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On the computation of characteristic polynomials and spectra of some balanced rooted trees

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ABSTRACT

A generalized Bethe tree is a rooted tree in which vertices at the same level have the same degree. In this paper we obtain an explicit formula for computation of the characteristic polynomials of the adjacency matrix and the Laplacian matrix of the rooted trees which is obtained from the union of some generalized Bethe trees joined at their respective root vertices.

1 Introduction.

Let G be a simple undirected graph. Throughout the paper we suppose that $A(G)$, $D(G)$, and $L(G) = D(G) - A(G)$, denote the adjacency matrix, the diagonal matrix of vertex degrees, and the Laplacian matrix, of G, respectively. The characteristic polynomials of $A(G)$ and $L(G)$ are denoted by $P_G(\lambda)$ and $Q_G(\lambda)$, respectively. Finally we denote the spectra of a matrix M by $\sigma(M)$.

A Bethe tree is an unweighted rooted tree of k levels such that the vertex root has degree d, the vertices in the intermediate levels have degree $d+1$ and the vertices in level k are the pendant vertices [1]. Also a generalized Bethe tree is a rooted unweighted tree in which the vertices in each of its levels have equal degree [2]. Recently computing the characteristic polynomials of adjacency and Laplacian matrices of some classes of trees have been the object of many papers (see for example [2]-[9]). Rojo in [2] characterize the eigenvalues of the adjacency matrix and of the Laplacian matrix of rooted weighted trees which obtained by join some generalized Bethe weighted trees at root vertices by calculation determinant of 3-diagonal matrices. In this paper we use the concept of rooted product of graphs and find an exact formula for characteristic polynomials of adjacency and Laplacian matrices of rooted unweighted trees which obtained by join some generalized Bethe trees at their root vertices by new method.

Suppose that $G = \{G_1, G_2, \ldots, G_k\}$ be a sequence of k rooted graphs and H be a labelled graph on k vertices. The rooted product of H by G, which is denoted by $H(G)$, is obtained by identifying the root vertex of G_i with ith vertex of H (see [6]). In [4] the characteristic polynomials of $A(G)$ and $L(G)$ are computed, in terms of the characteristic polynomials of the graphs H and G_i, $i = 1, 2, \ldots, k$. Now let T be a rooted tree of $k+1$ levels and $G_1, G_2, \ldots, G_k$ be rooted trees of k levels that are obtained by deletion the root vertex of T. If S_{k+1} denotes the star on $k+1$ vertices, then $T = S_{k+1}(S_1, G_1, G_2, \ldots, G_k)$, the rooted product of S_{k+1} by $\{S_1, G_1, G_2, \ldots, G_k\}$,

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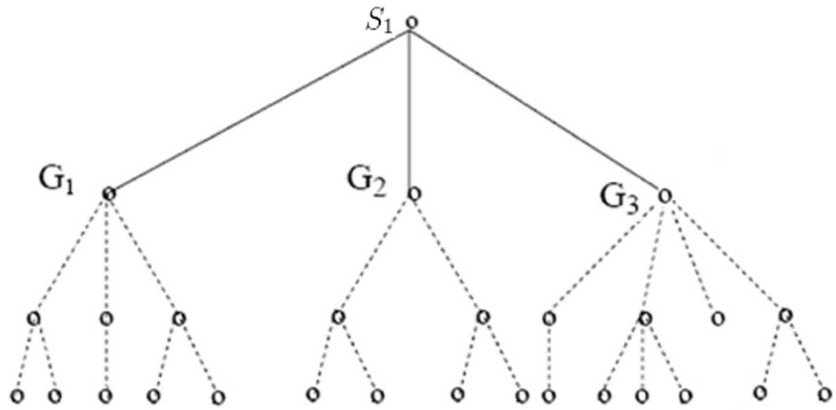

Figure 1: $S_4(S_1, G_1, G_2, G_3)$, where S_{n+1} is the star on $n+1$ vertices and G_1, G_2, G_3 are arbitrary rooted trees.

is equal to rooted tree which obtained from the union of $G_1, G_2, \ldots, G_k$ joined at their respective root vertices and s_1 is common root.

In order to state the main results of the paper we need some notations. Let β_k be a generalized Bethe tree of k levels. Suppose that n_{k-j+1} and d_{k-j+1} denote the number of vertices and degree of vertices in jth level of tree for $j = 1, 2, \ldots, k$. If e_{k-j+1} denotes the number of adjacent vertex in $j+1$th with a vertex in jth row of β_k, then

$$e_i = \begin{cases} d_i & \text{if} \quad i = k \\ d_i - 1 & \text{if} \quad j \neq k. \end{cases}$$

The characteristic polynomial of adjacency matrix and Laplacian matrix of generalized Bethe trees computed as follow (see [4] or [7]).

Theorem A. Let $P_0(\lambda) = 1$, $P_1(\lambda) = \lambda$ and

$$P_j(\lambda) = \lambda P_{j-1}(\lambda) - e_j P_{j-2}(\lambda) \quad \text{for} \;\; j = 2, 3, \ldots, k.$$

Then (a) $P_{\beta_k}(\lambda) = P_k \prod_{j=1}^{k-1} P_j^{n_j - n_{j+1}}$ and (b) $\sigma(A(\beta_k)) = \bigcup_{j\in\Omega}\{\lambda \in \mathbb{R} : P_j(\lambda) = 0\}$.

Theorem B. Let β_k be a generalized Bethe tree of k level. Let $Q_0(\lambda) = 1$, $Q_1(\lambda) = \lambda - 1$ and

$$Q_j(\lambda) = (\lambda - d_j)Q_{j-1}(\lambda) - e_j Q_{j-2}(\lambda), \qquad j = 2, 3, \ldots, k.$$

Then (a) $Q_{\beta_k}(\lambda) = Q_k \prod_{j=1}^{k-1} Q_j^{n_j - n_{j+1}}$ and (b) $\sigma(L(\beta_k)) = \bigcup_{j\in\Omega}\{\lambda \in \mathbb{R} \mid Q_j(\lambda) = 0\}$.

Now let T be a rooted tree which degree of root vertex is k and the subgraph obtained by deleting the root vertex of T be union of k generalized Bethe trees of levels r_i denoted by β_{r_i} for $i = 1, 2, \cdots, k$. Since T can be considered as rooted product of S_{k+1} by $G = \{S_1, \beta_{r_1}, \beta_{r_1}, \cdots, \beta_{r_k}\}$, characteristic polynomial of adjacency matrix and Laplacian matrix of T can be computed by using Theorems A and B.

2 Results.

Let $M^{1,1}$ denote the matrix obtained by delation of the first column and row of a matrix M, and put $\overline{P}_{G_i}(\lambda) = \det(\lambda I - A(G_i)^{1,1})$, $\overline{Q}_{G_i}(\lambda) = \det(\lambda I - L(G_i)^{1,1})$. If G_i is the graph of order 1 put $\overline{P}_{G_i}(\lambda) = \overline{Q}_{G_i}(\lambda) = 1$. We need the following Theorem which is proved in [4, Theorem 1]:

Theorem C. Let H and $G_1, G_2, \ldots, G_k$, $j = 1, 2, \ldots, k$, be simple graphs. If $K = H(G_1, G_2, \ldots, G_k)$ then $P_K(\lambda) = (-1)^k \det(M)$, where

$$M_{ij} = \begin{cases} -P_{G_i}(\lambda) & \text{if } i = j \\ \overline{P}_{G_i}(\lambda) & \text{if } i \neq j \text{ and } A(H)_{ij} = 1 \\ 0 & \text{if } i \neq j \text{ and } A(H)_{ij} = 0, \end{cases}$$

$1 \le i, j \le k$, and $Q_K(\lambda) = \det(N)$, where

$$N_{ij} = \begin{cases} Q_{G_i}(\lambda) & \text{if } i = j \\ \overline{Q}_{G_i}(\lambda) & \text{if } i \neq j \text{ and } A(H)_{ij}) = 1 \\ 0 & \text{if } i \neq j \text{ and } A(H)_{ij} = 0, \end{cases}$$

$1 \le i, j \le k$.

We also need the following lemma, which is proved by an easy induction on n.

Lemma 1. For $i = 1, 2, \ldots, n$, let x_i be an arbitrary variable. Then

$$\begin{vmatrix} x_1 & 1 & 1 & \ldots & 1 \\ 1 & x_2 & 0 & \ldots & 0 \\ 1 & 0 & x_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & x_n \end{vmatrix}_{n\times n} = \prod_{i=1}^{n} x_i - \sum_{i=2}^{n} \prod_{j=2, j\neq i}^{n} x_j.$$

Suppose for $j = 1, 2, \cdots, k$, β_{r_j} is a generalized Bethe tree of r_j levels which degree of root vertex is m_j. The induced subgraph obtained by deletion of root vertex of β_{r_j} is disjoint union of m_j generalized Bethe trees of $r_j - 1$ levels which are denoted by β'_{r_j}. Now the characteristic polynomial of adjacency matrix of the rooted tree in which degree of root vertex is k and β_{r_j} for $j = 1, 2, \cdots, k$ are its induced subtrees obtained by deleting the root vertex can be computed as follow.

Theorem 1. *Let $T = S_{k+1}(S_1, \beta_{r_1}, \beta_{r_2}, \cdots, \beta_{r_k})$. Then the characteristic polynomial of adjacency matrix of T is*

$$P_T(\lambda) = \prod_{i=1}^{k} P_{\beta_{r_i}}(\lambda)\Big(\lambda - \sum_{i=1}^{k} \prod_{j=1, j\neq i}^{k} \frac{P^{m_j}_{\beta'_{r_j}}(\lambda)}{P_{\beta_{r_j}}(\lambda)}\Big).$$

Proof. Suppose for $j = 1, 2, \cdots, k$ degree of root vertex of β_{r_j} is equal to m_j. If $\bar{\beta}_{r_j}$ denotes the induced subtree obtained by deleting the root vertex of β_{r_j}, then $\bar{\beta}_{r_j}$ contains m_j disjoint copy of β'_{r_j}. Thus characteristic polynomial of adjacency matrix of computed as $\bar{P}_{\beta_{r_j}}(\lambda) = P^{m_j}_{\beta'_{r_j}}(\lambda)$. By Theorem C and Lemma 1

$$P_T(\lambda) = (-1)^{k+1}\begin{vmatrix} -\lambda & 1 & 1 & \cdots & 1 \\ P^{m_1}_{\beta'_{r_1}}(\lambda) & -P_{\beta_{r_1}}(\lambda) & 0 & \cdots & 0 \\ P^{m_2}_{\beta'_{r_2}}(\lambda) & 0 & -P_{\beta_{r_2}}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P^{m_k}_{\beta'_{r_k}}(\lambda) & 0 & 0 & \dots & -P_{\beta_{r_k}}(\lambda) \end{vmatrix}$$

$$= (-1)^{k+1}\prod_{j=1}^{k} P^{m_j}_{\beta'_{r_j}}(\lambda)\begin{vmatrix} -\lambda & 1 & 1 & \cdots & 1 \\ 1 & -\frac{P_{\beta_{r_1}}(\lambda)}{P^{m_1}_{\beta'_{r_1}}(\lambda)} & 0 & \cdots & 0 \\ 1 & 0 & -\frac{P_{\beta_{r_1}}(\lambda)}{P^{m_2}_{\beta'_{r_2}}(\lambda)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & -\frac{P_{\beta_{r_1}}(\lambda)}{P^{m_k}_{\beta'_{r_k}}(\lambda)} \end{vmatrix}$$

$$= \prod_{j=1}^{k} P^{m_j}_{\beta'_{r_j}}(\lambda)\Big(\lambda\prod_{i=1}^{k}\frac{P_{\beta_{r_i}}(\lambda)}{P^{m_i}_{\beta'_{r_i}}(\lambda)} - \sum_{i=1}^{k}\prod_{j=1,j\neq i}^{k}\frac{P_{\beta_{r_j}}(\lambda)}{P^{m_j}_{\beta'_{r_j}}(\lambda)}\Big)$$

$$= \prod_{i=1}^{k} P_{\beta_{r_i}}(\lambda)\Big(\lambda - \sum_{i=1}^{k}\prod_{j=1,j\neq i}^{k}\frac{P^{m_j}_{\beta'_{r_j}}(\lambda)}{P_{\beta_{r_j}}(\lambda)}\Big).$$

Therefore Theorem is proved. □

Let $Q_{\beta_{r_i}}(\lambda)$ denote the characteristic polynomial of Laplacian matrix of generalized Bethe tree β_{r_i} and $\overline{Q}_{\beta_{r_i}}(\lambda) = \det(\lambda I - L(\beta_{r_i})^{1,1})$. Similar to Theorem 1 we can use of Theorem C and Lemma 1 to compute the characteristic polynomial of Laplacian matrix of $T = S_{k+1}(S_1, \beta_{r_1}, \beta_{r_2}, \cdots, \beta_{r_k})$.

Theorem 2. Let $T = S_{k+1}(S_1, \beta_{r_1}, \beta_{r_2}, \cdots, \beta_{r_k})$. Then characteristic polynomial of Laplacian matrix of T is

$$Q_T(\lambda) = \prod_{i=1}^{k} Q_{\beta_{r_i}}(\lambda)\Big(\lambda - k - \sum_{i=1}^{k}\prod_{j=1,j\neq i}^{k}\frac{\bar{Q}^{m_j}_{\beta_{r_j}}(\lambda)}{Q_{\beta_{r_j}}(\lambda)}\Big).$$

Proof. The proof is similar to that of Theorem 1. □

Example 1. Let T be a rooted tree of 4 levels as shown in figure 2. The induced subtree of T obtained by deleting the root vertex of T is equal to disjoint union of generalized Bethe trees $\beta_1, \beta_2, \beta_3$. By using Theorem A the

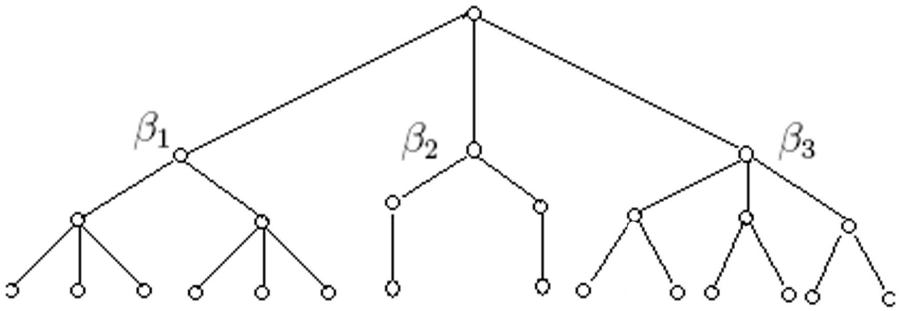

Figure 2: The rooted tree of example 1

characteristic polynomial of adjacency matrix of β_i for $i = 1, 2, 3$ can be computed as follow:

$$P_{\beta_1}(\lambda) = \lambda^5(\lambda^2 - 5)(\lambda^2 - 3)$$
$$P_{\beta_2}(\lambda) = \lambda(\lambda - 1)(\lambda + 1)(\lambda^2 - 3)$$
$$P_{\beta_3}(\lambda) = \lambda^4(\lambda^2 - 5)(\lambda^2 - 2)^2.$$

Now let S_{n+1} denote the star graph of order $n + 1$, then $P_{s_{n+1}}(\lambda) = \lambda^{n-1}(\lambda^2 - n)$. So $P_{\beta'_1}(\lambda) = \lambda^2(\lambda^2 - 3)$, $P_{\beta'_2}(\lambda) = \lambda^2 - 1$ and $P_{\beta'_3}(\lambda) = \lambda(\lambda^2 - 2)$. Therefore by using of Theorem 1 the characteristic polynomial of adjacency matrix of T can be computed as follow:

$$\begin{aligned} P_T(\lambda) &= P_{\beta_1}(\lambda)P_{\beta_2}(\lambda)P_{\beta_3}(\lambda)\left(\lambda - \frac{P^2_{\beta'_1}(\lambda)}{P_{\beta_1}(\lambda)} - \frac{P^2_{\beta'_2}(\lambda)}{P_{\beta_2}(\lambda)} - \frac{P^3_{\beta'_3}(\lambda)}{P_{\beta_3}(\lambda)}\right) \\ &= \lambda^9(\lambda^{16} - 24\lambda^{14} + 238\lambda^{12} - 1272\lambda^{10} + 4001\lambda^8 - 7568\lambda^6 \end{aligned}$$

$$+8384\lambda^4 - 4960\lambda^2 + 1200).$$

To compute of characteristic polynomial of Laplacian matrix of T we can use Theorem 2. Since $Q_{s_{n+1}}(\lambda) = (\lambda - 1)^{n-1}(\lambda^2 - (n+1)\lambda)$, so

$$\overline{Q}_{\beta_1}(\lambda) = (\lambda^2 - 5\lambda + 1)(\lambda - 1)^2$$
$$\overline{Q}_{\beta_2}(\lambda) = \lambda^2 - 3\lambda + 1$$
$$\overline{Q}_{\beta_3}(\lambda) = (\lambda - 1)(\lambda^2 - 4\lambda + 1).$$

Let $Q_{\beta_i}(\lambda)$ denote the characteristic polynomial of Laplacian matrix of β_i for $i = 1, 2, 3$. By using Theorem B we have

$$Q_{\beta_1}(\lambda) = (\lambda^2 - 5\lambda + 1)(\lambda^3 - 8\lambda^2 + 14\lambda - 1)(\lambda - 1)^4$$
$$Q_{\beta_2}(\lambda) = (\lambda^2 - 3\lambda + 1)(\lambda^3 - 6\lambda^2 + 8\lambda - 1)$$
$$Q_{\beta_3}(\lambda) = (\lambda^3 - 8\lambda^2 + 14\lambda - 1)(\lambda^2 - 4\lambda + 1)^2(\lambda - 1)^3.$$

Therefore by using of Theorem 1 we have

$$\begin{aligned} Q_T(\lambda) &= Q_{\beta_1}(\lambda)Q_{\beta_2}(\lambda)Q_{\beta_3}(\lambda)\Big(\lambda - 3 - \frac{\bar{Q}^2_{\beta_1}(\lambda)}{Q_{\beta_1}(\lambda)} - \frac{\bar{Q}^2_{\beta_2}(\lambda)}{Q_{\beta_2}(\lambda)} - \frac{\bar{Q}^3_{\beta_3}(\lambda)}{Q_{\beta_3}(\lambda)}\Big) \\ &= \lambda(\lambda-1)^7(\lambda^2-3\lambda+1)(\lambda^2-5\lambda+1)(\lambda^6-17\lambda^5+109\lambda^4 \end{aligned}$$

$$-328\lambda^3+465\lambda^2-263\lambda+25)(\lambda^3-8\lambda^2+14\lambda-1)(\lambda^2-4\lambda+1)^2.$$

Now suppose $\beta_{k_i,d}$ is the Bethe tree of k_i levels such that the vertex root has degree d, the vertices in the intermediate levels have degree $d+1$ and the vertices in level k are the pendant vertices for $i=1,2,\ldots,d$. So $B = S_{d+1}(S_1, \beta_{k_1,d}, \beta_{k_2,d}, \ldots, \beta_{k_d,d})$ is a rooted tree in which subtrees obtained by deletion its root vertex are Bethe trees where have not equal levels. In the two following corollaries we compute characteristic polynomials of adjacency and Laplacian matrices of B.

Corollary 1. Let $B = S_{d+1}(S_1, \beta_{k_1,d}, \beta_{k_2,d}, \ldots, \beta_{k_d,d})$. Then the characteristic polynomial of adjacency matrix of B is

$$P_B(\lambda) = \prod_{i=1}^{d} P_{k_i,d}(\lambda)\Big(\lambda - \sum_{i=1}^{d}\frac{P_{k_i-1}(\lambda)}{P_{k_i}(\lambda)}\Big).$$

Proof. Let $\beta'_{k_i,d}$ denote one of the d induced subtrees of $\beta_{k_i,d}$ obtained by delation it's root vertex for $i=1,2,\ldots,d$. Since number of vertices placed on ith level of $\beta_{k_i,d}$ is d^{i-1}, thus by Theorem A we have

$$P_{\beta_{k_i,d}}(\lambda) = P_{k_i}\prod_{j=1}^{k_i-1} P_j^{d^{k_i-j-1}(d-1)}.$$

$$P_{\beta'_{k_i,d}}(\lambda) = P_{k_i-1}\prod_{j=1}^{k_i-2} P_j^{d^{k_i-j-1}(d-1)}.$$

Thus by Theorem 1 the characteristic polynomial of adjacency matrix of B computed as follow:

$$\begin{aligned} P_B(\lambda) &= \prod_{i=1}^{d} P_{\beta_{k_i,d}}(\lambda)\Big(\lambda - \sum_{i=1}^{d}\prod_{j=1,j\neq i}^{d}\frac{P^d_{\beta'_{k_j,d}}(\lambda)}{P_{\beta_{k_j,d}}(\lambda)}\Big) \\ &= \prod_{i=1}^{d} P_{\beta_{k_i,d}}(\lambda)\Big(\lambda - \sum_{i=1}^{d}\frac{(P_{k_i-1}\prod_{j=1}^{k_i-2} P_j^{d^{k_i-j-1}(d-1)})^d}{P_{k_i}\prod_{j=1}^{k_i-1} P_j^{d^{k_i-j-1}(d-1)}} \\ &= \prod_{i=1}^{d} P_{k_i,d}(\lambda)\Big(\lambda - \sum_{i=1}^{k}\frac{P_{k_i-1}(\lambda)}{P_{k_i}(\lambda)}\Big). \end{aligned}$$

Therefore the Corollary is proved. □

Corollary 2. Let $B = S_{d+1}(S_1, \beta_{k_1,d}, \beta_{k_2,d}, \ldots, \beta_{k_d,d})$. Then the characteristic polynomial of Laplacian matrix of

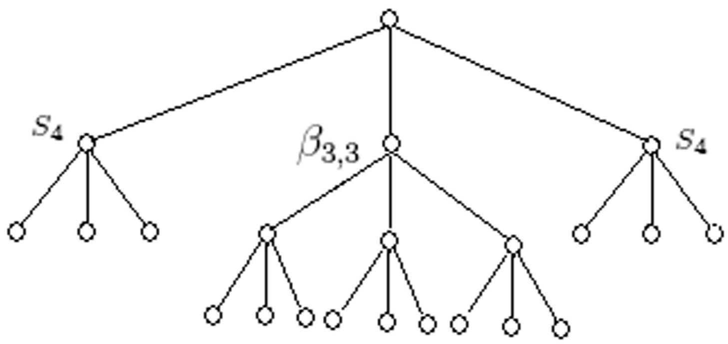

Figure 3: The rooted tree of example 3.

B is

$$Q_B(\lambda) = \prod_{i=1}^{d} Q_{k_i,d}(\lambda)\left(\lambda - d - \sum_{i=1}^{d} \frac{Q_{k_i-1}(\lambda)}{Q_{k_i}(\lambda)}\right).$$

Proof. The proof is similar to that of in Corollary 1. □

Example 2. Let B be a rooted tree as shown in figure 3. The induced subtree of B obtained by deletion of root vertex of B is equal to disjoint union of two star graph of order 4 and $\beta_{3,3}$. By using of Theorem A

$$\begin{aligned} P_{\beta_{3,3}}(\lambda) = P_3(\lambda)P_2^2(\lambda)P_1^6(\lambda) &= (\lambda^3 - 6\lambda)(\lambda^2 - 3)^2\lambda^6 \\ &= \lambda^{13}12\lambda^{11} + 45\lambda^9 - 54\lambda^7. \end{aligned}$$

Therefore by using Corollary 1 characteristic polynomial of adjacency matrix of B computed as follow:

$$\begin{aligned} P_B(\lambda) &= P_{S_4}^2(\lambda)P_{\beta_{3,3}}(\lambda)\left(\lambda - 2\frac{P_1(\lambda)}{P_2(\lambda)} - \frac{P_2(\lambda)}{P_3(\lambda)}\right) \\ &= \lambda^{10}(\lambda^3 - 6\lambda - 3)(\lambda^3 - 6\lambda + 3)(\lambda^2 - 3)^3. \end{aligned}$$

Now by using notation of Theorem B, $Q_0(\lambda) = 1$ and $Q_1(\lambda) = \lambda - 1$. So

$$\begin{aligned} Q_2(\lambda) &= (\lambda - 4)Q_1(\lambda) - 3Q_0(\lambda) = \lambda^2 - 5\lambda + 1. \\ Q_3(\lambda) &= (\lambda - 4)Q_2(\lambda) - 3Q_1(\lambda) = \lambda^3 - 9\lambda^2 + 18\lambda - 1. \end{aligned}$$

Therefore by using Corollary 2 characteristic polynomial of Laplacian matrix of B computed as follow:

$$\begin{aligned} Q_B(\lambda) &= \left(Q_3(\lambda)Q_3^2(\lambda)Q_1^6(\lambda)\right)\left(Q_1^2(\lambda)Q_2(\lambda)\right)^2\left(\lambda - 3 - 2\frac{Q_1(\lambda)}{Q_2(\lambda)} - \frac{Q_2(\lambda)}{Q_3(\lambda)}\right) \\ &= \lambda(\lambda - 1)^{10}(\lambda^2 - 5\lambda + 1)^3(\lambda^5 - 17\lambda^4 + 103\lambda^3 - 262\lambda^2 + 242\lambda - 22). \end{aligned}$$

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