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Using the Finite Differences Method for the Fredholm Integral Equations of the Second type

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ABSTRACT

In this paper, we want to solve the Fredholm integral equations of the second type using the numerical finite differences method. In this method, we use the forward, central and backward operator's to solve integral equations, and finally we compare these methods with the help of numerical examples.

1 Introduction

There has been a growing interest in the Fredholm integral equations in recent years. The use of the Fredholm integral equation has increased in many physical applications, e.g. potential theory and Dirichlet problems, electrostatic, mathematical problems of radiative equilibrium, the particle transport problems of astrophysics and reactor theory, and radiative heat problems. Some methods for solving Fredholm integral equation such as quadrature methods, single-term Walsh series method [11], Lagrange interpolation [10] and mixed interpolation collocation methods [4], Adomian's decomposition method [13, 5], etc have been developed. Recently, Mechanical algorithm method [12] for solving Fredholm integral equation had been developed. For further reading, you can read reference [9, 2, 7]. In this work, a new method is proposed to estimate the solution of a Fredholm integral equation of the second kind by using finite differences method. Linear and nonlinear Fredholm integral equations of the second kind is defined as follow,

$$F(s) = f(s) + \lambda \int_a^b k(s,t)F(t)dt, \quad a \leq s,t \leq b \tag{1.1}$$

$$F(s) = f(s) + \lambda \int_a^b k(s,t)G(F(t))dt, \quad a \leq s,t \leq b \tag{1.2}$$

where $k(s,t)$ is an arbitrary crisp kernel function over the square $a \leq s, t \leq b$, $\lambda \geq 0$ and G(F(t)) is nonlinear function of F(t).

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In section 2, we briefly present the necessary preliminaries. In sections 3, we propose a numerical method. Some numerical examples are given in section 4. Conclusion is given in section 5.

2 Preliminaries

Definition 2.1. *Newton's divided differences:*
The interpolating polynomial $F_n(x)$, at $n+1$ distinct points $x_0, x_1, \ldots, x_n$ can be written as,

$$F_n(x) = F[x_0] + \sum_{j=0}^{n-1} h_j(x) F[x_0, x_1, \ldots, x_{j+1}] \tag{2.1}$$

where,

$$h_j(x) = \prod_{i=0}^{j} (x - x_i),$$

and,

$$\begin{aligned} &F[x_0] = f(x_0), \\ &F[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \\ &\vdots \\ &F[x_0, \ldots, x_n] = \frac{f[x_0, x_1, \ldots, x_n] - f([x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0} \end{aligned}$$

Definition 2.2. *Finite Differences Operators:*
Let the tabular points $x_0, x_1, \ldots, x_n$ be equally spaced, that is, $x_i = x_0 + ih$, $i = 0, 1, \ldots, n$. The forward Δ, backward ∇ and central differences δ operators are defined as follows:

$$\begin{aligned} \Delta f(x_i) &= f(x_i + h) - f(x_i) \\ \nabla f(x_i) &= f(x_i) - f(x_i - h) \\ \delta f(x_i) &= f(x_i + \frac{h}{2}) - f(x_i - \frac{h}{2}). \end{aligned}$$

The mentioned nth order differences operations are as follows [8],

$$\Delta^n f(x_i) = \sum_{i=0}^{n} (-1)^k \binom{n}{k} f_{i+n-k}, \tag{2.2}$$

$$\nabla^n f(x_i) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f_{i-k}, \tag{2.3}$$

$$\delta^n f(x_i) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f_{i+\frac{n}{2}-k}, \tag{2.4}$$

where

$$f_i = f(x_i)$$

and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The Newton's divided differences in terms of forward, backward and central differences are as follows, which can be proved by induction [8].

$$F[x_0, x_1, \ldots, x_n] = \frac{1}{n!h^n}\Delta^n f_0, \tag{2.5}$$
$$F[x_0, x_1, \ldots, x_n] = \frac{1}{n!h^n}\nabla^n f_n, \tag{2.6}$$

and

$$F[x_0, x_1, \ldots, x_{2m}] = \frac{1}{(2m)!h^{2m}}\delta^{2m} f_m, \tag{2.7}$$
$$F[x_0, x_1, \ldots, x_{2m+1}] = \frac{1}{(2m+1)!h^{2m+1}}\delta^{2m+1} f_{m+1}. \tag{2.8}$$

3 Solving the linear Fredholm integral equations of the second kind by using Finite Differences Method

In this section, we propose a method for solving the linear and nonlinear Fredholm integral equations of the second kind by using Finite Differences Method. In order to, we first calculate the approximate solutions of $F(s)$, $k(s,t)F(t)$ by using Finite differences method as follows:

$$F(s) \simeq F_n(s) = F(s_0) + \sum_{j=0}^{n-1} h_j(s) F[s_0, ..., s_{j+1}] \tag{3.1}$$

and

$$k(s,t)F(t) \simeq k(s,t)F_n(t) = k(s,s_0)F(s_0) + \sum_{j=0}^{n-1} k(s,s_{j+1})F[s_0,...,s_{j+1}]h_j(t) \tag{3.2}$$

where, $s_j, 0 \leq j \leq n$, are distinct node points. By integration of Eq. (3.2) from a to b we have,

$$\int_a^b k(s,t)F_n(t)dt = \int_a^b k(s,s_0)F(s_0)dt + \sum_{j=0}^{n-1} k(s,s_{j+1})F[s_0,...,s_{j+1}]B_j \tag{3.3}$$

where,

$$B_j = \int_a^b h_j(t)dt, \qquad h_j(t) = \prod_{i=0}^{j}(t - t_i).$$

At last, the $(n+1)\times(n+1)$ system of linear equation is obtained. By solving this system of equations, $F[s_0], ..., F[s_0, ..., s_n]$ are given. We also obtain $F_n(s)$ which is the interpolation polynomial for $F(s)$. Then, the iterative procedure

$$F_{n,0}(s) = F_n(s),$$

$$F_{n,k+1}(s) = f(s) + \lambda \int_a^b k(s,t) F_{n,k}(t)dt, \tag{3.4}$$

converges to the solution of Eq.(1.1).

Nonlinear Fredholm integral equations is solved such as the linear Fredholm integral equations by the proposed method. But in solving nonlinear equations, the last system of nonlinear equations (14) is replaced ny nonlinear system and can be solved by Newton's method or other methods. In the sequel, the distance between approximate and exact solutions is considered as follows:

$$D(F_{exact}(s), F_{n,k}(s)) = |F_{exact}(s) - F_{n,k}(s)|, \tag{3.5}$$

where $F_{n,k}$ is the approximation of F.

In next subsections, we propose four methods for approach F. In order to, consider a uniform partition of the closed interval $[a,b]$ given by $s_i a + H$, $i = 1, \dots, n$ of step length $H = \dfrac{b-a}{n}$, $n \in N$, then we get

3.1 Numerical solution by Gregory-Newton's forward difference interpolation

By applying, (2.5) in (3.1), (3.3) and by replacing them in (1.1) we get,

$$\sum_{j=0}^{n} h_j(u_i)\Delta^j F(u_0) = f(s_i) + \sum_{j=0}^{n} \Delta^j F(u_0)\lambda k(s_i, s_j) B_j \quad i = 0(1)n, \tag{3.6}$$

where

$$h_j(u_i) = \begin{pmatrix} u_i \\ j \end{pmatrix}, \quad u = \frac{s - s_0}{h}, \quad u_i = \frac{s_i - s_0}{h}, \quad i \geq 0$$

and

$$B_j = \int_a^b h_j(u_i)du.$$

3.2 Numerical solution by Gregory-Newton's backward difference interpolation

By applying, (2.5) in (3.1), (3.3) and by replacing them in (1) we get,

$$\sum_{j=0}^{n} h_j(u_i)\nabla^j F(u_n) = f(s_i) + \sum_{j=0}^{n} \nabla^j F(u_n)\lambda k(s_i, s_j) B_i \quad i = 0(1)n, \tag{3.7}$$

where,

$$h_j(u_i) = (-1)^j \begin{pmatrix} -u_i \\ j \end{pmatrix}, \quad u = \frac{s - s_n}{h}, \quad u_i = \frac{s_i - s_n}{h} = (i - n)$$

and

$$B_j = \int_a^b h_j(u_i)du.$$

3.3 Numerical solution by Stirling's interpolation

For n even, we assume that the node points are $s_{-p}, s_{-(p-1)}, ..., s_{-1}, s_0, s_1, .$
$..,s_{p-1}, s_p$ and by applying, (2.7) in (3.1), (3.3) and by replacing them in (1.1) we get,

$$F(u_0)+\sum_{j=0}^{p-1}h_j(u_i)\delta^{2j}F(u_0)+\sum_{j=1}^{p-1}\widehat{h}_j(u_i)\frac{1}{2}[\delta^{2j-1}F_{\frac{1}{2}}+\delta^{2j-1}F_{-\frac{1}{2}}]=$$

$$f(s_i)+\lambda k(s_i,s_0)B_0F(u_0)+\sum_{j=0}^{p-1}\delta^{2j}F(u_0)\lambda k(s_i,s_j)B_j$$

$$+\sum_{j=1}^{p-1}\frac{1}{2}[\delta^{2j-1}F_{\frac{1}{2}}+\delta^{2j-1}F_{-\frac{1}{2}}]\lambda k(s_i,s_0)\widehat{B_j}\quad i=0(1)n, \tag{3.8}$$

where

$$h_j(u_i)=\frac{1}{2(j+1)!}\prod_{k=0}^{j}(u_i^2-k^2),\quad u_i=\frac{s_i-s_0}{h},$$

and

$$\widehat{h}_j(u_i)=\frac{1}{u_i(2j+1)!}\prod_{k=0}^{j}(u_i^2-k^2),\quad j=0,1,...,p-1$$

and
$B_j=\int_a^b h_j(u_i)du$ and $\widehat{B}_j=\int_a^b\widehat{h}_j(u_i)du,\quad u=\frac{s-s_0}{h}\quad j=0,1,2,...,p-1$

3.4 Numerical solution by the Bessel's interpolation

For n odd, we take the node points as $s_{-p},s_{-(p-1)},...,s_{-1},s_0,s_1,...,s_{p-1},s_p$ and by applying, (2.7) in (3.1), (3.3) by replacing them in (1.1) we get,

$$\frac{1}{2}[F(v_0)+F(v_1)]+\sum_{j=0}^{p}h_j(v_i)\frac{1}{2}[\delta^{2j}F(v_0)+\delta^{2j}F(v_1)]$$

$$+\sum_{j=0}^{p}\widehat{h}_j(v_i)\delta^{2j+1}F(\frac{v_0+v_1}{2})=f(s_i)+\lambda k(s_i,s_0)B_0\frac{1}{2}[F(v_0)+F(v_i)]$$

$$+\sum_{j=0}^{p}\frac{1}{2}[\delta^{2j}F(v_0)+\delta^{2j}F(v_1)]\lambda k(s_i,s_j)B_j+\lambda k(s_i,s_0)\widehat{B}_0\delta F(\frac{v_i+v_0}{2})$$

$$+\sum_{j=0}^{p}\delta^{2j+1}F(\frac{v_1+v_0}{2})\lambda k(s_i,s_j)\widehat{B}_j\qquad i=0(1)n, \tag{3.9}$$

where

$$h_j(v_i)=\frac{1}{(2j)!}\prod_{k=1}^{j}(v_i^2-(2k-1)^2/4)\quad,\quad\widehat{h}_j(v_i)=\frac{v_i}{(2j+1)!}\prod_{k=1}^{j}(v_i^2-(2k-1)^2/4),$$

$$v=u-\frac{1}{2}$$

$$B_j = \int_a^b h_j(v_i)dv \quad , \quad \widehat{B}_j = \int_a^b \widehat{h}_j(v_i)dv \quad j = 1, 2, ..., p, \quad i = 0, 1, 2, ..., n.$$

4 The Numerical example

Example 1. Consider the following Fredholm integral equation

$$F(s) = 2sin(\frac{s}{2}) + \int_0^{2\pi} 0.1sin(t)sin(\frac{s}{2})F(t)dt,$$

The exact solution in this case is given

$$F(s) = 2sin(\frac{s}{2}).$$

Table 1.1: Comparison between the exact solution and the approximate solutions in $n = 3$ and $1, 3, 5$ iterations

	s	$D(F_{exact}, F_{3,1})$	$D(F_{exact}, F_{3,3})$	$D(F_{exact}, F_{3,5})$
	$\pi/6$	$1.734940555 \times 10^{-1}$	$2.586908441 \times 10^{-10}$	$2.586908441 \times 10^{-10}$
Forward	$\pi/4$	$2.565277841 \times 10^{-1}$	$3.824994973 \times 10^{-10}$	$3.824994973 \times 10^{-10}$
	$\pi/3$	$3.351766941 \times 10^{-1}$	$4.997701026 \times 10^{-10}$	$4.997701026 \times 10^{-10}$
	$\pi - \pi/6$	$2.375431808 \times 10^{-1}$	$9.657538600 \times 10^{-10}$	$9.657538600 \times 10^{-10}$
Central	π	$2.459665090 \times 10^{-1}$	$9.999996829 \times 10^{-10}$	$9.999996829 \times 10^{-10}$
	$\pi + \pi/6$	$2.376445203 \times 10^{-1}$	$9.661658649 \times 10^{-10}$	$9.661658649 \times 10^{-10}$
	$2\pi - \pi/6$	$1.105189430 \times 10^{-1}$	$2.602289546 \times 10^{-10}$	$2.602289546 \times 10^{-10}$
Backward	$2\pi - \pi/4$	$1.630718599 \times 10^{-1}$	$3.839705529 \times 10^{-10}$	$3.839705529 \times 10^{10}$
	$2\pi - \pi/3$	$2.128373964 \times 10^{-1}$	$5.011489586 \times 10^{-10}$	$5.011489586 \times 10^{-10}$

Example 2. [5] Consider the following Fredholm integral equation .

$$F(s) = e^{3s} - \frac{1}{9}(2e^3 + 1)s + \int_0^1 stF(t)dt, \quad 0 \leq x \leq 1$$

with the exact solution of

$$F(s) = e^{3s}.$$

Table 2.1: Comparison between the exact solution and the approximate solutions in $n = 3$ and $5, 10, 15, 20$ iterations

s	$D(F_{exact}, F_{3,5})$	$D(F_{exact}, F_{3,10})$	$D(F_{exact}, F_{3,15})$	$D(F_{exact}, F_{3,20})$	
Forward	0.25	1.0100×10^{-2}	4.1717×10^{-5}	1.7200×10^{-6}	1.000×10^{-9}
Central	0.50	6.8400×10^{-2}	2.8155×10^{-4}	1.1595×10^{-6}	5.500×10^{-9}
Backward	0.75	1.0260×10^{-1}	4.2233×10^{-4}	1.7393×10^{-6}	5.500×10^{-9}

Table 2.2: Comparison between the exact solution and the approximate solutions in $n = 5$ and $5, 10, 15, 20$ iterations

s	$D(F_{exact}, F_{5,5})$	$D(F_{exact}, F_{5,10})$	$D(F_{exact}, F_{5,15})$	$D(F_{exact}, F_{5,20})$	
Forward	0.25	2.3427×10^{-4}	9.6450×10^{-7}	4.2500×10^{-9}	5.000×10^{-10}
Central	0.50	3.46×10^{-2}	1.4232×10^{-4}	5.865×10^{-7}	3.0000×10^{-9}
Backward	0.75	5.94×10^{-2}	2.4434×10^{-4}	1.0065×10^{-6}	5.250×10^{-9}

Example 3. Consider the following nonlinear Fredholm integral equation

$$F(s) = \cos(s) - \frac{\pi}{8} + \frac{1}{2}\int_0^{\frac{\pi}{2}} F^2(t)dt$$

with the exact solution of

$$F(s) = \cos(s)$$

Table 3.1: Comparison between the exact solution and the approximate solutions in $n = 2$ and $5, 10, 15, 20$ iterations

	s	$D(F_{exact}, F_{2,5})$	$D(F_{exact}, F_{2,10})$	$D(F_{exact}, F_{2,15})$	$D(F_{exact}, F_{2,20})$
Forward	$\pi/8$	0.0816610442	0.0641778920	0.0507725599	0.0420680048
Central	$\pi/4$	0.1486126275	0.0903591924	0.0655588336	0.0516225578
Backward	$3\pi/8$	0.1024923128	0.0715514996	0.0552159496	0.0393802415

5 Conclusion

In this work, we proposed a numerical method for solving linear Fredholm integral equations of the second kind by using finite differences method. The solution of the integral equation is approximated by an iterative method that the start point of this method is the achieved interpolation polynomial by solving the system of equation. The advantage of this method in relation to other methods is that the solution is approximated by having supported points.

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