii) Suppose $Max(R)=\{\mathsf{m}, \mathsf{m}'\}$ such that m is adjacent to every I which $I \nsubseteq J(R)$. Let $a \in \mathsf{m} \setminus J(R)$, then $Ra \subseteq \mathsf{m}$, $Ra \nsubseteq J(R)$. By above argument $\mathsf{m}+Ra=R$, contradiction. So $\mathsf{m}=Ra$. By this process we obtain $\mathsf{m}=Ra=Ra^2=\ldots$. There exists $r \in R$ such that $a=ra^2$. It easy to see that $b=ra$ is idempotent and $b=ra \in J(R)$. Thus for each $s \in R \setminus \mathsf{m}$, the equality $Rs+Rb=R$ are hold, so $(1-b)sR=(1-b)R$. Let $\phi: sR \longrightarrow (1-b)sR$ defined by $\phi(sr)=(1-b)sr$. It is clearly that ϕ

is an epimorphism and $Ker\phi = bR \cap sR$. Therefore $\frac{sR}{bR\cap sR} \cong (1-b)sR = (1-b)R$. On the other hand $\frac{sR}{bR\cap sR} \cong \frac{bR+sR}{bR} = \frac{R}{bR}$. Hence $(1-b)R \cong \frac{R}{bR}$. Since $\mathsf{m} = bR$ is a maximal ideal then $(1-b)R$ is a field. For each $b \in R$, $bR+(1-b)R = R$. Since b is an idempotent element $bR \cap (1-b)R = \{0\}$ and this implies $R \cong bR \times (1-b)R$, set $F = (1-b)R$ and $S = bR$. It is clear that $Max(S) = \{J(R)\}$. If $J(R) \neq 0$ then S with operations of ring R is quasi local ring. If $J(R) = 0$ then $S = bR$ is not contained any nonzero ideal. In this case identity of S is $1_S = b.1_R$, for each $br \in S$, $(br).1_S = (br).(b.1_R) = b^2r = br$. Therefore S is a commutative ring with identity and $\{0\}$ is maximal ideal of S, so S is a field.

Corollary 5 *Let R be a ring. Then $G(R) \cong K_n$ if and only if $n = 2$ and $R \cong F_1 \times F_2$, where F_1 and F_2 are fields.* $\square$

Recall that $r(0) = \cap\{P|\ P$ is prime ideal of $R\}$. In this section we look at the conditions on the $G(R)$ to show R is Noetherian or Artinian.

Theorem 6 *Let R be a ring. Assume that $J(R)$ is finitely generated and each maximal ideals of R as vertices of $G(R)$ have finite degree and $|Max(R)| \geq 2$. Hence R is Noetherian. Moreover if $Spec(R) = Max(R)$, then R is Artinian too.*

proof. Suppose that $I_1 \subseteq I_2 \subseteq ...I_n \subseteq ...$ is an ascending chain of ideals of R. Since $J(R)$ is finitely generated, there exists $k \in \mathbb{N}$ such that $I_n \nsubseteq J(R)$ for each $n \geq k$. Therefore, there exists $\mathsf{m} \in Max(R)$ such that $I_n \nsubseteq \mathsf{m}$ for each $n \geq k$. Hence $\mathsf{m} + I_n = R$ for each $n \geq k$. This is contradiction, because $deg_{G(R)}\mathsf{m} < \infty$.

Since R is Notherian and each prime ideal of R is maximal then by [7](Theorem 8.38), R is Artinian too.

In the ring $\mathbb{Z}$, the degree of all maximal ideals as vertices of $G(\mathbb{Z})$ is infinite, but $\mathbb{Z}$ is Noetherian ring. Hence the converse of above theorem is not true.

Recall that in this type of comaximal graph the clique number of $G(R)$ is equal to $|Max(R)|$.

Theorem 7 *The following statements are hold:*
i. Let R be an Artinian ring then $\omega(G(R)) < \infty$ and $\langle Spec(R)\rangle$ is a complete subgraph.
ii. Let R be an infinite integral domain such that $|U(R)| < \infty$. Then $G(R)$ doesn't have isolated vertex and $\omega(G(R)) = \infty$.

Proof. (i) Since R is Artinian, then the number of maxima ideals is finite. Let $\mathsf{m}_1, \mathsf{m}_2, ..., \mathsf{m}_k$ be maximal ideals of R. Since for each non-maximal ideal I of R, such that $I \nsubseteq J(R)$, there exist, $1 \leq i \leq k$, such that $I \subseteq \mathsf{m}_i$. Then the subgraph of $G(R)$, induced by maximal ideals of R as vertices of $G(R)$ is maximal complete subgraph of $G(R)$. Hence $\omega(G(R)) < \infty$. Also in Artinian ring, , each prime ideal is maximal, $Spec(R) = Max(R)$, then $\langle Spec(R)\rangle = \langle Max(R)\rangle$ is complete subgraph.

(ii) Let I be single vertex of $G(R)$, it is easy to see that $I \subseteq J(R)$. Let $a \in I$, for each $r \in R$, $1 - ra$ is unit. Since R is infinite integral domain, if $r_1 \neq r_2$, then $1 - r_1a \neq 1 - r_2a$. Hence the number of unital vertex i infinite and this is contradiction.

Now assume that $\omega(G(R)) = k < \infty$. Let $\mathsf{m}_1, \mathsf{m}_2, ..., \mathsf{m}_k$ be maximal ideal of R and for each $1 \leq i \leq k$, $0 \neq a_i \in \mathsf{m}_i$. Then $0 \neq a = a_1a_2...a_k \in J(R) = \cap \mathsf{m}_i$. Hence for each $r \in R$, $1 - ra$ is unit and by similar way in (i), there is contradiction.

A ring is said to be *clean* if all of its elements can be written as the sum of a unit and an idempotent [1]. For example, a quasi local ring is clean. The following theorem characterize clean rings.

Theorem 8 *For the ring R, the following are equivalent:*
i. R is a finite product of quasi local rings.
ii. R is clean and $\omega(G(R)) < \infty$.

Proof. $(i \Longrightarrow ii)$ Suppose $R = R_1 \times ... \times R_n$ where each R_i, $1 \leq i \leq n$, is quasi local ring with unique maximal ideal m_i. Obviously every maximal ideal has the form $\mathsf{n}_i = R_1 \times ... \times R_{i-1} \times \mathsf{m}_i \times R_{i+1} \times ... \times R_n$. Thus $|Max(R)| = n$. On the other hand each ideal of R is formed as $I_1 \times ... \times I_n$, where $I_i \trianglelefteq R_i$ for $1 \leq i \leq n$, then $\omega(G(R)) < \infty$. Every finite product of clean ring is clean, Since every quasi local ring is clean then R is clean.

($ii \implies i$) At first we show that a number of idempotent elements is finite. Assume the contrary. So the set of idempotent elements is infinite. Suppose that $e_1, e_2, ..., e_n, ...$ are different idempotent elements. Define $e'_1, e'_2, ..., e'_n, ...$ as follows:

$$e'_1 = e_1,\ \ e'_2 = 1 - e'_1 e_2,\ \ e'_3 = 1 - e'_1 e'_2 e_3,\ \ ..., e'_n = 1 - e'_1 ... e'_{n-1} e_n,\ \ ...$$

It is easy to see that $e'_1, e'_2, ..., e'_n, ...$ are idempotent. Every ideals in the form $I'_i = Re'_i$ are adjacent in $G(R)$ and this is contradiction.

In this section we completely characterize the diameter of $G(R)\backslash\Delta(R)$. The following result show that $G(R)\backslash\Delta(R)$ is connected and its diameter is not greater than 3.

Theorem 9 *Let R be a ring with $|Max(R)| \geq 2$. Then $G(R)\backslash\Delta(R)$ is connected and $diam(G(R)\backslash\Delta(R)) \leq 3$*

Proof. It is easy to see $G(R)\backslash\Delta(R)$ is connected. Suppose I and J are two arbitrary vertices of $G(R)\backslash\Delta(R)$. We consider two cases:

Case 1. There exists $\mathsf{m} \in Max(R)$ such that $I, J \subseteq \mathsf{m}$. It is clear that I and J are not adjacent. Since $I, J \nsubseteq J(R)$ then there exists $\mathsf{m}_1, \mathsf{m}_2 \in Max(R)$ such that $I \nsubseteq \mathsf{m}_1$ and $J \nsubseteq \mathsf{m}_2$. If $\mathsf{m}_1 = \mathsf{m}_2$, we have the path $I - \mathsf{m}_1 - J$ and if $\mathsf{m}_1 \neq \mathsf{m}_2$, there exists the path $I - \mathsf{m}_1 - \mathsf{m}_2 - J$. Therefore $d(I, J) \leq 3$

Case 2. I and J are not contained in the same maximal ideal. There exists different maximal ideals as m_1 and m_2 such that $I \subseteq \mathsf{m}_1$, $J \subseteq \mathsf{m}_2$, $I \nsubseteq \mathsf{m}_2$ and $J \nsubseteq \mathsf{m}_1$. In this case we have the path $I - \mathsf{m}_1 - \mathsf{m}_2 - J$, hence $d(I, J) \leq 3$. □

In the next theorem we characterize the ring R with $diam(G(R)\backslash\Delta(R)) = 1$, according to the structure of S and F in Theorem 2.4.

Theorem 10 *$diam(G(R)\backslash\Delta(R)) = 1$ if and only if $R \cong S \times F$.*

Proof. ($\Longrightarrow$) If $diam(G(R)\backslash\Delta(R)) = 1$ then $|Max(R)| \geq 2$. Suppose m is a maximal ideal of R, then m is adjacent to every other vertices of

$G(R)\backslash\Delta(R)$, by Theorem 2.4 the proof is completed.

($\Longleftarrow$) If S be a field then $Max(R) = \{S\times\{0\}, \{0\}\times F\}$. Hence $diam(G(R)\backslash\Delta(R)) = 1$. Similarly if S be a quasi local ring with unique maximal ideal m then $Max(R) = \{\mathsf{m} \times \{0\}, \mathsf{m} \times F\}$, thus $diam(G(R)\backslash\Delta(R)) = 1$. $\square$

References

[1] D.D. Anderson, V.P. Camillo, Commutative rings whose elements are a sum of a unit and idempotent, *J. Comm. Algebra* **30** (2002), 3327–3336.

[2] I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1998) 208–226.

[3] G. Chartrand, O.R. Oellermann, *Applied and Algorithmic Graph Theory*, McGraw-Hill, Inc., New York, 1993.

[4] H. R. Maimani, M. Salimi, A. Sattari, S. Yassemi, Comaximal graph of commutative rings, *J. Algebra* **319** (2008) 1801-1808.

[5] C. Mihăilă, *Comaximal Ideal Graphs of Commutative Rings*, Wellesley College, Honors Thesis Collection, 2012.

[6] P.K. Sharma, S.M. Bhatwadekar, A note on graphical representation of rings, *J. Algebra* **176** (1995) 124-127.

[7] R. Y. Sharp, *Steps in commutative algebra*, Londen Mathematical Society Student Texts51, Cambridge University Press, Cambridge, 2000.

[8] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Upper Saddle River, NJ, 1996.