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Quasi-orthogonal expansions for functions in BMO

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Abstract

For $\{\phi_n(x)\}_{n=1}^{\infty}, x \in [0,1]$ an orthonormal system of uniformly bounded functions, $\|\phi_n\|_\infty \leq M < \infty, n = 1, 2, \ldots$, we shall prove, there exists a subsystem $\{\phi_{n_k}(x)\}_{k=1}^{\infty}$ such that if $\sum_{k=1}^{\infty} a_k^2 < \infty$ then $\sum_{k=1}^{\infty} a_k \phi_{n_k}(x) \in BMO$.

1 Introduction

1.1 Concepts of orthonormal systems

A system of functions $\{\phi_n(x)\}_{n=1}^{\infty} \subset L^2(a,b)$ is orthonormal if for $m, n = 1, 2, \cdots$,

$$\int_a^b \phi_n(x)\phi_m(x)dx = \begin{cases} 1, & \text{if } n = m; \\ 0, & \text{if } n \neq m. \end{cases} \tag{1.1}$$

([**?**]) An orthonormal system $\Phi = \{\phi_n(x)\}_{n=1}^{\infty}, x \in (0,1)$, is called a convergence system if every series of the form $\sum_{n=1}^{\infty} a_n\phi_n(x), \sum_{n=1}^{\infty} a_n^2 < \infty$ converges a.e. on $(0,1)$. ([**?**]) For every orthonormal system $\Phi = \{\phi_n(x)\}_{n=1}^{\infty}, x \in (0,1)$, there is an increasing sequence $\{N_k\}_{k=1}^{\infty}$ of integers such that for every series of the form $\sum_{n=1}^{\infty} a_n\phi_n(x), \sum_{n=1}^{\infty} a_n^2 < \infty$ the sequence $S_{N_k}(x) = \sum_{n=1}^{N_k} a_n\phi_n(x), k = 1, 2, \cdots$, converges a.e. on $(0,1)$ and satisfies the inequality

$$\| \sup_{1\leq q<\infty} | S_{N_q}(x) | \|_2 \leq C(\sum_{n=1}^{\infty} a_n^2)^{1/2},$$

for some C. ([**?**]) For every orthonormal system $\Phi = \{\phi_n(x)\}_{n=1}^{\infty}, x \in (0,1)$, we can extract a subsystem $\Phi_1 = \{\phi_{n_k}(x)\}_{k=1}^{\infty}, n_1 < n_2 < \cdots$, which is a convergence system and for which, moreover,

$$\| \sup_{1\leq N<\infty} |\sum_{k=1}^{N} a_k\phi_{n_k(x)}| \|_2 \leq C \| a_k \|_{l^2}$$

where a_k is an arbitrary sequence in l^2 and C is a constant.

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1.2 The space of bounded mean oscillation functions

([**?**],[**?**])A locally integrable function f will be said to belong to BMO if the inequality

$$\frac{1}{|B|}\int_B |f(x)-f_B|dx \leq A \tag{1.2}$$

holds for all balls B; here $|B|$ is volume of B and $f_B = |B|^{-1}\int_B f dx$ denotes the mean value of f over the ball B. The inequality (2) asserts that over any ball B, the average oscillation of f is bounded. The smallest bound A in (2) is called the norm of f in this space, and is denoted by $||f||_{BMO}$.
A binary interval or dyadic interval is an interval of the form $((i-1)/2^k, i/2^k)$, where $i = 1, \cdots, 2^k, k = 0, 1, \cdots$. For $n = 2^k + i, i = 1, \cdots, 2^k, k = 0, 1, \cdots$, we write

$$\Delta_n = \Delta_k^i = ((i-1)/2^k, i/2^k), \Delta_1 = (0, 1).$$

If $\delta \subseteq (0,1)$ is any interval, we denote by δ^+ and δ^- the left hand and right hand halves of δ (not including the midpoint).
([**?**]) The Haar system is the system of functions $\chi = \{\chi_n(x)\}_{n=1}^{\infty}, x \in [0,1]$, where $\chi_1 \equiv 1$ and for $2^k < n \leq 2^{k+1}, k = 0, 1, \cdots, \chi_n(x)$ is defined as follows:

$$\chi_n(x) = \begin{cases} 2^{k/2}, & \text{if } x \in \Delta_n^+; \\ -2^{k/2}, & \text{if } x \in \Delta_n^-; \\ 0, & \text{Otherwise.} \end{cases} \tag{1.3}$$

[1,5,6] The Haar system is an orthonormal basis for $L^2(\mathbb{R})$. If $\chi_I(x)$ be the Haar function associated with the dyadic interval I then the Haar coefficient over I of f is

$$f_I = (f, \chi_I) := \int_I f(x)\chi_I(x)dx$$

([**?**]) An orthonormal system $\Phi = \{\phi_n(x)\}_{n=1}^{\infty}, x \in (0,1)$, is called a convergence system if every series of the form $\sum_{n=1}^{\infty} a_n\phi_n(x), \sum_{n=1}^{\infty} a_n^2 < \infty$ converges a.e. on $(0,1)$.

1.3 Quasi-orthogonal expansions

Our orthogonal decompositions (more precisely, "quasi-orthogonal" decompositions) will be given in terms of a family of "bump" functions; each such function will be associated to a dyadic cube. We fix our notation as follows: the letter Q will be reserved for a dyadic cube, and $B = B_Q$ will be the ball with the same center and twice the diameter (thus $B_Q \supset Q$); similarly the ball B_j will be associated to Q_j, etc. For each dyadic cube Q, we will be given a function ϕ_j, supported in B_Q, that satisfies certain natural size, regularity, and moment conditions. We shall assume that

$$|D^\alpha \phi_Q| \leq \frac{l(Q)^{-|\alpha|}}{|Q|^{1/2}}, \int x^\alpha \phi_Q(x)dx = 0, \quad 0 \leq |\alpha| \leq n \tag{1.4}$$

with $l(Q)$ denoting the length of a side of the cube $Q \subset R^n$.
We shall be dealing with functions f that can be represented in the form

$$f = \sum_{Q} a_Q \phi_Q, \tag{1.5}$$

where a_Q is a suitable collection of constants, and the summation in (5) is carried over all dyadic cubes.
Various extensions of the same ideas are possible, giving also characterizations of many other function spaces besides BMO, leading in addition to what are now known as "wavelet" decompositions.
[4,6](a) Suppose the coefficients a_Q satisfy the inequalities

$$\sum_{Q \subset Q_0} |a_Q|^2 \leq A|Q_0| \tag{1.6}$$

for all dyadic cubes Q_0, where the summation in (6) is taken over all dyadic subcubes of Q_0. Then the series (5) gives an $f \in BMO$ in the sense that

$$\lim_{\rho_1 \to 0, \rho_2 \to \infty} \sum_{\rho_1 \leq l(Q) \leq \rho_2} a_Q \phi_Q = f$$

exists in the weak topology of BMO.
(b) Conversely, suppose $f \in BMO$. Then there is a collection of functions ϕ_Q and a collection of coefficients a_Q that satisfy (4) and (6) respectively, so that f is representable as the sum (5), in the sense asserted in part (a). The smallest A for which (6) holds is comparable with $||f||^2_{BMO}$.
Remark. A simplified version of the system a_Q occurs in the dyadic context, and is given by the Haar basis. We describe the situation in one dimension. Suppose h is the function supported in the unit interval $[0, 1]$ that equals 1 in the left half and -1 in the right half. For any dyadic interval Q, set

$$h_Q = 2^{j/2} h(2^j x - k), Q = [k2^{-j}, (k + 1)2^{-j}].$$

While the h_Q satisfy only the size condition $|h_Q| \leq |Q|^{-1/2}$ and the moment condition $\int h_Q dx = 0$ (and not the full conditions (6)), they have the compensating merit of forming a complete orthonormal basis for $L^2(R^1)$. For $f = \sum a_Q h_Q$ and a constant c, the property

$$\sum_{Q \subseteq Q_0} |a_Q|^2 \leq c|Q_0|$$

is then equivalent with f being in BMO in the dyadic sense.

2 Main result

We will use the following two statements to prove and substantiate the main theorem. ([**?**]) For function f, the following statements are equivalent:
1- There exist the number c_1 such that $\int_0^1 exp(c_1|f(x)|^2)dx < \infty$
2- $\|f\|_p \leq c_2\sqrt{p}$, that c_2 is a constant number.
3- $|\{x; |f(x)| > \lambda\}| \leq c_1 e^{-c_3\lambda^2}$, that c_1 and c_3 are constant numbers. ([**?**]) Let f is a function on $[0, 1]$, then

$f \in BMO$ if and only if for every dyadic interval $J \subseteq [0,1]$ the inequality

$$\sum_{I\subseteq J} |f_I|^2 \leq A|J|$$

be satisfied, that I is dyadic.

Let $\{\phi_n(x)\}_{n=1}^{\infty}, x \in [0,1]$ be an orthonormal system of uniformly bounded functions, $\|\phi_n\|_\infty \leq M < \infty, n = 1,2,\ldots$. Then there exists a subsystem $\{\phi_{n_k}(x)\}_{k=1}^{\infty}$ s.t. if $\sum_{k=1}^{\infty} a_k^2 < \infty$ then $\sum_{k=1}^{\infty} a_k\phi_{n_k}(x) \in BMO$ and if $\sum_{k=1}^{\infty} a_k^2 < 1$, then for the function $f(x) = \sum_{k=1}^{\infty} a_k\phi_{n_k}(x)$ the following inequality is established:

$$|\{x; |f(x)| > t\}| \leq c_2 e^{-c_3 t^2}.$$

Proof. In fact, if $\{N_k\}_{k=1}^{\infty}$ is the sequence of numbers that was constructed for Φ in Theorem 1.3., it is sufficient we choose numbers n_k such that $N_k < n_k \leq N_{k+1}, k = 1,2,\ldots$. Now let $f(x) = \sum_{k=1}^{\infty} a_k\phi_{n_k}(x)$ then for every dyadic I with $|I| = \frac{1}{2^n}$ we have:

$|f_I| = |\int_0^1 f.\chi_I dx|$
$= |\int_0^1 (\sum_{k=1}^{\infty} a_k\phi_{n_k}(x)).\chi_I(x)dx|$
$= |\int_0^1 \sum_{k=1}^{\infty} a_k(\sum_{i=N_k+1}^{N_{k+1}} c_i\chi_i(x)).\chi_I(x)dx|$
$= |\sum_{k=1}^{\infty}\sum_{i=N_k+1}^{N_{k+1}} a_k c_i \int_0^1 \chi_i(x).\chi_I(x)dx|$
$\leq \sum_{k=1}^{\infty}\sum_{i=N_k+1}^{N_{k+1}} |a_k||c_i|| \int_0^1 \chi_i(x).\chi_I(x)dx|$
$= \sum_{k=1}^{\infty} |a_k| \sum_{i=N_k+1}^{N_{k+1}} |\int_0^1 \phi_{n_k}(x).\chi_i(x)|| \int_0^1 \chi_i(x).\chi_I(x)dx|$
$\leq |a_k|.M.\frac{1}{2^{n/2}}$.

So

$$|f_I|^2 \leq a_k^2.M^2.\frac{1}{2^n},$$

therefore if $|J| = \frac{1}{2^m}$ then

$$\sum_{I\subseteq J} |f_I|^2 = |f_J|^2 + |f_{J_1^{(1)}}|^2 + |f_{J_1^{(2)}}|^2 + \ldots + \sum_{i=1}^{2^k} |f_{J_k^{(i)}}|^2 + \ldots \leq M^2.\frac{1}{2^m}\|\{a_n\}\|_2^2 = A|J|.$$

Now with 4th remark next of Theorem 1.2. $f \in BMO$.

So if $\sum_{k=1}^{\infty} a_k^2 < 1$, then $\|f\|_{BMO} \leq M^2\|\{a_n\}\|_2^2 \leq M^2$ and $\|f\| \leq c_1\sqrt{p}$. Finally use Theorem 2.5 and Theorem 2.6, we have

$$|\{x; |f(x)| > t\}| \leq c_2 e^{-c_3 t^2}.$$

□

3 Conclusion and future works

In this paper with use Haar system, it was proved for every orthonormal system $\{\phi_n(x)\}_{n=1}^{\infty}, x \in [0,1]$ of uniformly bounded functions, $\|\phi_n\|_\infty \leq M < \infty, n = 1,2,\ldots$, there exists a subsystem $\{\phi_{n_k}(x)\}_{k=1}^{\infty}$ such that if $\sum_{k=1}^{\infty} a_k^2 < \infty$ then $\sum_{k=1}^{\infty} a_k\phi_{n_k}(x) \in BMO$. For future works, we intend to address the two issues. Firstly if $\{\phi_n(x)\} = \{\chi_{\sigma(n)}(x)\}$ is a rearrangement of Haar system, then for every $\varepsilon > 0$, does it exist a subsystem $\{\phi_{n_k}(x)\}$ which is convergence system and $n_k < k^{1+\varepsilon}$? Secondly For any positive integers $N_1 < N_2 < \cdots$, does it exist a rearrangement Haar system $\{\phi_n(x)\} = \{\chi_{\sigma(n)}(x)\}$, such that $\sum_{k=1}^{N_m} a_k\phi_k(x)$ is divergent a.e. for some sequence $\{a_k\}$ with $\sum a_k^2 < \infty$?

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