Many algorithms for approximation of restrained 2-rainbow domination in GP(n,5)

Mojtaba Ghanbari*

Department of Mathematics, Farahan Branch, Islamic Azad University, Farahan, Iran

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ABSTRACT

The concept of 2-rainbow domination of a graph G coincides with the ordinary domination of the prism $G\Box K_2$. Ghanbari and Mojdeh [6] initiated the concept of restrained 2-rainbow domination in graphs. In this paper is given many algorithms for good approximations of restrained 2-rainbow domination number of generalized Petersen Graph $GP(n,5)$.

1 Introduction and Preliminary

Throughout this paper, we consider G as a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [14] as a reference for terminology and notation which are not explicitly defined here.
In graph theory, the Cartesian product $G\Box H$ of graphs G and H is a graph such that the vertex set of $G\Box H$ is the Cartesian product $V(G)\times V(H)$; and two vertices (u,u') and (v,v') are adjacent in $G\Box H$ if and only if either $u=v$ and u' is adjacent to v' in H, or $u'=v'$ and u is adjacent to v in G. The Cartesian product of graphs is sometimes called the box product of graphs.
Domination and its variations in graphs have been extensively studied, cf. [8], [9] and [10]. For a graph $G=(V(G),E(G))$, a set $S\subseteq V(G)$ is called a *dominating set* if every vertex not in S has a neighbor in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality among all dominating sets of G. A *restrained dominating set* (RD set) in a graph G is a dominating set S in G for which every vertex in $V(G)\setminus S$ is adjacent to another vertex in $V(G)\setminus S$. The *restrained domination number* (RD number) of G, denoted by $\gamma_r(G)$, is the smallest cardinality of an RD set of G. This concept was formally introduced in [5] (Albeit, it was indirectly introduced in [13]). Domination presents a model for situations in which vertices from S guard neighboring vertices that are not in S. A generalization was proposed in cf. [1] where different types of guards are used, and vertices not in S must have all types of guards in their neighborhoods. Let G be a graph and $v\in V(G)$. The open neighborhood of v is the set $N(v)=\{u\in V(G)|uv\in E(G)\}$, and its closed neighborhood is the set $N[v]=N(v)\cup\{v\}$.

* Corresponding Author's E-mail: ghanbari543@yahoo.com

Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1, ..., k\}$; that is, $f : V(G) \rightarrow P(\{1, ..., k\})$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ we have $\cup_{u \in N(v)} f(u) = \{1, ..., k\}$, then f is called a *k-rainbow dominating function* (kRDF) of G. The weight, $\omega(f)$, of a function f is defined as $\omega(f) = \Sigma_{v \in V(G)} |f(v)|$. Given a graph G, the minimum weight of a kRDF is called the *k-rainbow domination number of G*, which we denote by $\gamma_{rk}(G)$. Clearly when $k = 1$ this concept coincides with the ordinary domination. The 2-rainbow domination in graphs have been studied by B. Bresar and T. K. Umenjak, cf. [2]. The concept of 2-rainbow domination of a graph G coincides with the ordinary domination of the prism $G \square K_2$.
Ghanbari and Mojdeh initiated the concept of *restrained 2- rainbow domination in graphs* cf. [6]. Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1, 2\}$; that is, $f : V(G) \rightarrow P(\{1, 2\})$. If for each vertex $v \in V(G)$, such that $f(v) = \emptyset$ we have $\cup_{u \in N(v)} f(u) = \{1, 2\}$, and v is adjacent to a vertex $w \in V(G)$ such that $f(w) = \emptyset$ then f is called a *restrained 2-rainbow dominating function* (R2RDF) of G. The weight, $\omega(f)$, of a function f is defined as $\omega(f) = \Sigma_{v \in V(G)} |f(v)|$. Given a graph G, the minimum weight of a R2RDF is called the *restrained 2-rainbow domination number of G*, which we denote by $\gamma_{rr2}(G)$. In this paper we give an algorithm for determinate values of 2-rainbow domination in the generalized Petersen graph $GP(n, 5)$.

Theorem 1.1. *[6] Restrained 2-rainbow dominating function is NP-complete.*

Theorem 1.2. *[6]*
(a) $\gamma_{rr2}(P_2) = 2$ *and* $\gamma_{rr2}(P_3) = 3$.
(b) For $n \geq 4$, $\gamma_{rr2}(P_n) = 2([\frac{n}{3}] + 1)$ *if* $n \equiv 0$ *or* $1 (mod 3)$.
(c) For $n \geq 4$, $\gamma_{rr2}(P_n) = 2[\frac{n}{3}] + 3$ *if* $n \equiv 2 (mod 3)$.
(d) For every $m, n \geq 2$; $\gamma_{rr2}(K_n) = 2$, $\gamma_{rr2}(K_{m,n}) = 4$ *and* $\gamma_{rr2}(K_{1,n}) = n + 1$.

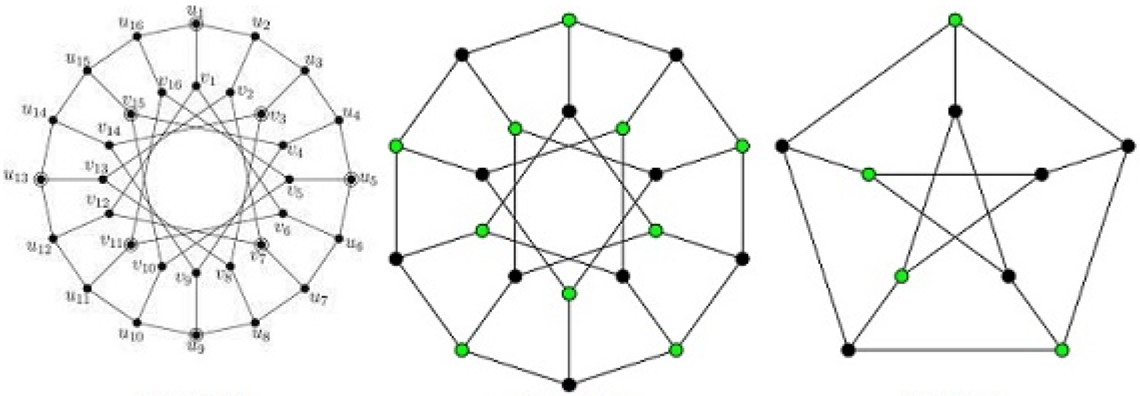
Theorem 1.3. *[6] For* $n \geq 3$
(a) $\gamma_{rr2}(C_n) = \frac{2n}{3}$ *if* $n \equiv 0 (mod 3)$.
(b) $\gamma_{rr2}(C_n) = 2([\frac{n}{3}] + 1)$ *if* $n \equiv 1 (mod 3)$.
(c) $\gamma_{rr2}(C_n) = 2[\frac{n}{3}] + 3$ *if* $n \equiv 2 (mod 3)$.

2 Main Result

The domination invariants of generalized Petersen graphs were studied. Let us recall what a generalized Petersen graph is, cf. also [3].
Let $n \geq 3$ and k be relatively prime natural numbers and $k < n$. The generalized Petersen graph $GP(n, k)$ is defined as follows. Let C_n, C'_n be two disjoint cycles of length n. Let the vertices of C_n be $u_1, ..., u_n$ and edges $u_i u_{i+1}$ for $i = 1, ..., n-1$ and $u_n u_1$. Let the vertices of C'_n be $v_1, ..., v_n$ and edges $v_i v_{i+k}$ for $i = 1, ..., n$, the sum $i + k$ being taken modulo n (throughout this section). The graph $GP(n, k)$ is obtained from the union of C_n and C'_n by adding the edges $u_i v_i$ for $i = 1, ..., n$. Its obvious that $GP(n, k) = GP(n, n-k)$. The graph $GP(5, 2)$ or $GP(5, 3)$ is the well-known Petersen graph.

Theorem 2.1. *[6]*
(a) For $n \geq 5$ *and* $n \equiv 0 (mod 4)$*, the inequality* $\gamma_{rr2}(GP(n, 1)) = \gamma_{rr2}(GP(n, n-1)) \leq n$ *is satisfied.*
(b) For $n \geq 5$ *and* $n \equiv i (mod 4)$, $i = 1, 2, 3$*, the inequality* $\gamma_{rr2}(GP(n, 1)) = \gamma_{rr2}(GP(n, n-1)) \leq n + 1$ *is satisfied.*

Theorem 2.2. *[6] For* $n \geq 5$
(a) If $n \equiv 0 (mod 5)$*, the inequality* $\gamma_{rr2}(GP(n, 2)) = \gamma_{rr2}(GP(n, n-2)) \leq \frac{4n}{5} + 2$ *is satisfied.*



GP(16,5) GP(10,3) GP(5,2)

(b) If $n \equiv 1(mod5)$*, the inequality* $\gamma_{rr2}(GP(n,2)) = \gamma_{rr2}(GP(n,n-2)) \leq 4\lfloor \frac{n}{5} \rfloor + 2$ *is satisfied.*
(c) If $n \equiv 2(mod5)$*, the inequality* $\gamma_{rr2}(GP(n,2)) = \gamma_{rr2}(GP(n,n-2)) \leq 4(\lfloor \frac{n}{5} \rfloor + 1)$ *is satisfied.*
(d) If $n \equiv 3(mod5)$*, the inequality* $\gamma_{rr2}(GP(n,2)) = \gamma_{rr2}(GP(n,n-2)) \leq 4(\lfloor \frac{n}{5} \rfloor + \frac{3}{2})$ *is satisfied.*
(e) If $n \equiv 4(mod5)$*, the inequality* $\gamma_{rr2}(GP(n,2)) = \gamma_{rr2}(GP(n,n-2)) \leq 4(\lfloor \frac{n}{5} \rfloor + \frac{3}{2})$ *is satisfied.*

Theorem 2.3. *[6] For* $n \geq 5$
(a) If $n = 5, 7, 8$ *then* $\gamma_{rr2}(GP(n,3)) = \gamma_{rr2}(GP(n,n-3)) \leq n+1$ *is satisfied.*
(b) If $n \geq 10$*,* $(n,3) = 1$ *and* n *is even, then the inequality* $\gamma_{rr2}(GP(n,3)) = \gamma_{rr2}(GP(n,n-3)) \leq n+2$ *is satisfied.*
(c) If $n \geq 10$*,* $(n,3) = 1$ *and* n *is odd, then the inequality* $\gamma_{rr2}(GP(n,3)) = \gamma_{rr2}(GP(n,n-3)) \leq n+3$ *is satisfied.*

Theorem 2.4. *For* $n \geq 5$
(a) If n *is an odd number and* $(n,5) = 1$*, then* $\gamma_{rr2}(GP(n,5)) = \gamma_{rr2}(GP(n,n-5)) \leq n+5$.
(b) If n *is an even number,* $(n,5) = 1$*,* $5 \leq [\frac{n}{4}]$ *and* $t = [\frac{n}{10}]$ *is even number, then* $\gamma_{rr2}(GP(n,5)) = \gamma_{rr2}(GP(n,n-5)) \leq \frac{3n}{2} - 5t$ *if* $n \equiv 0(mod4)$ *and* $\gamma_{rr2}(GP(n,5)) = \gamma_{rr2}(GP(n,n-5)) \leq \frac{3n}{2} - 5t + 1$ *if* $n \equiv 2(mod4)$.
(c)If n *is an even number,* $(n,5) = 1$*,* $5 \leq [\frac{n}{4}]$ *and* $t = [\frac{n}{10}]$ *is odd number, then* $\gamma_{rr2}(GP(n,5)) = \gamma_{rr2}(GP(n,n-5)) \leq \frac{n}{2} + 5(t+1)$ *if* $n \equiv 0(mod4)$ *and* $\gamma_{rr2}(GP(n,5)) = \gamma_{rr2}(GP(n,n-5)) \leq \frac{n}{2} + 5(t+1) + 1$ *if* $n \equiv 2(mod4)$.
(d)If n *is an even number,* $(n,5) = 1$*,* $[\frac{n}{4}] < 5 \leq [\frac{n}{2}]$*) and* $t = [\frac{n}{n-10}]$ *is even number, then* $\gamma_{rr2}(GP(n,5)) = \gamma_{rr2}(GP(n,n-5)) \leq \frac{3n}{2} - \frac{t(n-10)}{2} - 1$ *if* $n \equiv 0(mod4)$ *and* $\gamma_{rr2}(GP(n,5)) = \gamma_{rr2}(GP(n,n-5)) \leq \frac{3n}{2} - \frac{t(n-10)}{2}$ *if* $n \equiv 2(mod4)$.
(e)If n *is an even number,* $(n,5) = 1$*,* $[\frac{n}{4}] < 5 \leq [\frac{n}{2}]$*) and* $t = [\frac{n}{n-10}]$ *is odd number, then* $\gamma_{rr2}(GP(n,5)) = \gamma_{rr2}(GP(n,n-5)) \leq \frac{n+(t+1)(n-10)}{2}$ *if* $n \equiv 0(mod4)$ *and* $\gamma_{rr2}(GP(n,5)) = \gamma_{rr2}(GP(n,n-5)) \leq \frac{n+(t+1)(n-10)}{2} + 1$ *if* $n \equiv 2(mod4)$.

Proof. (a) We use the following algorithm and define the function f on $GP(n,5)$:
Step 1) $f(u_i) = f(v_i) = \emptyset$ for every even integer $1 < i < n$.
Step 2) $f(u_i) = \{1\}$, for every $1 \leq i \leq n$ such that $i \equiv 1(mod4)$.
Step 3) $f(u_i) = \{2\}$, for every $1 \leq i \leq n$ such that $i \equiv 3(mod4)$.
Step 4) For even integer $1 < i < 5$, $f(v_{i+5}) = f(v_{n-10+i}) = \{1,2\}$.
Step 5) If $5 \leq [\frac{n}{4}]$, for every even integer $1 < i < n$, $f(v_{i-5}) = \{1\}$ and $f(v_{i+5}) = \{2\}$ such that $1 < i \leq 10$ or

$20m < i \leq (4m+2)5$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
Step 6) If $5 \leq [\frac{n}{4}]$), for every even integer $1 < i < n$, $f(v_{i-5}) = \{2\}$ and $f(v_{i+5}) = \{1\}$ such that $2(2m-1)5 < i \leq 20m$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
Step 7) If $[\frac{n}{4}] < 5 \leq [\frac{n}{2}]$, for every even integer $1 < i \leq n$, $f(v_{i-5}) = \{1\}$ and $f(v_{i+5}) = \{2\}$ such that $1 < i \leq n - 10$ or $2m(n-10) < i \leq (2m+1)(n-10)$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
Step 8) If $[\frac{n}{4}] < 5 \leq [\frac{n}{2}]$), for every even integer $1 < i \leq n$, $f(v_{i-5}) = \{2\}$ and $f(v_{i+5}) = \{1\}$ such that $(2m-1)(n-10) < i \leq 2m(n-10)$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
We now claim that the function f defines a R2RDF on $GP(n, 5)$ and calculate $\omega(f)$.
Firstly according definition of f (step 1), each vertex with a label $\emptyset$ is adjacent to the other vertex with a label $\emptyset$. Now if w is a vertex of $GP(n, 5)$ and $f(w) = \emptyset$, then the following cases has happened.
case 1) There exist an even integer $1 < i < n$ such that $w = u_i$ and according step 2, step 3 and step 4, we have $f(u_{i-1}) \bigcup f(u_{i+1}) = \{1, 2\}$.
case 2) There exist an even integer $1 < i \leq n$ such that $w = v_i$. and according steps 4, 5, 6 and 7, we have $f(v_{i-5}) \bigcup f(v_{i+5}) = \{1, 2\}$.
Finally according to step 1, the number of vertices with empty label is equal to $n - 1$. By step 4, for every even integer $1 < i < 5$, there exist two vertices, such that their labels are $\{1, 2\}$ and by steps 2, 3, 5, 6, 7 and 8, the label of other vertices is $\{1\}$ or $\{2\}$. Then will have $\gamma_{rr2}(GP(n, 5)) = \gamma_{rr2}(GP(n, n-5)) \leq n + 5$.
(b) We use the following algorithm and define the function f on $GP(n, 5)$:
step 1) $f(u_i) = f(v_i) = \emptyset$ for every even integer $1 < i \leq n$.
step 2) If $n \equiv 2(mod4)$, then $f(u_{n-1}) = \{1, 2\}$.
step 3) $f(u_i) = \{1\}$, for every $1 \leq i \leq n$ such that $i \equiv 1(mod4)$ (The label defined in step 2 does not change).
step 4) $f(u_i) = \{2\}$, for every $1 \leq i \leq n$ such that $i \equiv 3(mod4)$ (The label defined in step 2 does not change).
step 5) For even integer $10t < j \leq n$, $f(v_{j+5}) = \{1, 2\}$.
step 6) For every even integer $1 < i \leq n$, $f(v_{i-5}) = \{1\}$ and $f(v_{i+5}) = \{2\}$ such that $1 < i \leq 10$ or $20m < i \leq (4m+2)5$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
step 7) For every even integer $1 < i \leq n$, $f(v_{i-5}) = \{2\}$ and $f(v_{i+5}) = \{1\}$ such that $2(2m-1)5 < i \leq 20m$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
We now claim that the function f defines a R2RDF on $GP(n, 5)$ and calculate $\omega(f)$.
Firstly according definition of f (step 1), each vertex with a label $\emptyset$ is adjacent to the other vertex with a label $\emptyset$. Now if w is a vertex of $GP(n, 5)$ and $f(w) = \emptyset$, then the following cases has happened.
case 1) There exist an even integer $1 < i \leq n$ such that $w = u_i$ and since t is an even number, according step 2, step 3 and step 4, we have $f(u_{i-1}) \bigcup f(u_{i+1}) = \{1, 2\}$.
case 2) There exist an even integer $1 < i \leq n$ such that $w = v_i$. and according steps 5, 6, and 7, we have $f(v_{i-5}) \bigcup f(v_{i+5}) = \{1, 2\}$.
Finally according to step 1, the number of vertices with empty label is equal to n. By step 2, if $n \equiv 2(mod4)$, then $f(u_{n-1}) = \{1, 2\}$, by step 5, for every even integer $10t < j \leq n$, the label of v_{j+5}, is $\{1, 2\}$ and by steps 3, 4, 6, and 7, the label of other vertices is $\{1\}$ or $\{2\}$. Then if $n \equiv 0(mod4)$ we will have $\gamma_{rr2}(GP(n, 5)) = \gamma_{rr2}(GP(n, n-5)) \leq \frac{n}{2} + \frac{n}{2} + (\frac{n-10t}{2}) = \frac{3n}{2} - 5t$ and if $n \equiv 2(mod4)$ we will have $\gamma_{rr2}(GP(n, 5)) = \gamma_{rr2}(GP(n, n-5)) \leq \frac{n}{2} + 1 + \frac{n}{2} + (\frac{n-10t}{2}) = \frac{3n}{2} - 5t + 1$.
(c) We use the following algorithm and define the function f on $GP(n, 5)$:
step 1) $f(u_i) = f(v_i) = \emptyset$ for every even integer $1 < i \leq n$.
step 2) If $n \equiv 2(mod4)$, then $f(u_{n-1}) = \{1, 2\}$.

step 3) $f(u_i) = \{1\}$, for every $1 \leq i \leq n$ such that $i \equiv 1(mod4)$ (The label defined in step 2 does not change).
step 4) $f(u_i) = \{2\}$, for every $1 \leq i \leq n$ such that $i \equiv 3(mod4)$ (The label defined in step 2 does not change).
step 5) For even integer $10t - l < j \leq 10t$, $f(v_{j+5}) = \{1, 2\}$, such that $l = 10(t+1) - n$
step 6) For every even integer $1 < i \leq n$, $f(v_{i-5}) = \{1\}$ and $f(v_{i+5}) = \{2\}$ such that $1 < i \leq 10$ or $20m < i \leq (4m+2)5$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
step 7) For every even integer $1 < i \leq n$, $f(v_{i-5}) = \{2\}$ and $f(v_{i+5}) = \{1\}$ such that $2(2m-1)5 < i \leq 20m$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
We now claim that the function f defines a R2RDF on $GP(n, 5)$ and calculate $\omega(f)$.
Firstly according definition of f (step 1), each vertex with a label $\emptyset$ is adjacent to the other vertex with a label $\emptyset$. Now if w is a vertex of $GP(n, 5)$ and $f(w) = \emptyset$, then the following cases has happened.
case 1) There exist an even integer $1 < i \leq n$ such that $w = u_i$ and since t is an even number, according step 2, step 3 and step 4, we have $f(u_{i-1}) \bigcup f(u_{i+1}) = \{1, 2\}$.
case 2) There exist an even integer $1 < i \leq n$ such that $w = v_i$. and according steps 5, 6, and 7, we have $f(v_{i-5}) \bigcup f(v_{i+5}) = \{1, 2\}$.
Finally according to step 1, the number of vertices with empty label is equal to n. By step 2, if $n \equiv 2(mod4)$, then $f(u_{n-1}) = \{1, 2\}$, and by step 5, for every even integer $10t - l < j \leq 10t$, the label of v_{j+5}, is $\{1, 2\}$ and by steps 3, 4, 6, and 7, the label of other vertices is $\{1\}$ or $\{2\}$. Then if $n \equiv 0(mod4)$ we will have $\gamma_{rr2}(GP(n, 5)) = \gamma_{rr2}(GP(n, n-5)) \leq \frac{n}{2} + \frac{n}{2} + (\frac{10(t+1)-n}{2}) = \frac{n}{2} + 5(t+1)$ and if $n \equiv 2(mod4)$ we will have $\gamma_{rr2}(GP(n, 5)) = \gamma_{rr2}(GP(n, n-5)) \leq \frac{n}{2} + 5(t+1) + 1$.
(d) We use the following algorithm and define the function f on $GP(n, 5)$:
step 1) $f(u_i) = f(v_i) = \emptyset$ for every even integer $1 < i \leq n$.
step 2) If $n \equiv 2(mod4)$, then $f(u_{n-1}) = \{1, 2\}$.
step 3) $f(u_i) = \{1\}$, for every $1 \leq i \leq n$ such that $i \equiv 1(mod4)$ (The label defined in step 2 does not change).
step 4) $f(u_i) = \{2\}$, for every $1 \leq i \leq n$ such that $i \equiv 3(mod4)$ (The label defined in step 2 does not change).
step 5) For even integer $t(n-10) < j \leq n$, $f(v_{j+5}) = \{1, 2\}$.
step 6) For every even integer $1 < i \leq n$, $f(v_{i-5}) = \{1\}$ and $f(v_{i+5}) = \{2\}$ such that $1 < i \leq n-10$ or $2m(n-10) < i \leq (2m+1)(n-10)$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
step 7) For every even integer $1 < i \leq n$, $f(v_{i-5}) = \{2\}$ and $f(v_{i+5}) = \{1\}$ such that $(2m-1)(n-10) < i \leq 2m(n-10)$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
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case 1) There exist an even integer $1 < i \leq n$ such that $w = u_i$ and since t is an even number, according step 2, step 3 and step 4, we have $f(u_{i-1}) \bigcup f(u_{i+1}) = \{1, 2\}$.
case 2) There exist an even integer $1 < i \leq n$ such that $w = v_i$. and according steps 5, 6, and 7, we have $f(v_{i-5}) \bigcup f(v_{i+k}) = \{1, 2\}$.
Finally according to step 1, the number of vertices with empty label is equal to n. By step 2, if $n \equiv 2(mod4)$, then $f(u_{n-1}) = \{1, 2\}$, and by step 5, for every even integer $t(n-10) < j \leq n$, the label of v_{j+5}, is $\{1, 2\}$ and by steps 3, 4, 6, and 7, the label of other vertices is $\{1\}$ or $\{2\}$. Then if $n \equiv 0(mod4)$ we will have $\gamma_{rr2}(GP(n, 5)) = \gamma_{rr2}(GP(n, n-5)) \leq \frac{n}{2} + \frac{n}{2} + (\frac{n-t(n-10)}{2}) = \frac{3n}{2} - \frac{t(n-10)}{2}$ and if $n \equiv 2(mod4)$ we will have $\gamma_{rr2}(GP(n, 5)) = \gamma_{rr2}(GP(n, n-5)) \leq \frac{3n}{2} - \frac{t(n-10)}{2} + 1$.
(e) We use the following algorithm and define the function f on $GP(n, 5)$:

step 1) $f(u_i) = f(v_i) = \emptyset$ for every even integer $1 < i \leq n$.
step 2) If $n \equiv 2(mod4)$, then $f(u_{n-1}) = \{1, 2\}$.
step 3) $f(u_i) = \{1\}$, for every $1 \leq i \leq n$ such that $i \equiv 1(mod4)$ (The label defined in step 2 does not change).
step 4) $f(u_i) = \{2\}$, for every $1 \leq i \leq n$ such that $i \equiv 3(mod4)$ (The label defined in step 2 does not change).
step 5) For even integer $t(n-10) - l < j \leq t(n-10)$, $f(v_{j+5}) = \{1, 2\}$, such that $l = (t+1)(n-10) - n$.
step 6) For every even integer $1 < i \leq n$, $f(v_{i-5}) = \{1\}$ and $f(v_{i+5}) = \{2\}$ such that $1 < i \leq n-10$ or $2m(n-10) < i \leq (2m+1)(n-10)$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
step 7) For every even integer $1 < i \leq n$, $f(v_{i-5}) = \{2\}$ and $f(v_{i+5}) = \{1\}$ such that $(2m-1)(n-10) < i \leq 2m(n-10)$, $m = 1, 2, \cdots$. (The labels defined in previous steps do not change)
We now claim that the function f defines a R2RDF on $GP(n, 5)$ and calculate $\omega(f)$.
Firstly according definition of f (step 1), each vertex with a label $\emptyset$ is adjacent to the other vertex with a label $\emptyset$. Now if w is a vertex of $GP(n, 5)$ and $f(w) = \emptyset$, then the following cases has happened.
case 1) There exist an even integer $1 < i \leq n$ such that $w = u_i$ and since t is an even number, according step 2, step 3 and step 4, we have $f(u_{i-1}) \bigcup f(u_{i+1}) = \{1, 2\}$.
case 2) There exist an even integer $1 < i \leq n$ such that $w = v_i$. and according steps 5, 6, and 7, we have $f(v_{i-5}) \bigcup f(v_{i+5}) = \{1, 2\}$.
Finally according to step 1, the number of vertices with empty label is equal to n. By step 2, if $n \equiv 2(mod4)$, then $f(u_{n-1}) = \{1, 2\}$, and by step 5, for every even integer $t(n-10) - l < j \leq t(n-10)$, the label of v_{j+5}, is $\{1, 2\}$ and by steps 3, 4, 6, and 7, the label of other vertices is $\{1\}$ or $\{2\}$. Then if $n \equiv 0(mod4)$ we will have $\gamma_{rr2}(GP(n, 5)) = \gamma_{rr2}(GP(n, n-5)) \leq \frac{n}{2} + \frac{n}{2} + (\frac{(t+1)(n-10)-n}{2}) = \frac{(t+1)(n-10)-n}{2}$ and if $n \equiv 2(mod4)$ we will have $\gamma_{rr2}(GP(n, 5)) = \gamma_{rr2}(GP(n, n-5)) \leq \frac{(t+1)(n-10)-n}{2} + 1$.

□

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