Best Co-approximation and Worst Approximation by Closed Unit Balls

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Abstract

A kind of approximation, called best coapproximation was introduced and discussed in normed linear spaces by C. Franchetti and M. Furi in 1972. Subsequently, this study was taken up by several researchers in different abstract spaces. In this paper, we consider best coapproximation by closed unit balls. We define qcoproximinal and coremotal, and find some theorems.

1 Introduction

As a counter part to best approximation, a kind of approximation called best coapproximation was introduced in normed linear spaces by C. Franchetti and M. Furi [3] to study some characteristic properties of real Hilbert spaces. Subsequently, this theory has been developed to a large extent in normed linear spaces and in Hilbert spaces by C. Franchetti and M. Furi, H. Mazaheri, P.L. Papini and I. Singer, Geetha S. Rao and by many others (see e.g. [3, 5, 6, 12, 13] and references cited therein). In a series of papers, G. Albinus, G.G. Lorentz, T.D. Narang, G. Pantelidis, K. Schnatz, A.I. Vasilev and others (see e.g. [1,4, 7, 11, 14, 16, 19] and references cited therein) have tried to extend various results on best approximation available in normed linear spaces to metric linear spaces. The situation in case of best coapproximation is somewhat different. Whereas some attempts have been made to discuss best coapproximation in metric linear spaces (see e.g. [9, 10]) but still in these spaces this theory is less developed as compared to the theory of best approximation. The present paper is also a step in this direction. The paper mainly deals with some results on the existence and uniqueness of best coapproximation in quotient spaces when the underlying spaces are metric linear spaces. We also show how coproximinality is transmitted to and from quotient spaces. The results proved in the paper extend and generalize various known results on the subject.

Let $(X, \|.\|)$ be a normed linear space, W a non-empty subset of X. A point $y_0 \in W$ is said to be a best coapproximation point for $x \in X$, if

$$\|y - y_0\| \le \|x - y\|,$$

Suppose $g \in W$, we set

$$R_g = \{x \in X : \|y - g\| \le \|x - y\| \; for \; y \in W)\},$$

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Let $(X, \|.\|)$ be a normed linear space, W a non-empty subset of X, $x \in X$ and $0 \in W$. We set

$$R_W(x) = \{g_0 \in W : \ ||y - g_0|| \leq \|x - y\| \ for \ all \ y \in W\}.$$

The set W is coproximinal if for all $x \in X$, $R_W(x)$ is non-empty. The set W is cochebyshev if for all $x \in X$, $R_W(x)$ is singelton.

Suppose $x \in X$, we set

$$H_{d_x} = W + B[0, d_x],$$

and

$$H^{\oplus}_{d_x} = W \bigoplus B[0, d_x],$$

where $\bigoplus$ means that the sum decomposition of each element $x \in X$ is unique , for $x \in X$, $d_x = dist(x, W) := \inf_{g \in W} \|x - g\|$ and for $r > 0$ and $x \in X$, $B[0, r] = \{z \in X : \ \|x - z\| \leq r\}$.

Let X be a normed linear space and W a bounded non-empty subset of X. A point $q(x) \in W$ is said to be a farthest point for $x \in X$, if

$$\|x - q(x)\| \geq \|x - y\|,$$

for each $y \in W$. For each $x \in X$, put

$$F_W(x) = \{y_0 \in W : \ \|x - y_0\| = \delta(x, W) := \sup_{y \in W} \|x - y\| = \delta_x\}.$$

For each $x \in X$, if $F_W(x)$ is non-empty (a singleton), we say that W is remotal (uniquely remotal). Suppose $g \in W$, we set

$$F_g = \{x \in X : \ g \in F_W(x)\},$$

If for $x \in X$, there exists an unique $g \in W$, we set F_g instead by $F^{\oplus}_g$.

Suppose $x \in X$, we set

$$K_{\delta_x} = W + B^c[0, \delta_x],$$

and

$$K^{\oplus}_{\delta_x} = W \bigoplus B^c[0, \delta_x],$$

where $\bigoplus$ means that the sum decomposition of each element $x \in X$ is unique , for $r > 0$ and $x \in X$, $B^c[0, r] = \{z \in X : \ \|x - z\| \geq r\}$.

Let $(X, \|.\|)$ be a normed linear space, W a non-empty subset of X. A point $y_0 \in W$ is said to be a best cofarthest point for $x \in X$, if

$$\|y - y_0\| \geq \|x - y\|,$$

for each $y \in W$. For each $x \in X$, put

$$G_W(x) = \{y_0 \in W : \ \|y - y_0\| \geq \|x - y\| \ for \ all \ y \in W\}.$$

For each $x \in X$, if $G_W(x)$ is non-empty (a singleton), we say that W is coremotal (uniquely coremotal). Suppose $g \in W$, we set $G_g = \{x \in X : \ g \in G_W(x)\}$. If for $x \in X$, there exists an unique $g \in W$, we set G_g instead by $G^{\oplus}_g$. (see [4,5,7,9,10,11,14,15,16,20,21]).

Example 1.1. *Let $X = R^3$ with euclidean norm, $W = \{(x, x, x) : x \in R, 0 \leq x \leq 2\}$ a subset of X and $(x, y, z) \in X \backslash W$. then $z = (\frac{x+y+z}{\sqrt{3}}, \frac{x+y+z}{\sqrt{3}}, \frac{x+y+z}{\sqrt{3}}) \in R_W(x, y, z)$ and $z = (-\frac{x+y+z}{\sqrt{3}}, -\frac{x+y+z}{\sqrt{3}}, -\frac{x+y+z}{\sqrt{3}}) \in G_W(x, y, z)$. Becasue for $w = (a, a, a) \in W$ we have*

$$\begin{aligned}
\|w - z\| &= \|(\frac{(x-a) + (y-a) + (z-a}{\sqrt{3}} \\
&, \quad \frac{(x-a) + (y-a) + (z-a}{\sqrt{3}}, \frac{(x-a) + (y-a) + (z-a}{\sqrt{3}})\| \\
&\leq |(x-a) + (y-a) + (z-a)| \\
&\leq \|(x, y, z) - (a, a, a)\|.
\end{aligned}$$

Therefore $(\frac{x+y+z}{\sqrt{3}}, \frac{x+y+z}{\sqrt{3}}, \frac{x+y+z}{\sqrt{3}}) \in R_W(x, y, z)$ and supose $x, y, z \geq 0$, then

$$\begin{aligned}
\|w - z\| &= \|(\frac{(-x-a) + (-y-a) + (-z-a}{\sqrt{3}} \\
&, \quad \frac{(-x-a) + (-y-a) + (-z-a}{\sqrt{3}}, \frac{(-x-a) + (-y-a) + (-z-a}{\sqrt{3}})\| \\
&= |(x+a) + (y+a) + (z+a)| \\
&\geq (x+a) + (y+a) + (z+a) \\
&\geq (x-a) + (y-a) + (z-a) \\
&= \|(x, y, z) - (a, a, a)\|.
\end{aligned}$$

Therefore $(\frac{x+y+z}{\sqrt{3}}, \frac{x+y+z}{\sqrt{3}}, \frac{x+y+z}{\sqrt{3}}) \in G_W(x, y, z)$

Definition 1.1. *Let X be a normed linear space, W a subset of X.*
(i) W is called qremotal if for every $x \in X$, the set $(x - W) \cap F_0$ is a non-empty compact subset of X.
(ii) W is call qcoproximinal if for every $x \in X$, the set $(x - W) \cap R_0$ is a non-empty compact subset of X.

2 BEST COAPPROXIMATION BY CLOSED UNITE BALLS

In this section we obtain some results on best coapproximation and worst approximation by closed unite balls.

Theorem 2.1. *Let $(X, \|.\|)$ be a normed space and W a subspace of X. Then*
(i) The set W is a coproximinal, if and only if for $x \in X$

$$W \subseteq \bigcap_{w \in W} H_{\|x-w\|}.$$

(ii) The set W is a cochebyshev, if for $x \in X$

$$W \subseteq \bigcap_{w \in W} H^{\oplus}_{\|x-w\|}.$$

Proof. (i) Suppose W is coproximminal and $x \in X$, then there exists $g_0 \in W$ such that $g_0 \in R_W(x)$ Therefore for all $w \in W \ \ \|g_0 - w\| \leq \|x-w\|$, and for all $w \in W$ s.t. $w \in H_{\|x-w\|}$. It is follows that

$$W \subseteq \bigcap_{w \in W} H_{\|x-w\|}.$$

conversely, if for$x \in X$ and $W \subseteq \bigcap_{w\in W} H_{\|x-w\|}$. Then for $x \in X$ and $z \in W$, we show that $z \in R_W(x)$, For all $w \in W$, that is $\|w - z\| \leq \|x-w\|$ $w \in \bigcap_{w\in W} H_{\|x-w\|}$ and

$$w \in H_{\|x-w\|}.$$

For some $w' \in W$, $\|w-w'\| \leq \|x-w\|$, then $w' \in R_W(x)$, that W is coproximinal. (ii) Suppose W is cochebyshev, and $x \in X$, then W is coprximinal, from (i) we have

$$W \subseteq \bigcap_{w \in W} H_{\|x-w\|}.$$

and there exists an unique $g_0 \in W$ such that $g_0 \in R_W^{\oplus}(x)$ Therefore for all $w \in W \ \ \|g_0 - w\| \leq \|x-w\|$, and for all $w \in W$ s.t. $w \in H^{\oplus}_{\|x-w\|}$. It is follows that

$$W \subseteq \bigcap_{w \in W} H^{\oplus}_{\|x-w\|}.$$

conversely, if for $x \in X$ and $W \subseteq \bigcap_{w\in W} H^{\oplus}_{\|x-w\|}$. For $x \in X$ and $w \in W$, $w \in \bigcap_{w\in W} H^{\oplus}_{\|x-w\|}$, it follows

$$w \in H^{\oplus}_{\|x-w\|}.$$

For an unique $w' \in W$, $\|w-w'\| \leq \|x-w\|$, then $w' \in R_W(x)$, that W is cochebyshev. □

Theorem 2.2. *Let $(X, \|.\|)$ be a normed space. Then*
(i) the set W is coproximinal if and only if $X = \bigcup_{g\in W} R_g$;
(ii) the set W is cochebyshev if and only if $X = \bigcup_{g\in W} R_g^{\oplus}$.

Proof. (i) If W is coproximnal,

$$\begin{aligned} u \in X \quad &\Longleftrightarrow \quad \exists g \in W s.t. g \in R_W(u) \\ &\Longleftrightarrow \quad \exists g \in W s.t. u \in R_g \\ &\Longleftrightarrow \quad u \in \bigcup_{g \in W} R_g. \end{aligned}$$

conversely, if $X = \bigcup_{g\in W} R_g$ and $x \in X$. There exsists a $g_0 \in W$ such that $x \in R_{g_0}$ and $g_0 \in R_W(x)$. It follows that W is coproximinal. (ii)

$$\begin{aligned} u \in X \quad &\Longleftrightarrow \quad \exists ! g \in W s.t. g \in R_W(u) \\ &\Longleftrightarrow \quad \exists ! g \in W s.t. u \in R_g \\ &\Longleftrightarrow \quad \exists ! g \in W s.t. u \in \bigcup_{g \in W} R_g^{\oplus}. \\ &\Longleftrightarrow \quad X = \bigcup_{g \in W} R_g^{\oplus} \end{aligned}$$

conversely, if $X = \bigcup_{g \in W} R_g$ and $x \in X$. There exsists an unique $g_0 \in W$ such that $x \in R_{g_0}$ and $g_0 \in R_W(x)$. It follows that W is cochebyshev. □

Theorem 2.3. *Let $(X, \|.\|)$ be a normed space and W a coproximinal subspace of X. The set W is qcoproximinal if and only if for for all $x \in X$ and for all $w \in W$, the sequence $\{x_n\}_{n \geq 1} \subseteq H_{\|x-w\|}$ has a convergent subsequence.*

Proof. Suppose $x \in X$, we must show that the set $(x - W) \cap R_0$ is a non-empty compact subset of X. If $\{y_n\}_{n \geq 1}$ is a sequence in $(x - W) \cap R_0$, then $\{x - y_n\}_{n \geq 1} \subseteq W$ and $\{y_n\}_{n \geq 1} \subseteq R_0$. For all $n \geq 1$ and $w \in W$, $x - y_n - w \in B[0, \|x - w\|]$. , we have $x - y_n = w + x - y_n - w \in W + B[0, \|x - w\|] = H_{\|x-w\|}$. That is the sequence $\{x - y_n\}_{n \geq 1}$ has a convergent subsequence. It is follows that W is qcoproximinal.

Suppose W is qcoproximinal, $x \in X$, for all $w \in W$ and $\{x_n\}_{n \geq 1}$ is any sequence in $H_{\|x-w\|}$. There exists a sequece $\{g_n - w\}_{n \geq 1} \subseteq W$ and for $w \in W$, $\|g_n - w\| \leq \|x - w\|$ and $g_n \in (0 - W) \cap R_0$. It follows that the sequence $\{g_n\}_{n \geq 1}$ has a convergent subsequence and by relation $\|x_n - y_n\| \leq \|x - w\|$. the sequence $\{x_n\}_{n \geq 1}$ has a convergent subsequence. □

Definition 2.1. *Let X be a normed space. The set X is said to have the sequential Kadec-Klee property if weak and norm sequential convergence coincide on $S_X = \{x \in X : \|x\| = 1\}$.*

Theorem 2.4. *Let X be a normed linear space, X is a reflexive space and has the Kadec-Klee property. Then in every closed linear subspace of W of X is qcoproximinal.*

Proof. Suppose $x \in X \setminus W$ and $\|x\| = 1$, we must show that the set $(x - W) \cap R_0$ is a non-empty compact subset of X. Since X is reflexive, the closed unit ball B_X is weakly compact., Consider the sequence $\{x_n\}_{n \geq 1} \subseteq (x - W) \cap R_0$. then $\{x - x_n\}_{n \geq 1} \subseteq W$ and $\{x_n\}_{n \geq 1} \subseteq R_0$. For all $n \geq 1$. Therefore $x_n \in B_X$, because

$$\begin{aligned} \|x_n\| &\leq \|x_n - w\| \\ &\leq \|x_n - (x - x_n)\| \\ &\leq \|x\| \\ &\leq 1. \end{aligned}$$

there exists a subsequence $\{x_{n_k}\}_{n \geq 1}$ and $x_0 \in B_X$ such that $x_{n_k} \rightharpoonup x_0$. Since X has Kadec-Klee property, $x_{n_k} \to x_0$.Then W is qcoproximinal.

□

Conclusion 2.1. *Let X be a reflexive normed linear space, closed linear subspace of W of X has Kadec-Klee property, Then W is qcoproximinal.*

3 Birkhoff orthogonality and farthest orthogonality in best coapproximation and worst approximation

In this section we show that Birkhoff orthogonality in best coapproximationy by closed unite balls.

Definition 3.1. *[10] Let X be a normed linear space, W a subspace of X and $x \in X$. We say that x is Birkhoff orthogonality with W and denoted by $x \perp^B W$ if and only if $\|x\| \leq \|x + \alpha y\|$ for every $y \in W$ and for every scaler α.*

Definition 3.2. *[10] Let X be a normed linear space, W a subspace of X, $x \in X$ and $\epsilon > 0$. We say that $x \perp^B_\epsilon W$ if and only if $\|x\| \leq \|x + \alpha y\| + \epsilon$ for every $y \in W$ and for every scaler α.*

It was observed in [15] that

$$g_0 \in R_W(x) \iff W \perp x - g_0.$$

Definition 3.3. *[11] A finite or infinite sequence $\{x_n\}_{n \in L}$ in a Banach space X is said to be farthest orthogonal if*

$$\|x_0\| \geq \| \sum_{n \in L} (-1)^n x_n \|.$$

Denoted by $x_0 \perp^F \{x_n\}_{n \in L}$. Where $L := \{0, 1, 2, ..., N\}$, or $L := \{0, 1, 2, , ...\}$.
Note that for $x, y \in X$, $x \perp^F y$ if and only if $\|x\| \geq \|x - y\|$.

Let X be a normed linear space, W a subset of X, It should be noted that if $0 \in W$, then

$$0 \in F_W(x_0) \iff x_0 \perp^F W$$

.

Corollary 3.1. *Let X be a normed linear space, W a linear subspace of X and $x \in X$.*

$$x \perp^B W \iff x \in B[0, \|x + w\|] \; \forall w \in W.$$

Corollary 3.2. *Let X be a normed linear space, W a linear subspace of X, $x \in X$ and $\epsilon > 0$.*

$$x \perp^B_\epsilon W \iff x \in B[0, \|x + w\| + \epsilon] \; \forall w \in W.$$

Let $(X, \|.\|)$ be a normed linear space, W a non-empty subset of X, $x \in X$ and $0 \in W$. We set

$$R_{W,\epsilon}(x) = \{g_0 \in W : \; ||y - g_0|| \leq \|x - y\| + \epsilon \; for \; all \; y \in W\}.$$

Corollary 3.3. *Let X be a normed linear space, W a subspace of X, $x \in X$ and $\epsilon > 0$. Then*

$$g_0 \in R_{W,\epsilon}(x) \iff W \perp^B_\epsilon x - g_0.$$

Corollary 3.4. *Let X be a normed linear space, W a bounded subset of X, $x \in X$ and $0 \in W$.*

$$x \perp^F W \iff x \in B[0, \delta_x].$$

Proof.

$$\begin{aligned} x\perp^F W &\iff 0\in F_W(x)\\ &\iff \|x\|=\delta_x\\ &\iff x\in B^c[0,\delta_x]. \end{aligned}$$

□

4 Worst approximationsion by closed unit balls

In this section we obtain some results on worst approximationsion by closed unit balls.

Theorem 4.1. *Let $(X,\|.\|)$ be a normed space and W a subset of X.*
(i) The set W is a remotal, if and only if

$$X=\bigcup_{x\in X}\bigcap_{g\in W}K_{\|x-g\|}.$$

(ii) The set W is a uniquely remotal, if and only if

$$X=\bigcup_{x\in X}\bigcap_{g\in W}K^{\oplus}_{\|x-g\|}.$$

(iii) The set W is a coremotal, if and only if

$$X=\bigcup_{a\in W}\bigcap_{g\in W}K_{\|g-a\|}.$$

(iv) The set W is a uniquely coremotal, if and only if

$$X=\bigcup_{a\in W}\bigcap_{g\in W}K^{\oplus}_{\|g-a\|}.$$

Proof. (i) Since W is remtal, for $x\in X$, there exists a $a\in W$ such that $a\in F_W(x)$.

$$\begin{aligned} x\in X &\iff \exists a\in W\forall g\in W\ \ \|x-a\|\geq\|x-g\|\\ &\iff \exists a\in W\forall g\in W\ \ s.t.\ \ x-a\in B^c[0,\|x-g\|]\\ &\iff \forall g\in W\ \ s.t.\ \ x\in K_{\|x-g\|}\\ &\iff x\in\bigcup_{x\in X}\bigcap_{g\in W}K_{\|x-g\|}. \end{aligned}$$

(ii) Since W is uniquely remotal, for $x \in X$, there exists an unique $a \in W$ such that $a \in F_W(x)$.

$$\begin{aligned} x \in X &\iff \exists! a \in W \forall g \in W \;\; \|x-a\| \geq \|x-g\| \\ &\iff \exists! a \in W \forall g \in W \;\; s.t. \;\; x-a \in B^c[0, \|x-g\|] \\ &\iff \forall g \in W \;\; s.t. \;\; x \in K^{\oplus}_{\|x-g\|} \\ &\iff x \in \bigcup_{x \in X} \bigcap_{g \in W} K^{\oplus}_{\|x-g\|}. \end{aligned}$$

(iii) Since W is coremtal, for $x \in X$, for $x \in X$, there exists a $a \in W$ such that $a \in G_W(x)$.

$$\begin{aligned} x \in X &\iff \exists a \in W \forall g \in W \;\; \|g-a\| \geq \|x-g\| \\ &\iff \exists a \in W \forall g \in W \;\; s.t. \;\; x-g \in B^c[0, \|g-a\|] \\ &\iff \exists a \in W \forall g \in W \;\; s.t. \;\; x \in K_{\|g-a\|} \\ &\iff x \in \bigcup_{a \in W} \bigcap_{g \in W} K_{\|g-a\|}. \end{aligned}$$

(iv) Since W is uniquely coremtal, for $x \in X$, there exists an unique $a \in W$ such that $a \in F_W(x)$.

$$\begin{aligned} x \in X &\iff \exists! a \in W \forall g \in W \;\; \|g-a\| \geq \|x-g\| \\ &\iff \exists! a \in W \forall g \in W \;\; s.t. \;\; x-g \in B^c[0, \|g-a\|] \\ &\iff \exists! a \in W \forall g \in W \;\; s.t. \;\; x \in K^{o}plus_{\|g-a\|} \\ &\iff x \in \bigcup_{a \in W} \bigcap_{g \in W} K^{\oplus}_{\|g-a\|}. \end{aligned}$$

□

Theorem 4.2. *Let $(X, \|.\|)$ be a normed space. Then*

(i) The set W is remotal if and only if $X = \bigcup_{g \in W} F_g$ and for all $a \in W$ and for all $x \in X$, we have $\bigcup_{g \in W} K_{\|x-g\|} = W + \bigcap_{g \in W} B^c[0, \|x-g\|]$;

(ii) The set W is uniquely remotal if and only if $X = \bigcup_{g \in W} F^{\oplus}_g$ and for all $a \in W$ and for all $x \in X$, we have $\bigcup_{g \in W} K^{\oplus}_{\|x-g\|} = W \bigoplus \bigcap_{g \in W} B^c[0, \|x-g\|]$;

(iii) The set W is coremotal if and only if $X = \bigcup_{g \in W} G_g$;

(iv) The set W is remotal if and only if $X = \bigcup_{g \in W} G^{\oplus}_g$.

Proof. (i)

$$\begin{aligned} W\text{`}is\ remotal &\iff \forall u \in X \leftrightarrow \exists g \in W s.t. g \in F_W(u) \\ &\iff \forall u \in X \leftrightarrow \exists g \in W s.t. u \in F_g \\ &\iff \forall u \in X \leftrightarrow u \in \bigcup_{g \in W} F_g. \end{aligned}$$

and for all $x \in X$

$$\begin{aligned} x \in \bigcup_{g \in W} K_{\|x-g\|} &\iff \exists g \in W \;\; s.t. \;\; x \in K_{\|x-g\|} \\ &\iff \exists g \in W \exists h \in W \;\; s.t. \;\; x-h \in B^c[0, \|x-g\|] \\ &\iff \exists h \in W \exists g \in W \;\; s.t. \;\; x-h \in B^c[0, \|x-g\|] \\ &\iff \exists h \in W \;\; s.t. \;\; x-h \in \bigcap_{g \in W} B^c[0, \|x-g\|] \\ &\iff x \in W + \bigcap_{g \in W} B^c[0, \|x-g\|]. \end{aligned}$$

(ii)

$$\begin{aligned} W \text{ `}is\ uniquely\ remotal &\iff \forall u \in X \leftrightarrow \exists! g \in W s.t. g \in F_W(u) \\ &\iff \forall u \in X \leftrightarrow \exists! g \in W s.t. u \in F_g \\ &\iff \forall u \in X \leftrightarrow u \in \bigcup_{g \in W}^{\oplus} F_g. \end{aligned}$$

and for all $x \in X$

$$\begin{aligned} x \in \bigcup_{g \in W} K^{\oplus}_{\|x-g\|} &\iff \exists! g \in W \;\; s.t. \;\; x \in K_{\|x-g\|} \\ &\iff \exists! g \in W \exists! h \in W \;\; s.t. \;\; x-h \in B^c[0, \|x-g\|] \\ &\iff \exists! h \in W \exists! g \in W \;\; s.t. \;\; x-h \in B^c[0, \|x-g\|] \\ &\iff \exists! h \in W \;\; s.t. \;\; x-h \in \bigcap_{g \in W} B^c[0, \|x-g\|] \\ &\iff x \in W \bigoplus \bigcap_{g \in W} B^c[0, \|x-g\|]. \end{aligned}$$

(iii)

$$\begin{aligned} W\ is\ coremotal &\iff \forall u \in X \leftrightarrow \exists g \in W s.t. g \in G_W(u) \\ &\iff \forall u \in X \leftrightarrow \exists g \in W s.t. u \in G_g \\ &\iff \forall u \in X \leftrightarrow u \in \bigcup_{g \in W} G_g. \end{aligned}$$

(iv)

$$\begin{aligned} W\ is\ uniquely\ coremotal &\iff \forall u \in X \leftrightarrow \exists! g \in W s.t. g \in G_W(u) \\ &\iff \forall u \in X \leftrightarrow \exists! g \in W s.t. u \in G_g \\ &\iff \forall u \in X \leftrightarrow u \in \bigcup_{g \in W} G^{\oplus}_g. \end{aligned}$$

□

Theorem 4.3. *Let $(X, \|.\|)$ be a normed space and W a remotall subspace of X. If for $x \in X$ and for all $w \in W$, the sequence $\{x_n\}_{n\geq 1} \subseteq \bigcup_{a\in W}\bigcap_{g\in W} K_{\|g-a\|}$ has a convergent subsequence. W is qremtal if and only*

Proof. Suppose $x \in X$, we must show that the set $(x - W) \cap F_0$ is a non-empty compact subset of X. If $\{y_n\}_{n\geq 1}$ is a sequence in $(x-W)\cap R_0$, then $\{x-y_n\}_{n\geq 1} \subseteq W$ and $\{y_n\}_{n\geq 1} \subseteq F_0$. For all $n \geq 1$, $x - y_n \in B^c[0, \|x-w\|]$. Since $\{y_n\}_{n\geq 1} \subseteq F_0$, therefore for all $w \in W$, we have $\|w\| \geq \|y_n - w\|$. It follows that $\|x - y_n\| \leq \|y_n - (x - y_n - w)\| = \|x - w\|$. For $n \geq 1$ and for all $w \in W$, we have $0 + x - y_n \in W + B^c[0, \|x - w\|] = K_{\|x-w\|}$. That is the sequence $\{x - y_n\}_{n\geq 1}$ has a convergent subsequence. It is follows that W is qcoremotal. □

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