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A Novel Method to Solve Fuzzy Volterra Integral Equations Using Collocation Method

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Abstract

Fuzzy Volterra integral equations, especially the second kind is interested for researchers to be solved with numerical methods since analytical methods are not applicable. Here a new study based on Fibonacci polynomials collocation method in order to solve them is introduced. Some properties of these polynomials are considered to implement a collocation method in order to approximate the solution of Fuzzy Volterra integral equations of the second kind. The existence and uniqueness of the solution also convergence and error analysis of proposed method are proved thoroughly. The results showed the calculations of the method are simple and low cost.

1 Introduction

Fuzzy systems have been used in a various of problems ranging from fuzzy metric spaces [1], fuzzy topological spaces [2], control chaotic systems [3, 4], fuzzy differential equations [5, 6, 7] and particle physics [8, 9, 10, 11]. The topics of fuzzy integral equations (FIE) which attracted growing interest for some times in relation with fuzzy control, have been developed in recent years.
The concept of integration of fuzzy functions was firstly introduced by Dubois and Prade [12]. Alternative approaches were later suggested by Goetschel and Voxman [13], Kaleva [14], Matloka [15], Seikkala [16].
Recently, some numerical methods have been introduced to solve fuzzy integral equation of the second kind in one-dimensional space FIE-2. For instance Babolian et al [17] used the Adomian decomposition method (ADM) to solve Fredholm fuzzy integral equations of the second kind (FFIE-2). Also Allahviranloo et al [18] applied the homotopy perturbation method for solving fuzzy Volterra integral equations (FVIE). Allahviranloo et al [19] obtained numerical solution of FFIE-2 by modified trapezoidal method. Allahviranloo and Behzadi [20] used airfoil and Chebyshev polynomials methods to solve fuzzy Fredholm integro-differential equations with Cauchy kernel. After Mirzaee and hosieni [21] solved systems of linear Fredholm integro-differential equations with Fibonacci polynomials. Besides Behzadi et al [22] used fuzzy collocation methods for solving second–order fuzzy Abel–Volterra integro–differential equations. Mirzaee and hosieni [30] solved a class of Fredholm–Volterra integral equations in two-dimensional spaces by Fibonacci collocation method. Recently Paripour and Kamyar [36] used new basis functions to solve nonlinear Volterra-Fredholm integral equations.

In this work, we use the Fibonacci polynomials collocation method in order to find a solution for the Fuzzy Volterra integral equations of the second kind. The existence and uniqueness of this solution are proved to remove hesitation about the method. Also convergence and error analysis of proposed method are examined thoroughly.

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The results showed the calculations of the method respect to others are simple and low cost indeed. The rest of the paper is organized as follows: In Section 2, the basic notations and concepts of fuzzy numbers, fuzzy functions and fuzzy integrals have been illustrated. In Section 3, we point out some properties of the Fibonacci polynomials and collocation method which is used for solving Fuzzy Volterra integral equations of the second kind. In Section 4, existence and uniqueness of the solution, convergence and error analysis of the proposed method are discussed. In Section 5, numerical results with the exact solution for some examples have been compared.

2 Preliminaries

In this section the basic definitions of a fuzzy calculus is as follows.

Definition 2.1. *[23] A fuzzy number is a fuzzy set $u : R^1 \longrightarrow I = [0, 1]$ which satisfies*

(i) u *is upper semicontinuous.*

(ii) $u(x) = 0$ *outside some interval* $[c, d]$.

(iii) There are real numbers a,b: $c \leq a \leq b \leq d$ *for which*

(1) $u(x)$ *is monotonic increasing on* $[c, d]$,

(2) $u(x)$ *is monotonic decreasing on* $[b, d]$,

(3) $u(x) = 1$, $a \leq x \leq b$.

The set of all fuzzy numbers is denoted by E^1. An equivalent definition or parametric form of fuzzy numbers which yields the same E^1 is given by Kaleva [19].

Definition 2.2. *[17] An arbitrary fuzzy number in parametric form is represented by an ordered pair of functions* $(\underline{u}(\alpha), \overline{u}(\alpha))$, $0 \leq \alpha \leq 1$, *which satisfy the following requirements:*

(1) $\underline{u}(\alpha)$ *is a bounded monotonic increasing left continuous function,*

(2) $\overline{u}(\alpha)$ *is a bounded monotonic decreasing left continuous function,*

(3) $\underline{u}(\alpha) \leq \overline{u}(\alpha), 0 \leq \alpha \leq 1.$

Lemma 2.1. *[20] Suppose* $(\underline{u}(r), \bar{u}(r))$, $0 \leq r \leq 1$ *is a given family of non-empty intervals. If*

(1) $(\underline{u}(r_1), \bar{u}(r_1)) \supseteq (\underline{u}(r_2), \bar{u}(r_2))$ *for* $0 \leq r_1 \leq r_2 \leq 1.$

(2) $(\lim_{k\to\infty} \underline{u}(r_k), \lim_{k\to\infty} \bar{u}(r_k)) = (\underline{u}(r), \bar{u}(r))$*, whenever* $\{r_k\}$ *is a non-decreasing converging sequence converges to* $0 \leq r \leq 1$*,*

then the family $(\underline{u}(r), \bar{u}(r))$*,* $0 \leq r \leq 1$*, represent the* r*-cut sets of a fuzzy number* $u \in E^1$*.*
On the contrary, suppose $(\underline{u}(r), \bar{u}(r))$*,* $0 \leq r \leq 1$*, are the* r*-cut sets of a fuzzy number* $u \in E^1$*, then the conditions* (1) *and* (2) *hold.*

Definition 2.3. *[18] For arbitrary* $u = (\underline{u}(\alpha), \overline{u}(\alpha))$ *,* $v = (\underline{v}(\alpha), \overline{v}(\alpha))$ *and* $k \in R$*, we define addition and multiplication by* k *as follows:*

$$
\begin{aligned}
(\underline{u+v}(\alpha)) &= (\underline{u}(\alpha) + \underline{v}(\alpha)), \\
(\overline{u+v}(\alpha)) &= (\overline{u}(\alpha) + \overline{v}(\alpha)), \\
(\underline{ku}(\alpha)) = k\underline{u}(\alpha), (\overline{ku}(\alpha)) &= k\overline{u}(\alpha) \quad if \quad k \geq 0, \\
(\overline{ku}(\alpha)) = k\underline{u}(\alpha), (\underline{ku}(\alpha)) &= k\overline{u}(\alpha) \quad if \quad k < 0.
\end{aligned}
$$

Definition 2.4. *[26] For arbitrary* $\tilde{u} = (\underline{u}, \overline{u})$ *,* $\tilde{v} = (\underline{v}, \overline{v})$ *the distance between* u, v *is define as follows:*

$$
D(u, v) = \sup_{0 \leq \alpha \leq 1} \max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\} \tag{2.1}
$$

and also metric space (D, E^1) *is a complete metric space [25], and the following properties are well known:*

$$
\begin{aligned}
D(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}) &= D(\tilde{u}, \tilde{v}), \qquad\qquad \forall \tilde{u}, \tilde{v} \in E \\
D(k\tilde{u}, k\tilde{v}) &= |k| D(\tilde{u}, \tilde{v}), \qquad\quad \forall k \in R, \tilde{u}, \tilde{v} \in E \\
D(\tilde{u} + \tilde{v}, \tilde{w} + \tilde{e}) &\leq D(\tilde{u}, \tilde{w} + D(\tilde{v}, \tilde{e})), \forall \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in E
\end{aligned}
$$

Definition 2.5. *[27] Consider* $\tilde{u}, \tilde{v} \in E$*. If there exist* $\tilde{w} \in E$ *such that* $\tilde{u} = \tilde{v} + \tilde{w}$ *then* $\tilde{w}$ *is called the H-difference of* $\tilde{u}$ *and* $\tilde{v}$*, and is denoted by* $\tilde{u} \ominus \tilde{v}$*.*

Definition 2.6. *[22] The mapping* $\tilde{f} : T \to E$ *for some interval* T *is called a fuzzy process. Therefore, its* α*–level set can be written as follows:*

$$
[f(t)]^\alpha = [f_-^\alpha(t), f_+^\alpha(t)]
$$

Definition 2.7. *[22] Let* $\tilde{f} : T \to E$ *be Hukuhara differentiable and denote by* $[f(t)]^\alpha = [f_-^\alpha, f_+^\alpha]$*. Then, the boundary functions* f_-^α *and* f_+^α *are differentiable (or Seikkala differentiable) and*

$$
[f'(t)]^\alpha = [(f_-^\alpha)'(t), (f_+^\alpha)'(t)], t \in T, \alpha \in [0, 1].
$$

3 Fuzzy integral equations

The fuzzy Fredholm integral equation of the second kind is displayed as follows:

$$\tilde{u}(s) = \tilde{f}(s) + \lambda \int_a^b k(s,t)\tilde{u}(t)dt, \quad a \leq s \leq b, \tag{3.1}$$

If the kernel function satisfies

$$k(s,t) = 0, \quad s > t,$$

we obtain the fuzzy Volterra integral equation

$$\tilde{u}(s) = \tilde{f}(s) + \lambda \int_a^s k(s,t)\tilde{u}(t)dt, \quad s \geq a, \tag{3.2}$$

where $\tilde{u}(s)$ and $\tilde{f}(s)$ are fuzzy functions on $[a,b]$ and $k(s,t)$ is an arbitrary kernel function over $[a,b] \times [a,b]$, λ is a crisp constant and u is unknown on $[a,b]$ [24].

4 The Fibonacci polynomials and collocation method

4.1 The Fibonacci polynomials and their properties

A sequence of polynomials is a Fibonacci polynomials, $\{F_n(x)\}$, are defined by the recursion

$$F_{n+2}(x) = xF_{n+1} + F_n(x); \quad n \geq 1$$

with initial values $F_1(x) = 1$ and $F_2(x) = x$.

Definition 4.1. *[21] For any positive real number k, the $k-$Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$ is defined recurrently by*

$$F_{k,n+1} = kF_{k,n} + F_{k,n+1}, n \geq 1$$

with initial conditions

$$F_{k,0} = 0, \quad F_{k,1} = 1.$$

The Fibonacci polynomials are given by the explicit formula

$$F_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{n-2i}, \quad n \geq 0, \tag{4.1}$$

where $\lfloor \frac{n}{2} \rfloor$ denotes the greatest integer in $\frac{n}{2}$.
Note that $F_{2n}(0) = 0$ and $x = 0$ is the only real root, while $F_{2n+1}(0) = 1$ with no real roots. Also for $x = k \in N$, we obtain the elements of the k-Fibonacci sequence [26].
The Fibonacci polynomials have generating function [27]

$$G(x,t) = \frac{t}{1 - t^2 - tx} = \sum_{n-1}^{\infty} F_n(x) t^n = t + xt^2 + (x^2+1)t^3 + (x^3 + 2x)t^4 + \dots.$$

The Fibonacci polynomials are normalized so that $F_n(1) = F_n$, where the F_n is the nth Fibonacci number.
The equation for the Fibonacci polynomials can be written in matrix form as

$$F(x) = BX(x),$$

where $F(x) = [F_1(x), F_2(x), \dots, F_{N+1}(x)]^T$, $X(x) = [1, x, x^2, \dots, x^N]^T$, and B is the lower triangular matrix with entrances the coefficients appearing in expansion of Fibonacci polynomials in increasing powers of x, for example, for $N = 6$ we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 1 & 0 \\ 1 & 0 & 6 & 0 & 5 & 0 & 1 \end{bmatrix}.$$

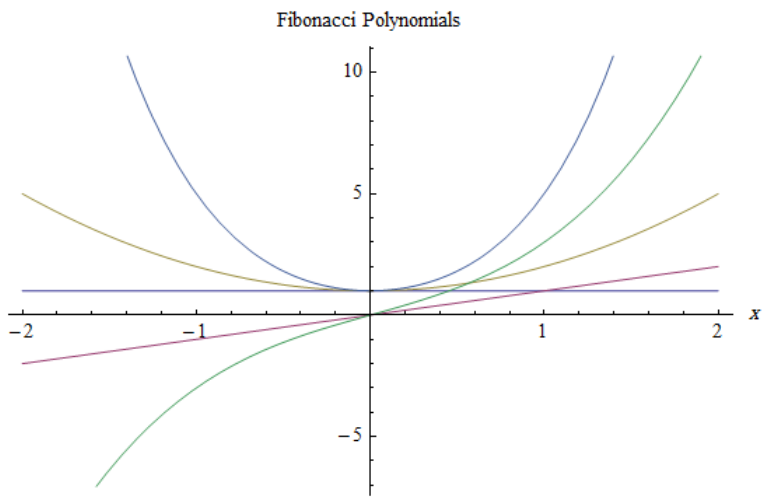
Note that in martix B the non–zero entrances construct precisely the diagonals of the pascal triangle and the sum of the elements in the same row gives the classical Fibonacci sequence. In addition, matrix B is invertible. So x^n may be written as linear of Fibonacci polynomials [31].
These expansions are given in closed form in theorem as follows

Theorem 4.1. *[31] For every integer $n \geq 1$, x^{n-1} may be written in a unique way as linear combination of the n first Fibonacci polynomials as*

$$x^{n-1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \left[\binom{n}{i} - \binom{n}{i-1} \right] F_{n-2i}(x), \quad where \quad \binom{n}{-1} = 0.$$

4.2 Description of the method

In this section the equation (3.2) is solved using the Fibonacci polynomials and collocation method.
To obtain the approximation solution of (3.2) we can write



$$y(x) = \sum_{n=1}^{N+1} c_n F_n(x), \quad 0 \leq a \leq x \leq b, \tag{4.2}$$

where $c_n, n = 1, 2, \ldots, N+1$, are unknown Fibonacci coefficients, N is an arbitrary positive integer and $F_n(x), n = 1, 2, \ldots, N+1$ are Fibonacci polynomials. The aim of the method is to get solution as Fibonacci series defined by

$$\tilde{u} \cong \tilde{u}_N = \sum_{n=1}^{N+1} c_n F_n(s) = F(s)C, \tag{4.3}$$

where $c_n, n = 1, 2, \ldots, N+1$ are unknown Fibonacci coefficients,

$$C = [c_1, c_2, \ldots, c_{N+1}]^T,$$
$$F(s) = [F_1(s), F_2(s), \ldots, F_{N+1}(s)]^T,$$

where N is an arbitrary positive integer.

5 Existence and Uniqueness of the solution and Error analysis

In this section, uniqueness and existence of the solution are proved. Then error in the method is illustrated.

Theorem 5.1. *In the Equation* (3.2)*, assume that* $\tilde{f}(s)$*,* $s \in [a, b]$ *is a fuzzy continuous function and* $k(s, t)$ *is continuous for* $s \in [a, b]$*. Moreover, assume that*

$$p = \max_{a \leq s \leq b} D(\tilde{u}(s), \tilde{0}) \ ,$$
$$M = \max_{a \leq s,t \leq b} \mid k(s,t) \mid .$$

If $L = M(b-a)^a p < 1$*, then the integral equation* (3.2) *has a unique solution.*

Proof. At first, we investigate the conditions of the Banach fixed point principle.
We define the operator $K : X \to X$ by

$$K(\tilde{u}(s)) = \tilde{f}(s) + \int_a^s k(s,t)\tilde{u}(t)dt, \quad s \in [a,b].$$

We show that the operator K is uniformly continuous. Since $\tilde{f}(s)$ is continuous on compact set of $[a,b]$, we deduce that it is uniformly continuous and hence for $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$D(\tilde{f}(s_1), \tilde{f}(s_2)) < \epsilon_1, \quad whenever \quad \mid s_1 - s_2 \mid < \delta_1, \quad s_1, s_2 \in [a,b].$$

As described above, $k(s,t)$ also is uniformly continuous, thus for $\epsilon_2 > 0$, exists $\delta_2 > 0$ such that

$$\mid k(s_1,t) - k(s_2,t) \mid < \epsilon_2, \quad whenever \quad \mid s_1 - s_2 \mid < \delta_1, \quad s_1, s_2 \in [a,b].$$

Take $\delta = \min\{\delta_1, \delta_2\}$, therefore we have

$$D(K(\tilde{u}(s_1)), K(\tilde{u})(s_2)) =$$
$$D\bigg(\tilde{f}(s_1) + \int_a^s k(s_1,t)\tilde{u}(t)dt, \tilde{f}(s_2) + \int_a^s k(s_2,t)\tilde{u}(t)dt\bigg), \tag{5.1}$$

by using Definition (2.4), we have

$$
\begin{aligned}
&D(K(\tilde{u}(s_1)),K(\tilde{u}(s_2)))\leq\\
&D(\tilde{f}(s_1),\tilde{f}(s_2))+D\bigg(\int_a^s k(s_1,t)\tilde{u}(t)dt,\int_a^s k(s_2,t)\tilde{u}(t)dt\bigg)\leq\\
&\epsilon_1+\max_{s\in[a,b]}\bigg\{\sup_{0\leq\alpha\leq1}\bigg|\int_a^s k(s_1,t)\underline{u}(t)dt-\int_a^s k(s_2,t)\underline{u}(t)dt\bigg|,\sup_{0\leq\alpha\leq1}\bigg|\int_a^s k(s_1,t)\overline{u}(t)dt-\int_a^s k(s_2,t)\overline{u}(t)dt\bigg|\bigg\}\leq\\
&\epsilon_1+\max_{s\in[a,b]}\bigg\{\sup_{0\leq\alpha\leq1}\int_a^s\mid k(s_1,t)-k(s_2,t)\mid\mid\underline{u}(t)\mid dt,\sup_{0\leq\alpha\leq1}\int_a^s\mid k(s_1,t)-k(s_2,t)\mid\mid\overline{u}(t)dt\mid\bigg\}\leq\\
&\epsilon_1+\epsilon_2\int_a^s(\tilde{u}(t),\tilde{0})dt\leq\epsilon_1+M\epsilon_2,
\end{aligned}
$$

now let $\epsilon_1=\epsilon$ and $\epsilon_2=\frac{\epsilon}{M}$, then we derive

$$
D(K(\tilde{u}(s_1)),K(\tilde{u}(s_2)))\leq\epsilon,
$$

this show that K is uniformly continuous for any $\tilde{u}(s)$, and so continuous on $[a,b]$, and hence $K(X)\subset X$. Now, we prove that the operator K is contraction map. So, for $\tilde{u}_1(s),\tilde{u}_2(s)\subset X$ and $s\in[a,b]$, now

$$
D(K(\tilde{u}_1)(x),\tilde{u}_2)(x))\leq D\bigg(\int_a^s k(s,t)\tilde{u}_1(t)dt,\int_a^s k(s,t)\tilde{u}_2(t)dt\bigg).
$$

By using Definition (2.4), we have

$$
\begin{aligned}
&D(K(\tilde{u}_1)(s),\tilde{u}_2)(s))\leq\\
&\max_{s\in[a,b]}\bigg\{\sup_{0\leq\alpha\leq1}\bigg|\int_a^s k(s,t)\underline{u}_1(t)dt-\int_a^s k(s,t)\underline{u}_2(t)dt\bigg|,\sup_{0\leq\alpha\leq1}\bigg|\int_a^s k(s,t)\overline{u}_1(t)dt-\int_a^s k(s,t)\overline{u}_2(t)dt\bigg|\bigg\}\leq\\
&\max_{s\in[a,b]}\bigg\{\sup_{0\leq\alpha\leq1}\bigg|\int_a^s\mid k(s,t)\mid\mid\underline{u}_1(t)-\underline{u}_2(t)\mid dt\bigg|,\sup_{0\leq\alpha\leq1}\bigg|\int_a^s\mid k(s,t)\mid\mid\overline{u}_1(t)-\overline{u}_2(t)\mid dt\bigg|\bigg\}\leq\\
&M\int_a^s D(\tilde{u}_1(t),\tilde{u}_2(t))dt+M\int_a^s D(\tilde{u}_1(t),\tilde{u}_2(t))dt\leq\\
&2M\int_a^s D(\tilde{u}_1,\tilde{u}_2)dt,
\end{aligned}
$$

and thus according to the above, K is the contraction on the Banach space. Consequently, the Banach fixed point principle implies that Equation (3.2) has a unique solution in X. □

Theorem 5.2. *According to the assumptions of the theorem* (5.1)*, the series solution of the Equation* (3.2) *using Fibonacci polynomials method converges.*

Proof. Let the $\tilde{u}_m(s)$ and $\tilde{u}_n(s)$, are the approximate(with $m>n$) of Equation (3.2). By using Equation (4.3) we can write

$$\tilde{u}_m(s) = \sum_{i=1}^{m} F_i(s)\tilde{U}_i,$$

$$\tilde{u}_n(s) = \sum_{i=1}^{n} F_i(s)\tilde{U}_i.$$

Assume $\tilde{u}_m(s)$ and $\tilde{u}_n(s)$ be two arbitrary partial sums with $m > n$. Now, we are going to prove that $\tilde{u}_m(s)$ is a Cauchy sequence in Banach space X.

$$D(\tilde{u}_m(s), \tilde{u}_n(s)) \leq D\Bigg(\bigg(\int_a^s k(s,t)\tilde{u}_m(t)dt\bigg), \bigg(\int_a^s k(s,t)\tilde{u}_m(t)dt\bigg)\Bigg).$$

Then

$$\begin{aligned}
&D(\tilde{u}_m(s), \tilde{u}_n(s)) \leq \\
&D\Bigg(\bigg(\int_a^s k(s,t)\sum_{i=1}^{m} F_i(t)\tilde{U}_i dt\bigg), \bigg(\int_a^s k(s,t)\sum_{i=1}^{n} F_i(t)\tilde{U}_i dt\bigg)\Bigg) = \\
&\max_{s\in[a,b]}\Bigg\{\sup_{0\leq\alpha\leq 1}\bigg|\int k(s,t)\sum_{i=1}^{m} F_i(t)\underline{U}_i dt - \int k(s,t)\sum_{i=1}^{n} F_i(t)\underline{U}_i dt\bigg|, \\
&\sup_{0\leq\alpha\leq 1}\bigg|\int k(s,t)\sum_{i=1}^{m} F_i(t)\overline{U}_i dt - \int k(s,t)\sum_{i=1}^{n} F_i(t)\overline{U}_i dt\bigg|\Bigg\} \leq \\
&\max_{s\in[a,b]}\Bigg\{\sup_{0\leq\alpha\leq 1}\bigg|\bigg(\int_a^s \mid k(s,t) \mid\mid \sum_{i=1}^{m} F_i(t)\underline{U}_i - 0 \mid dt\bigg)\bigg|, \sup_{0\leq\alpha\leq 1}\bigg|\bigg(\int_a^s \mid k(s,t) \mid\mid \sum_{i=1}^{n} F_i(t)\underline{U}_i - 0 \mid dt\bigg)\bigg|\Bigg\} + \\
&\max_{s\in[a,b]}\Bigg\{\sup_{0\leq\alpha\leq 1}\bigg|\bigg(\int_a^s \mid k(s,t) \mid\mid \sum_{i=1}^{m} F_i(t)\overline{U}_i - 0 \mid dt\bigg)\bigg|, \sup_{0\leq\alpha\leq 1}\bigg|\bigg(\int_a^s \mid k(s,t) \mid\mid \sum_{i=1}^{n} F_i(t)\overline{U}_i - 0 \mid dt\bigg)\bigg|\Bigg\} \leq \\
&M\bigg(\int_a^s D(\tilde{u}_m(t), 0)\bigg) + M\bigg(\int_a^s D(\tilde{u}_n(t), 0)\bigg),
\end{aligned} \tag{5.2}$$

it is clear that

$$D(\tilde{u}_m(t), 0) \to 0 \quad \textit{and} \quad D(\tilde{u}_n(t), 0) \to 0,$$

now, we have

$$D(\tilde{u}_m(s), \tilde{u}_n(s)) \to 0.$$

By choosing $m = n + 1$, we have

$$D(\tilde{u}_{n+1}(s), \tilde{u}_n(s)) = 0.$$

□

Now, we obtain error analysis for the Equation (3.2), by the following theorem.

Theorem 5.3. *Suppose that $\tilde{u}_N(s) = \sum_{n=1}^{N} F(s)a_n$ and $\tilde{u}(s)$ are the approximate and exact solution of Equation (3.2), respectively. If the assumption of Definition (2.4) is satisfied, then $\tilde{u}_N(s) \to \tilde{u}(s)$ as $N \to \infty$.*

Proof. Due to the Theorem (5.1), Defenition (2.4) and replace $\tilde{u}_m(s)$ by $\tilde{u}(s)$ in the mentioned theorem, we have

$$D(\tilde{u}(s), \tilde{u}_N(s)) \leq CD(\tilde{u}(s), \sum_{n=1}^{N} F(s)a_n),$$

now, we can write

$$\tilde{u}(s) = \lim_{N\to\infty} \sum_{n=1}^{N} F(s)a_n \Rightarrow \lim_{N\to\infty} D(\tilde{u}(s), \sum_{n=1}^{N} F(s)a_n) \to 0,$$

we will have

$$\lim_{N\to\infty} D(\tilde{u}(s), \tilde{u}_N(s)) \leq C \lim_{N\to\infty} D(\tilde{u}(s), \sum_{n=1}^{N} F(s)a_n),$$

as a result

$$\lim_{N\to\infty} D(\tilde{u}(s), \tilde{u}_N(s)) \to 0.$$

□

6 Numerical examples

By using Some examples and comparing numerical results with other methods in this section, in last section we are able to conclude something about new method.

Example 6.1. *[35]. Consider the fuzzy Volterra integral equation*

$$\underline{f}(x;\alpha) = 2x(\alpha^5 + 2\alpha)[3 - 3\cos(x) - x^2],$$
$$\overline{f}(x;\alpha) = 6x(2 - \alpha^3)[3 - 3\cos(x) - x^2],$$

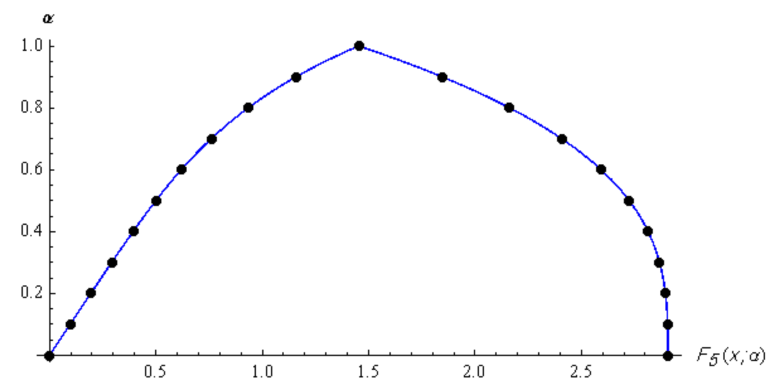


Figure 1: Comparison between the exact solution and the approximate solution of the Example 6.1

and kernel

$$k(x,t) = x\cos(t-x), \quad 0 \le t \le x, \quad 0 \le x \le \frac{\pi}{2},$$

and $a = 0, b = \frac{\pi}{4}, \lambda = 1$*, the exact solution is*

$$\underline{F}(x;\alpha) = x^3(\alpha^5 + 2\alpha),$$
$$\overline{F}(x;\alpha) = x^3(6 - 3\alpha^3).$$

In this example, $k(x,t) \ge 0$ *for each* $0 \le t \le x$.
we have

$$\underline{f}(x,\alpha) = \underline{F}(x,\alpha) - \lambda \int_0^x t\cos(x-t)\underline{F}(t,\alpha)dt$$
$$\overline{f}(x,\alpha) = \overline{F}(x,\alpha) - \lambda \int_0^x t\cos(x-t)\overline{F}(t,\alpha)dt$$

and

$$\underline{F}(x) = \sum_{i=0}^{n} \underline{U}_i F_i(x) - \underline{f}(x,\alpha) - \sum_{i=0}^{n} \underline{U}_i \int_0^x t\cos(x-t)F_i(t)dt$$
$$\overline{F}(x) = \sum_{i=0}^{n} \overline{U}_i F_i(x) - \overline{f}(x,\alpha) - \sum_{i=0}^{n} \overline{U}_i \int_0^x t\cos(x-t)F_i(t)dt$$

Example 6.1 *has been solved by assuming* $N = 5$ *and* $x = \frac{\pi}{4}$*. In figures* 2 *and* 3 *the absolute error of the above example has been showed.*

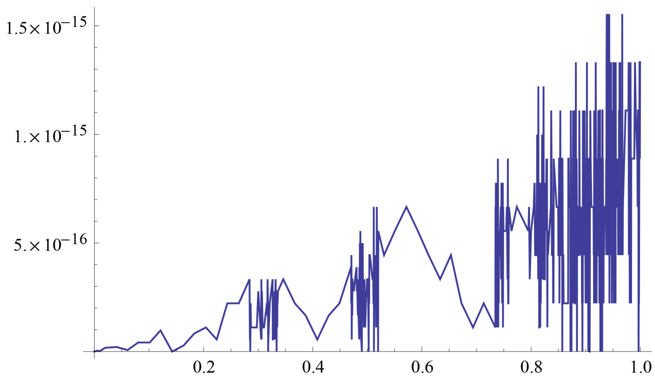


Figure 2: Absolute error $|\underline{F}(x,\alpha) - \underline{F}_5(x,\alpha)|$

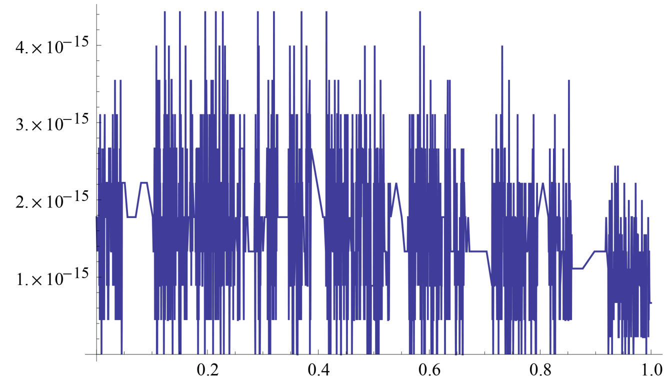


Figure 3: Absolute error $|\overline{F}(x,\alpha) - \overline{F}_5(x,\alpha)|$

Table 1: Numerical results of Example 6.1

α	Exact solution $(\underline{F}(x;\alpha), \overline{F}(x;\alpha))$	Approximate solution $x=\frac{\pi}{4}$ and $N=5$	Absolute error for presented method $x=\frac{\pi}{4}$ and $N=5$	Absolute error for VIM in [30] $x=\frac{\pi}{4}$ and $N=7$
0.0	(0.0000 , 2.9068)	(0.0000 , 2.9068)	(0.0000e-00 , 1.7763e-15)	(0.0000e-00 , 6.3733e-10)
0.1	(0.0968 , 2.9053)	(0.0968 , 2.9053)	(4.1633e-17 , 1.3322e-15)	(2.1245e-11 , 6.3701e-10)
0.2	(0.1939 , 2.8952)	(0.1939 , 2.8952)	(1.1102e-16 , 8.8817e-16)	(4.2522e-11 , 6.3478e-10)
0.3	(0.2918 , 2.7686)	(0.2918 , 2.8676)	(2.7755e-16 , 4.4408e-16)	(6.3991e-11 , 6.2872e-10)
0.4	(0.3925 , 2.8138)	(0.3925 , 2.8138)	(5.5511e-17 , 2.2204e-15)	(8.6065e-11 , 6.1693e-10)
0.5	(0.4996 , 2.7251)	(0.4996 , 2.7251)	(2.7755e-16 , 8.8817e-16)	(1.0954e-10 , 5.9749e-10)
0.6	(0.6190 , 2.5929)	(0.6190 , 2.5929)	(1.1102e-16 , 1.7763e-15)	(1.3572e-10 , 5.6850e-10)
0.7	(0.7596 , 2.4083)	(0.7596 , 2.4083)	(3.3306e-16 , 808817e-16)	(1.6656e-10 , 5.2803e-10)
0.8	(0.9339 , 2.1626)	(0.9339 , 2.1626)	(3.3306e-16 , 1.3322e-15)	(2.0476e-10 , 4.7417e-10)
0.9	(1.1581 , 1.8473)	(1.1581 , 1.8473)	(2.2204e-16 , 1.3322e-15)	(2.5392e-10 , 4.0502e-10)

Example 6.2. *[37]. Consider the fuzzy Volterra integral equation*

$$\underline{f}(x;\alpha) = \alpha + \alpha^2 - (\alpha+\alpha^2)x\sinh(x),$$
$$\overline{f}(x;\alpha) = 4 - \alpha - \alpha^3 - (4-\alpha-\alpha^3)x\sinh(x),$$

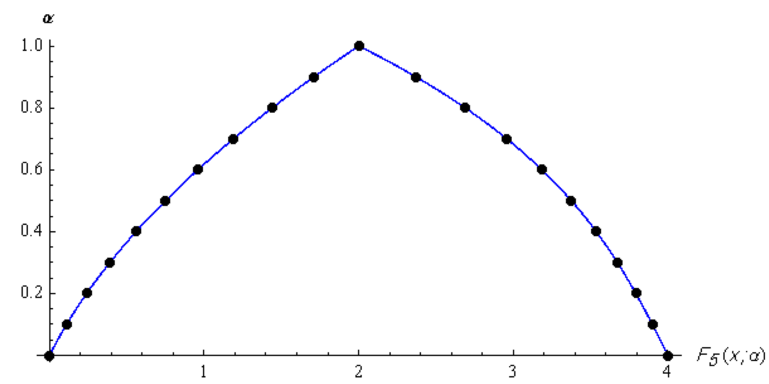


Figure 4: Comparison between the exact solution and the approximate solution of the Example 6.2

and kernel

$$k(x,t) = \sinh(x),$$

and $a = 0, b = 1, \lambda = 1$*, the exact solution is*

$$\underline{F}(x;\alpha) = \alpha^2 + \alpha,$$
$$\overline{F}(x;\alpha) = 4 - \alpha^3 - \alpha.$$

we have

$$\underline{f}(x,\alpha) = \underline{F}(x,\alpha) - \lambda \int_0^s \sinh(t)\underline{F}(t,\alpha)dt$$
$$\overline{f}(x,\alpha) = \overline{F}(x,\alpha) - \lambda \int_0^s \sinh(t)\overline{F}(t,\alpha)dt$$

and

$$\underline{F}(x) = \sum_{i=0}^{n} \underline{U}_i F_i(x) - \underline{f}(x,\alpha) - \sum_{i=0}^{n} \underline{U}_i \int_0^x \sinh(t)F_i(t)dt$$
$$\overline{F}(x) = \sum_{i=0}^{n} \overline{U}_i F_i(x) - \overline{f}(x,\alpha) - \sum_{i=0}^{n} \overline{U}_i \int_0^x \sinh(t)F_i(t)dt$$

Example 6.2 *has been solved by assuming* $N = 5$ *and* $x = 0.3$*. The absolute error of example has been showed in Figures* 5 *and* 6*.*

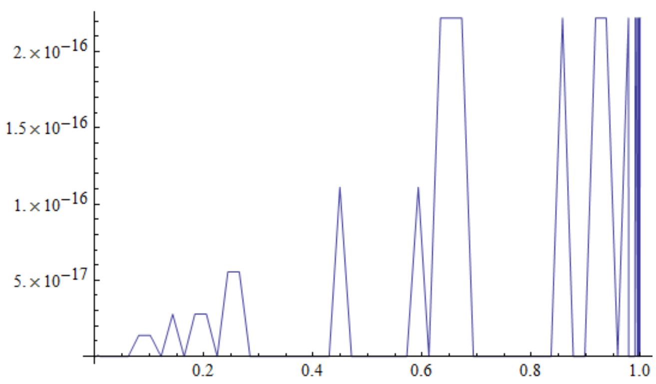


Figure 5: Absolute error $|\underline{F}(x,\alpha) - \underline{F}_5(x,\alpha)|$

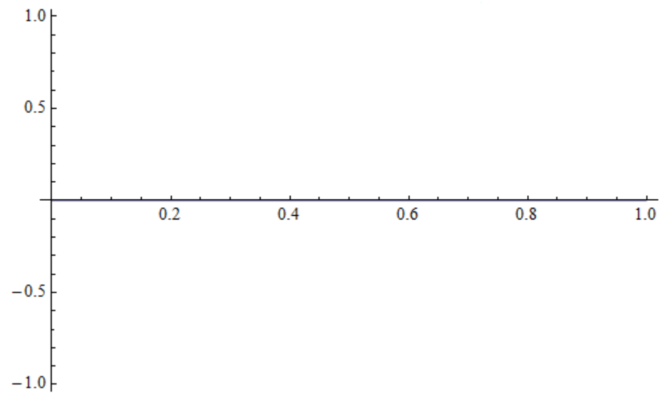


Figure 6: Absolute error $|\overline{F}(x,\alpha) - \overline{F}_5(x,\alpha)|$

Table 2: Numerical results of Example 6.2

α	Exact solution $(\underline{F}(x;\alpha), \overline{F}(x;\alpha))$	Approximate solution $x = 0.3$ and $N = 5$	Absolute error $x = 0.3$ and $N = 5$
0.0	(0.000 , 4.000)	(0.000 , 4.000)	(0.000e-00 , 0.000e-00)
0.1	(0.110 , 3.899)	(0.110 , 3.899)	(0.000e-00 , 0.000e-00)
0.2	(0.240 , 3.792)	(0.240 , 3.792)	(2.775e-17 , 0.000e-00)
0.3	(0.390 , 3.673)	(0.390 , 3.673)	(0.000e-00 , 0.000e-00)
0.4	(0.560 , 3.536)	(0.560 , 3.536)	(1.110e-16 , 0.000e-00)
0.5	(0.750 , 3.375)	(0.750 , 3.375)	(0.000e-00 , 0.000e-00)
0.6	(0.960 , 3.184)	(0.960 , 3.184)	(0.000e-00 , 0.000e-00)
0.7	(1.190 , 2.957)	(1.190 , 2.957)	(0.000e-00 , 0.000e-00)
0.8	(1.440 , 2.688)	(1.440 , 2.688)	(0.000e-00 , 0.000e-00)
0.9	(1.710 , 2.371)	(1.710 , 2.371)	(0.000e-00 , 0.000e-00)

7 Conclusion

As an alternate method for solving second-order fuzzy Volterra integral equations based on collocation method of polynomial Fibonacci, a numerical method has been introduced. Existence and uniqueness of the solution of method along with its convergence clearly has been illustrated. Using some numerical examples, the rational and tiny amount of error expressed for suggested method. As future work, we are about to develop this method in speed and exactness respect to others.

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