An Explicit Midpoint Algorithm for Nonexpansive Semigroup in smooth Banach Spaces

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ABSTRACT

In this paper, we consider **"Nearest points"** and **"Farthest points"** in normed linear spaces. For normed space $(X, \|.\|)$, the set $W \subseteq X$, we define P_g, F_g, R_g where $g \in W$. We obtion results about on P_g, F_g, R_g. We find new results on Chebyshev centers in normed spaces. In finally we define remotal center in normed spaces.

1 Introduction

Throughout this paper, let E be Banach space, C be closed convex subset of E. Let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$, where E^* denotes the dual space of E and , $\langle ., . \rangle$ denotes the generalized duality pairing. We also denote by B_r the closed ball in E with center 0 and radius r.

The function $\delta_E(\varepsilon) : [0, 2] \to [0, 1]$ is said to be the modulus of convexity of Banach space E, where $\delta_E(\varepsilon) = \inf\{1 - \frac{\|x-y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}$, $0 \leq \varepsilon \leq 1$. E is said to be uniformly convex if for each $\delta_E(\varepsilon) > 0$.
Let $U = \{x \in E : \|x\| = 1\}$. E is said to be smooth if the limit

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit exists uniformly for $x \in U$. We know that if E is smooth the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping is norm to weak star uniformly continuous on each bounded subset of E. When J is single-valued, we use instead of $J(x)$, $j(x)$.

A mapping $T : C \to C$ is said to be contraction if there exists a constant $\alpha \in (0, 1)$ such that $\|T(x) - T(y)\| \leq \alpha\|x - y\|$, for all $x, y \in C$. If $\alpha = 1$, T is called nonexpansive on C.

The fixed point problem (FPP) for a nonexpansive mapping T is: To find $x \in C$ such that $x \in \text{Fix}(T)$, where $\text{Fix}(T)$ is the fixed point set of the nonexpansive mapping T.

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The explicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations. For related works, we refer to [2, 3, 4, 7, 8, 10, 12, 11] and the references cited therein. For instance, consider the initial value problem for the differential equation $y^{'}(t) = f(y(t))$ with the initial condition $y(0) = y_0$, where f is a continuous function from $\mathbb{R}^d$ to $\mathbb{R}^d$. The explicit midpoint rule in which generates a sequence $\{y_n\}$ by the following the recurrence relation

$$\frac{1}{h}(y_{n+1} - y_n) = f(\frac{y_{n+1} - y_n}{2}).$$

In 2017, Luo et al. [9] introduced the following iterative method for the explicit midpoint rule of nonexpansive mappings in uniformly smooth Banach space:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}).$$

In 2018, Aibinu et al. [1] introduced the following iterative method for the explicit midpoint rule of nonexpansive mappings in a uniformly smooth Banach space:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\frac{x_n + x_{n+1}}{2}).$$

A family $S := \{T(s) : 0 \leq s < \infty\}$ of mapping from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

1. $T(0)x = x$ for all $x \in C$
2. $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$
3. $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$
4. For all $x \in C, s \to T(s)x$ is continuous.

Chen and Song [6] introduced and studied the following iterative method to prove a strong convergence theorem for FPP in a uniformly convex Banach space:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\tfrac{1}{s_n} \textstyle\int_0^{s_n} T(s)x_n ds, \qquad \forall n \in \mathbb{N}.$$

where f is a contraction mapping and $\{\alpha_n\}$ is the sequences in $(0, 1)$ and $\{s_n\}$ is a positive real divergent sequence. Motivated and inspired by the results mentioned and related literature in [1, 6], we propose an iterative midpoint algorithm based on the viscosity method for finding a common element of the set of solutions of nonexpansive semigroup in a uniformly smooth Banach space. Then we prove strong convergence theorems that extend and improve the corresponding results of Aibinu et al. [1], Chen and Song [6], Luo et al. [9] and others. Finally, we give a example and numerical result to illustrate our main result.

The rest of paper is organized as follows. The next section presents some preliminary results. Section 3 is devoted to introduce midpoint algorithm for solving it. The last section presents a numerical example to demonstrate the proposed algorithms.

2 Preliminaries

Lemma 2.1. *[5] Let E be Banach space, for each $x, e \in E$, $j(x) \in J(x)$, $j(x+y) \in J(x+y)$, the following inequalities hold:*

$$\|x\|^2 + 2\langle y, j(x)\rangle \le \|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \qquad \forall x, y \in E.$$

Let μ be a continuous linear functional on l^∞ and let $(a_0, a_1, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \ldots))$. We call μ a Banach limit when μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for each $(a_0, a_1, \ldots) \in l^\infty$. For a Banach limit μ, we know that $\liminf_{n\to\infty} a_n \le \mu_n(a_n) \le \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \ldots) \in l^\infty$.

Lemma 2.2. *[14] Let C be a convex subset of a Banach space E whose norm is uniformly Gâteaux differentiable. Let $\{x_n\}$ be a bounded subset of E, let z be a element of C and μ be a Banach limit. Then $\mu_n\|x_n - z\|^2 = \min_{y\in C} \mu_n\|x_n - y\|^2$ if and only if $\mu_n\langle y - z, j(x_n - z)\rangle \le 0$, $\forall y \in C$.*

Lemma 2.3. *[13] Let E be a reflexive Banach space and C be a closed convex subset of E. Let g be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$ and suppose that $g(x_n) \to \infty$ as $\|x_n\| \to \infty$. Then, there exists $x_0 \in C$ such that $g(x_0) = \inf\{g(x) : x \in C\}$.*

Definition 2.1. *A function $\omega = \mathbb{R}^+ \to \mathbb{R}^+$ is said to belong to $\Im$ if it satisfies the following conditions:*

1. $\omega(0) = 0$;
2. $r > 0 \Rightarrow \omega(r) > 0$;
3. $t \le s \Rightarrow \omega(t) \le \omega(s)$.

Lemma 2.4. *[13] Let E be a uniformly convex Banach space. Then, for any $r > 0$, there exists $\omega_r \in \Im$ such that for each $x, y \in B_r$, $x^* \in j(x)$, $y^* \in j(y)$ have $\langle x - y, x^* - y^*\rangle \ge \omega_r(\|x-y\|)\|x-y\|$.*

Lemma 2.5. *[6] Let C be a nonempty bounded closed convex subset of uniformly convex Banach space E and let $S := \{T(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C such that $F(S)$ is nonempty. Then for each $h, r > 0$,*

$$\lim_{t\to\infty} \sup_{x\in C\cap B_r} \|\frac{1}{t}\int_0^t T(s)xds - T(h)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

Lemma 2.6. *[15] Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \le (1-\alpha_n)a_n + \delta_n,\ \ n \ge 0$ where α_n is a sequence in $(0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that*

(i) $\Sigma_{n=1}^\infty \alpha_n = \infty$; *(ii)* $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\Sigma_{n=1}^\infty \delta_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Viscosity Nonlinear Midpoint Algorithm

In this section, we prove a strong convergence theorem based on the explicit iterative for fixed point of nonexpansive semigroup. We firstly present the following unified algorithm.
Let C be a nonempty closed convex subset of real Banach space E. Let $S = \{T(s) : s \in [0, +\infty)\}$ be a nonexpansive semigroup on C such that $\text{Fix}(S) \neq \emptyset$. Also $f : C \to C$ be a α-contraction mapping.

Algorithm 3.1. *For given $x_0 \in C$ arbitrary, let the sequence $\{x_n\}$ be generated by:*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2})ds. \tag{3.1}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$ are the sequence in $(0,1)$ and $\{\beta_n\}$ is the sequence in $[0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ and $\{s_n\} \subset [s, \infty)$ with $s > 0$.

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0, \ \Sigma_{n=1}^{\infty}\alpha_n = \infty;$

$(C2)$ $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty \quad or \quad \lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 1;$

$(C3)$ $\lim_{n\to\infty} s_n = \infty, \ \sup_{n\in\mathbb{N}} |s_{n+1} - s_n| \ is\ bounded.$

Lemma 3.1. *Let E be a uniformly smooth Banach space and C be a nonempty closed convex subset of E and $p \in$ Fix(S). Then the sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded.*

Proof. Let $p \in \text{Fix}(S)$, we obtain

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\tfrac{x_n + x_{n+1}}{2})ds - p\| \\
&\le \alpha_n\|\gamma f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\tfrac{x_n + x_{n+1}}{2}) - T(s)p\|ds \\
&\le \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + \beta_n\|x_n - p\| + \gamma_n\|\tfrac{x_n + x_{n+1}}{2} - p\| \\
&\le \alpha_n\alpha\|x_n - p\| + \alpha_n\|f(p) - Bp\| + \beta_n\|x_n - p\| + \tfrac{\gamma_n}{2}(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}$$

which implies that

$$(1 - \tfrac{\gamma_n}{2})\|x_{n+1} - p\| \le (\alpha_n\alpha + \beta_n + \tfrac{\gamma_n}{2})\|x_n - p\| + \alpha_n\|f(p) - p\|.$$

Then

$$\begin{aligned}
\|x_{n+1} - p\| &\le (1 - \tfrac{2(1-\alpha)\alpha_n}{1+\beta_n+\alpha_n})\|x_n - p\| + \tfrac{2\alpha_n(1-\alpha)}{1+\beta_n+\alpha_n}\tfrac{\|f(p)-p\|}{1-\alpha} \\
&\le \max\{\|x_n - p\|, \tfrac{\|f(p)-p\|}{1-\alpha}\} \\
&\ \ \vdots \\
&\le \max\{\|x_0 - p\|, \tfrac{\|f(p)-p\|}{1-\alpha}\}.
\end{aligned} \tag{3.2}$$

Hence $\{x_n\}$ is bounded. □

Now, set $t_n := \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2})ds$. Then $\{t_n\}$ and $\{f(x_n)\}$ are bounded.

Lemma 3.2. *The following properties are satisfying for the Algorithm 3.1*

$P1.$ $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$

$P2.$ $\lim_{n\to\infty}\|x_n-t_n\|=0.$

$P3.$ $\lim_{n\to\infty}\|T(s)t_n-t_n\|=0.$

Proof. P1:We have

$$
\begin{aligned}
&\|t_{n+1}-t_n\|\\
&=\|\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}}T(s)(\tfrac{x_{n+1}+x_{n+2}}{2})ds-\tfrac{1}{s_n}\int_0^{s_n}T(s)(\tfrac{x_n+x_{n+1}}{2})ds\|\\
&=\tfrac{1}{2}\|\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}}T(s)x_{n+1}ds+\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}}T(s)x_{n+2}ds\\
&\quad-\tfrac{1}{s_n}\int_0^{s_n}T(s)x_nds-\tfrac{1}{s_n}\int_0^{s_n}T(s)x_{n+1}ds\|\\
&=\tfrac{1}{2}\|\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}}(T(s)x_{n+1}-T(s)x_n)ds+(\tfrac{1}{s_{n+1}}-\tfrac{1}{s_n})\int_0^{s_n}(T(s)x_n-T(s)p)ds\\
&\quad+\tfrac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}}(T(s)x_n-T(s)p)ds+\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}}(T(s)x_{n+2}-T(s)x_{n+1})ds\\
&\quad+(\tfrac{1}{s_{n+1}}-\tfrac{1}{s_n})\int_0^{s_n}(T(s)x_{n+1}-T(s)p)ds+\tfrac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}}(T(s)x_{n+1}-T(s)p)ds\|\\
&\le\tfrac{1}{2}\|x_{n+1}-x_n\|+\tfrac{|s_{n+1}-s_n|}{s_{n+1}}\|x_n-p\|+\tfrac{1}{2}\|x_{n+2}-x_{n+1}\|+\tfrac{|s_{n+1}-s_n|}{s_{n+1}}\|x_{n+1}-p\|\\
&=\tfrac{1}{2}(\|x_{n+1}-x_n\|+\|x_{n+2}-x_{n+1}\|)+\tfrac{|s_{n+1}-s_n|}{s_{n+1}}(\|x_n-p\|+\|x_{n+1}-p\|).
\end{aligned}
\tag{3.3}
$$

Next, we show that the sequence $\{x_n\}$ is asymptotically regular, i.e.,
$\lim_{n\to\infty}\|x_{n+2}-x_{n+1}\|=0.$
By (3.3) we estimate that

$$
\begin{aligned}
&\|x_{n+2} - x_{n+1}\| \\
&= \|(\alpha_{n+1} f(x_{n+1}) + \beta_{n+1} x_{n+1} + \gamma_{n+1} \tfrac{1}{s_{n+1}} \int_0^{s_{n+1}} T(s)(\tfrac{x_{n+1}+x_{n+2}}{2}) ds) \\
&\quad -(\alpha_n f(x_n) + \beta_n x_n + \gamma_n \tfrac{1}{s_n} \int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2}) ds)\| \\
&= \|\gamma_{n+1}(\tfrac{1}{s_{n+1}} \int_0^{s_{n+1}} T(s)(\tfrac{x_{n+1}+x_{n+2}}{2}) ds - \tfrac{1}{s_n} \int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2}) ds) \\
&\quad +(\gamma_n - \gamma_{n+1}) \tfrac{1}{s_n} \int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2}) ds + (\alpha_{n+1} - \alpha_n) f(x_n) \\
&\quad +\alpha_{n+1}(f(x_{n+1}) - f(x_n)) + (\beta_{n+1} - \beta_n) x_n + \beta_{n+1}(x_{n+1} - x_n)\| \\
&\leq (1 - \alpha_{n+1} - \beta_{n+1})\|t_{n+1} - t_n\| + |\alpha_{n+1} - \alpha_n| M + \alpha_{n+1}\|f(x_{n+1}) - f(x_n)\| \\
&\quad +|\beta_{n+1} - \beta_n| N + \beta_{n+1}\|x_{n+1} - x_n\| \\
&\leq \tfrac{1-\alpha_{n+1}-\beta_{n+1}}{2}(\|x_{n+1} - x_n\| + \|x_{n+2} - x_{n+1}\|) \\
&\quad +(1 - \alpha_{n+1} - \beta_{n+1}) \tfrac{|s_{n+1}-s_n|}{s_{n+1}}(\|x_n - p\| + \|x_{n+1} - p\|) + |\alpha_{n+1} - \alpha_n| M \\
&\quad +\alpha_{n+1}\alpha\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| N + \beta_{n+1}\|x_{n+1} - x_n\|,
\end{aligned}
$$

where
$M := \sup\{\|\frac{1}{s_n} \int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2}) ds\| + \|f(x_n)\|\}$,
$N := \sup\{\|\frac{1}{s_n} \int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2}) ds\| + \|x_n\|\}$.

Then

$$
\begin{aligned}
(1 + \alpha_{n+1} + \beta_{n+1})\|x_{n+2} - x_{n+1}\| &\leq (1 + \beta_{n+1} + (2\alpha - 1)\alpha_{n+1})\|x_{n+1} - x_n\| \\
&\quad +(1 - \alpha_{n+1} - \beta_{n+1}) \tfrac{2|s_{n+1}-s_n|}{s_{n+1}}(\|x_n - p\| \\
&\quad +\|x_{n+1} - p\|) + 2|\alpha_n - \alpha_{n+1}| M + 2|\beta_n - \beta_{n+1}| N.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\leq (1 - \tfrac{2(1-\alpha)\alpha_{n+1}}{1+\alpha_{n+1}+\beta_{n+1}})\|x_{n+1} - x_n\| \\
&\quad +(\tfrac{1-\beta_{n+1}-\alpha_{n+1}}{1+\alpha_{n+1}+\beta_{n+1}})(\tfrac{2|s_{n+1}-s_n|}{s_{n+1}})(\|x_n - p\| + \|x_{n+1} - p\|) \\
&\quad +\tfrac{2M}{1+\alpha_{n+1}+\beta_{n+1}}|\alpha_n - \alpha_{n+1}| + \tfrac{2N}{1+\alpha_{n+1}+\beta_{n+1}}|\beta_n - \beta_{n+1}|.
\end{aligned}
$$

Hence, it follows by Lemma 2.6 and (C1)-(C3) that

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.4}
$$

And similarly, we have

$$\lim_{n\to\infty} \|x_{n+2} - x_{n+1}\| = 0. \tag{3.5}$$

Also by (3.3), (3.4),(3.5) and (C3) we have $\lim_{n\to\infty} \|t_{n+1} - t_n\| = 0$.

P2: We can write

$$\begin{aligned}
\|x_n - t_n\| &\leq \|x_{n+1} - x_n\| + \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n t_n - t_n\| \\
&= \|x_{n+1} - x_n\| + \|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)t_n - t_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - t_n\| + \beta_n \|x_n - t_n\|,
\end{aligned}$$

then

$$(1 - \beta_n)\|x_n - t_n\| \leq \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - t_n\|.$$

By (C1) and (3.4), we obtain

$$\lim_{n\to\infty} \|x_n - t_n\| = 0. \tag{3.6}$$

P3: Let $K := \{w \in C : \|w - p\| \leq \|x_0 - p\|, \frac{1}{1-\alpha}\|f(p) - p\|\}$. Then K is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $s \in [0, +\infty)$ and contains $\{x_n\}$. So, without loss of generality, we may assume that $S := \{T(s) : s \in [0, +\infty)\}$ is a nonexpansive semigroup on K. Since $\{x_n\} \subset K$ and K is bounded, there exists $r > 0$ such that $K \subset B_r$.

$$\begin{aligned}
\|T(s)x_n - x_n\| &= \|T(s)x_n - T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds + T(s)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds \\
&\quad -\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds + \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_n\| \\
&\leq \|T(s)x_n - T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_n\| \\
&\leq \|x_n - \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_n\| \\
&= 2\|\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_n\| \\
&\quad +\|T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\|
\end{aligned}$$

Since $\frac{x_n+x_{n+1}}{2} \in C$, from (3.6) and Lemma 2.5, we obtain $\lim_{n\to\infty} \|T(s)x_n - x_n\| = 0$. Therefore

$$\begin{aligned}\|T(s)t_n - t_n\| &\leq \|T(s)t_n - T(s)x_n\| + \|T(s)x_n - x_n\| + \|x_n - t_n\| \\ &\leq \|t_n - x_n\| + \|T(s)x_n - x_n\| + \|x_n - t_n\|.\end{aligned}$$

Then we have

$$\lim_{n\to\infty} \|T(s)t_n - t_n\| = 0. \tag{3.7}$$

□

4 Convergence Algorithm

Theorem 4.1. *Let E The Algorithm defined by (4.4) convergence strongly to $z \in$ Fix(S), which is a unique solution in of the variational inequality $\langle (I-f)z, j(x-z)\rangle \leq 0, \quad \forall x \in$ **Fix**(S).*

Proof. Set $t_{n_i} := t_n$ and let $\acute{K} = \{q \in C : \mu_i\|t_{n_i} - q\|^2 = \min_{x\in C} \mu_i\|t_{n_i} - x\|^2\}$ such that μ be a Banach limit. we claim that $\acute{K}$ consists of one point. Indeed, let $g(x) = \mu_i\|t_{n_i} - x\|^2$ for each $x \in C$ and $r_0 = \inf\{g(x) : x \in C\}$. Since the function g on C is convex and continuous and $g(t_n) \to \infty$ as $\|t_n\| \to \infty$, from Lemma 2.3, there exists $z \in C$ with $g(z) = r_0$, i.e.,$\acute{K}$ is nonempty. From Lemma 2.2, we know that $z \in \acute{K}$ if and only if

$$\mu_i\langle x - z, j(t_{n_i} - z)\rangle \leq 0, \qquad \forall x \in C \tag{4.1}$$

Suppose $\acute{z} \in \acute{K}$ and $z \neq \acute{z}$. By Lemma 2.4, there exists a positive number k such that $\langle t_{n_i} - z - (t_{n_i} - \acute{z}), j(t_{n_i} - z) - j(t_{n_i} - \acute{z})\rangle \geq k$ for each $i \in \mathbb{N}$. Therefore, $\mu_i\langle \acute{z} - z, j(t_{n_i} - z) - j(t_{n_i} - \acute{z})\rangle \geq k > 0$.
On the other hand, since $z, \acute{z} \in \acute{K}$, we have $\mu_i\langle \acute{z} - z, j(t_{n_i} - z)\rangle \leq 0$ and $\mu_i\langle z - \acute{z}, j(t_{n_i} - \acute{z})\rangle \leq 0$. Then we have

$$\mu_i\langle \acute{z} - z, j(t_{n_i} - z) - j(t_{n_i} - \acute{z})\rangle \leq 0$$

This is a contradiction. Therefore $z = \acute{z}$, that is, $\acute{K}$ consists of one point.
Noting (3.7), we have for each $s \geq 0$,

$$g(T(s)z) = \mu_i\|t_{n_i} - T(s)z\|^2 = \mu_i\|T(s)t_{n_i} - T(s)z\|^2 \leq \mu_i\|t_{n_i} - z\|^2 = g(z).$$

Since $\acute{K}$ consists of a point, $T(s)z = z, \forall s \geq 0$, i.e., $z \in \mathrm{Fix}(S)$.
We show that $\limsup_{n\to\infty}\langle f(z) - z, j(x_{n+1} - z)\rangle \leq 0$.

From $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2})ds$, we have

$(I-f)x_n = \frac{1}{\alpha_n}((\beta_n + \alpha_n)x_n + (1-\beta_n-\alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_{n+1}),$

then

$$\langle (I-f)x_n, j(x_{n+1}-z)\rangle$$

$$= \frac{1}{\alpha_n}\langle (\beta_n+\alpha_n)x_n + (1-\beta_n-\alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_{n+1}, j(x_{n+1}-z)\rangle$$

$$= \frac{1}{\alpha_n}(\frac{1-\beta_n-\alpha_n}{2}\langle \frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - x_n, j(x_{n+1}-z)\rangle$$

$$+\frac{1-\beta_n-\alpha_n}{2}\langle \frac{1}{s_n}\int_0^{s_n} T(s)x_{n+1} ds - x_{n+1}, j(x_{n+1}-z)\rangle$$

$$-\frac{1+\beta_n+\alpha_n}{2}\langle x_{n+1}-x_n, j(x_{n+1}-z)\rangle)$$

$$\leq \frac{1-\beta_n-\alpha_n}{2\alpha_n}(\|t_n-x_n\|+\|t_{n+1}-x_{n+1}\|)\|x_{n+1}-z\| - \frac{1+\beta_n+\alpha_n}{2\alpha_n}\|x_{n+1}-x_n\|\|x_{n+1}-z\|$$

By (3.4)and (3.6), we obtain

$$\limsup_{n\to\infty}\langle f(z)-z, j(x_{n+1}-z)\rangle \leq 0. \tag{4.2}$$

Now we prove that x_n is strongly convergence to z.

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \alpha_n\langle f(x_n)-z, j(x_{n+1}-z)\rangle + \beta_n\langle x_n-z, j(x_{n+1}-z)\rangle \\
&\quad +\langle (1-\beta_n-\alpha_n)(t_n-z), j(x_{n+1}-z)\rangle \\
&\le \alpha_n(\langle f(x_n)-f(z), j(x_{n+1}-z)\rangle + \langle f(z)-z, j(x_{n+1}-z)\rangle) \\
&\quad +\beta_n\|x_n-z\|\|x_{n+1}-z\| + (1-\beta_n-\alpha_n)\|t_n-z\|\|x_{n+1}-z\| \\
&\le \alpha_n\alpha\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, j(x_{n+1}-z)\rangle \\
&\quad +\beta_n\|x_n-z\|\|x_{n+1}-z\| + (1-\beta_n-\alpha_n)\|\tfrac{x_n+x_{n+1}}{2}-z\|\|x_{n+1}-z\| \\
&\le \alpha_n\alpha\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, j(x_{n+1}-z)\rangle \\
&\quad +\beta_n\|x_n-z\|\|x_{n+1}-z\| + \tfrac{1-\beta_n-\alpha_n}{2}(\|x_n-z\|+\|x_{n+1}-z\|)\|x_{n+1}-z\| \\
&= \tfrac{1+\beta_n-\alpha_n(1-2\alpha)}{2}\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, j(x_{n+1}-z)\rangle \\
&\quad +\tfrac{1-\beta_n-\alpha_n}{2}\|x_{n+1}-z\|^2 \\
&\le \tfrac{1+\beta_n-\alpha_n(1-2\alpha)}{4}(\|x_n-z\|^2+\|x_{n+1}-z\|^2) + \alpha_n\langle f(z)-z, j(x_{n+1}-z)\rangle \\
&\quad +\tfrac{1-\beta_n-\alpha_n}{2}\|x_{n+1}-z\|^2 \\
&\le \tfrac{1+\beta_n-\alpha_n(1-2\alpha)}{4}\|x_n-z\|^2 + \tfrac{3-\beta_n-\alpha_n(3-2\alpha)}{4}\|x_{n+1}-z\|^2 \\
&\quad +\alpha_n\langle f(z)-z, j(x_{n+1}-z)\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \tfrac{1+\beta_n-\alpha_n(1-2\alpha)}{1+\beta_n+\alpha_n(3-2\alpha)}\|x_n-z\|^2 + \tfrac{4\alpha_n}{1+\beta_n+\alpha_n(3-2\alpha)}\langle f(z)-z, j(x_{n+1}-z)\rangle \\
&\le (1-\tfrac{4(1-\alpha)\alpha_n}{1+\beta_n+\alpha_n(3-2\alpha)})\|x_n-z\|^2 + \tfrac{4\alpha_n}{1+\beta_n+\alpha_n(3-2\alpha)}\langle f(z)-z, j(x_{n+1}-z)\rangle \qquad (4.3)\\
&= (1-k_n)\|x_n-z\|^2 + 4\alpha_n l_n,
\end{aligned}$$

where $k_n = \frac{4(1-\alpha)\alpha_n}{1+\beta_n+\alpha_n(3-2\alpha)}$ and $l_n = \langle f(z)-z, j(x_{n+1}-z)\rangle$.
Since $\lim_{n\to\infty}\alpha_n = 0$ and $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, it is easy to see that $\lim_{n\to\infty} k_n = 0$, $\Sigma_{n=0}^{\infty}k_n = \infty$ and $\limsup_{n\to\infty} l_n \le 0$. Hence, from (4.2) and (4.3) and Lemma 2.6, we deduce that $x_n \to z$. □

Corollary 4.1. *Let E be a uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $S = \{T(s) : s \in [0,+\infty)\}$ be a nonexpansive semigroup on C such that Fix$(S) \neq \emptyset$. Also $f : C \to C$ be a*

α-contraction mapping. For given $x_0 \in C$ arbitrary, let the sequence $\{x_n\}$ be generated by:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2})ds. \tag{4.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{s_n\} \subset [s, \infty)$ with $s > 0$.

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0, \ \Sigma_{n=1}^{\infty}\alpha_n = \infty;$

$(C2)$ $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty \quad or \quad \lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1;$

$(C2)$ $\lim_{n\to\infty} s_n = \infty, \ \sup_{n\in\mathbb{N}} |s_{n+1} - s_n| \ is \ bounded.$

Then the sequence $\{x_n\}$ converges strongly to $z \in$ Fix(S), which is a unique solution in of the variational inequality $\langle (I-f)z, j(x-z)\rangle \leq 0, \quad \forall x \in$ Fix(S).

5 Numerical example

In this section, we give some examples and numerical results for supporting our main theorem. All the numerical results have been produced in Matlab 2017 on a Linux workstation with a 3.8 GHZ Intel annex processor and 8 Gb of memory

Example 5.1. Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy, \ \forall x, y \in \mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C = [-2, 1]$; Let $f(x) = \frac{1}{10}(x-3)$ and let, for each $x \in C, \ T(s)x = \frac{1}{1+2s}x$. Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$ generated by the iterative scheme

$$x_{n+1} = \frac{1}{30n+20}(x_n - 3) + \frac{2n+1}{3n+2}x_n + \frac{n}{3n+2}\frac{1}{s_n}\int_0^{s_n}\frac{1}{1+2s}(\frac{x_n + x_{n+1}}{2})ds \tag{5.1}$$

where $\alpha_n = \frac{1}{3n+2}, \ \beta_n = \frac{2n+1}{3n+2}, \ \gamma_n = \frac{n}{3n+2}$ and $s_n = 2n$. Then $\{x_n\}$ converges to $\{0\} \in \text{Fix}(S)$. f is contraction mapping with constant $\alpha = \frac{1}{9}$. It is easy to observe that $\text{Fix}(S) = \{0\} \neq \emptyset$. After simplification, scheme (5.1) reduce to

$$x_{n+1} = \frac{(\frac{20n+11}{30n+20} + \frac{n}{24n^2+16n}\ln(1+4n))x_n - \frac{3}{30n+20}}{1 - \frac{n}{24n^2+16n}\ln(1+4n)}.$$

Following the proof of Theorem 4.1, we obtain that $\{x_n\}$ converges strongly to $w = \{0\} \in \text{Fix}(S)$.

Example 5.2. Let $H = \mathbb{R}^2$, the set of all real numbers, with the inner product defined by $\langle (x,y),(z,t)\rangle = xz + yt, \ \ \forall (x,y),(z,t) \in \mathbb{R}^2$, and induced usual norm $\|(x,y)\| = (x^2+y^2)^{\frac{1}{2}}$. Let $C = [0,4] \times [-2,1]$; Let for each $(x,y) \in \mathbb{R}^2$, we define $f(x,y) = (\frac{1}{5}x, \frac{1}{6}y)$ and let, for each $(x,y) \in C, \ T(s)(x,y) = e^{-2s}(x,y)$. Then there exist unique sequences $\{(x_n, y_n)\} \subset \mathbb{R}^2$ generated by the iterative scheme

$$(x_{n+1}, y_{n+1}) = \frac{3}{6n+2}(\frac{1}{5}x_n, \frac{1}{6}y_n) + (\frac{2}{3} - \frac{n+1}{6n+2})(x_n, y_n) + \frac{15n-2}{18n+6}\frac{1}{s_n}\int_0^{s_n} e^{-2s}(\frac{(x_{n+1}, y_{n+1}) + (x_n, y_n)}{2})ds \tag{5.2}$$

where $\alpha_n = \frac{3}{6n+2}, \ \beta_n = \frac{2}{3} - \frac{n+1}{6n+2}, \ \gamma_n = \frac{15n-2}{18n+6}$ and $s_n = n$. Then $\{(x_{n+1}, y_{n+1})\}$ converges to $\{(0,0)\} \in \text{Fix}(S)$. f is contraction mapping with constant $\alpha = \frac{1}{7}$. It is easy to observe that $\text{Fix}(S) = \{(0,0)\} \neq \emptyset$. After simplification, scheme (5.2) reduce to

$$x_{n+1} = \frac{\frac{45n+14}{90n+30} - \frac{15n-2}{72n^2+24n}e^{-2n}}{1 + \frac{15n-2}{72n^2+24n}e^{-2n}}x_n,$$

$$y_{n+1} = \frac{\frac{18n+5}{36n+12} - \frac{15n-2}{72n^2+24n} e^{-2n}}{1 + \frac{15n-2}{72n^2+24n} e^{-2n}} y_n,$$

Following the proof of Theorem 4.1, we obtain that $\{(x_n, y_n)\}$ converges strongly to $w = \{(0,0)\} \in \text{Fix}(S)$.

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