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A New Method for Solving Two-Dimensional Fuzzy Fredholm Integral Equations of The Second Kind

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Abstract

In this work, we introduce a novel method for solving two-dimensional fuzzy Fredholm integral equations of the second kind (2D-FFIE-2). We use new representation of parametric form of fuzzy numbers and convert a two-dimensional fuzzy Fredholm integral equation to system of two-dimensional Fredholm integral equations of the second kind in crisp case. We can use Adomian decomposition method for finding the approximation solution of the each equation, hence obtain an approximation for fuzzy solution of 2D-FFIE-2. We prove the convergence of the method and finally apply the method to some examples.

1 Introduction

Fuzzy systems are now used to study a different of problems ranging from fuzzy metric spaces [1], fuzzy topological spaces [2], to control chaotic systems [3, 4], fuzzy differential equations [5, 6, 7] and particle physics [8, 9, 10, 11].The topics of fuzzy integral equations (FIE) which attracted growing interest for some time, in particular, in relation to fuzzy control, have been developed in recent years.The concept of integration of fuzzy functions was first presented by Dubois and Prade (1982). Alternative approaches were later suggested by Goetschel and Voxman (1986), Kaleva (1987), Matloka (1987), Seikkala (1987).Recently, some numerical methods have been introduced to solve fuzzy Fredholm integral equation of the second kind in one-dimensional space FFIE-2 and two-dimensional space 2D-FFIE-2. For example Babolian et al [12] used the Adomian decomposition method (ADM) to solve FFIE-2. Abbasbandi et al [13] obtained numerical solution of FFIE-2 using Nystrom method. Ezzati et al [14] applied the fuzzy Bernstein polynomials to solve fuzzy integrals. Rivaz et al [15] used the modified homotopy perturbation method to solve 2D-FFIE-2. Mirzaee et al [16] obtained numerical solution of 2D-FFIE-2 by using triangular functions.In this work, we use the Adomian decomposition method for solving 2D-FFIE-2.The rest of the paper is organized as follows: In section 2, the basic notations of fuzzy numbers, fuzzy functions and fuzzy integrals have been presented. In section 3, Adomian decomposition method is used for solving 2D-FFIE-2. In section 4, the convergence of the method have been proved. In section 5, numerical results with the exact solution for some examples have been compared.

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2 Preliminaries

In this section the basic concepts about fuzzy calculus are introduced.

Definition 2.1. *[22] A fuzzy number is a fuzzy set $u : R^1 \longrightarrow I = [0,1]$ which satisfies*

(i) u *is upper semicontinuous.*

(ii) $u(x) = 0$ *outside some interval* $[a_2, b_2]$.

(iii) There are real numbers a,b: $a_2 \leq a_1 \leq b_1 \leq b_2$ *for which*

- *(1)* $u(x)$ *is monotonic increasing on* $[a_2, b_2]$,
- *(2)* $u(x)$ *is monotonic decreasing on* $[b_1, b_2]$,
- *(3)* $u(x) = 1$, $a_1 \leq x \leq b_1$.

The set of all fuzzy numbers is denoted by E^1. An equivalent definition or parametric form of fuzzy numbers which yields the same E^1 is given by Kaleva [26].

Definition 2.2. *[12] An arbitrary fuzzy number in parametric form is represented by an ordered pair of functions* $(\underline{u}(\alpha), \overline{u}(\alpha))$, $0 \leq \alpha \leq 1$, *which satisfy the following requirements:*

(1) $\underline{u}(\alpha)$ *is a bounded monotonic increasing left continuous function,*

(2) $\overline{u}(\alpha)$ *is a bounded monotonic decreasing left continuous function,*

(3) $\underline{u}(\alpha) \leq \overline{u}(\alpha), 0 \leq \alpha \leq 1$.

Lemma 2.1. *[18] Suppose* $(\underline{u}(r), \bar{u}(r))$, $0 \leq r \leq 1$ *is a given family of non-empty intervals. If*

(1) $(\underline{u}(r_1), \bar{u}(r_1)) \supseteq (\underline{u}(r_2), \bar{u}(r_2))$ *for* $0 \leq r_1 \leq r_2 \leq 1$.

(2) $(\lim_{k\to\infty} \underline{u}(r_k), \lim_{k\to\infty} \bar{u}(r_k)) = (\underline{u}(r), \bar{u}(r))$, *whenever* $\{r_k\}$ *is a non-decreasing converging sequence converges to* $0 \leq r \leq 1$,

then the family $(\underline{u}(r), \bar{u}(r))$, $0 \leq r \leq 1$, *represent the* r*-cut sets of a fuzzy number* $u \in E^1$.
On the contrary, suppose $(\underline{u}(r), \bar{u}(r))$, $0 \leq r \leq 1$, *are the* r*-cut sets of a fuzzy number* $u \in E^1$, *then the conditions* (1) *and* (2) *hold.*

Definition 2.3. *[13] For arbitrary* $u = (\underline{u}(\alpha), \overline{u}(\alpha))$, $v = (\underline{v}(\alpha), \overline{v}(\alpha))$ *and* $k \in R$ *we define addition and multiplication by* k *as follows:*

$$
\begin{aligned}
(\underline{u+v}(\alpha)) &= (\underline{u}(\alpha) + \underline{v}(\alpha)), \\
(\overline{u+v}(\alpha)) &= (\overline{u}(\alpha) + \overline{v}(\alpha)), \\
(\underline{ku}(\alpha)) = k\underline{u}(\alpha), (\overline{ku}(\alpha)) &= k\overline{u}(\alpha) \quad if \quad k \geq 0, \\
(\overline{ku}(\alpha)) = k\underline{u}(\alpha), (\underline{ku}(\alpha)) &= k\overline{u}(\alpha) \quad if \quad k < 0.
\end{aligned}
$$

Definition 2.4. *[21] For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ the distance between u, v is define as follows:*

$$D(u, v) = \sup_{0\leq\alpha\leq1} \max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\} \tag{2.1}$$

and also metric space (D, E^1) is a complete metric space [20].

Definition 2.5. *[23] The fuzzy Fredholm integral equation of the second kind is*

$$u(x) = f(x) + \lambda \int_{a_1}^{b_1} k(x, t)u(t)dt, \quad x \in D, \tag{2.2}$$

where $u(x)$ and $f(x)$ are fuzzy functions on $D = [a_1, b_1]$ and $k(x, t)$ is an arbitrary kernel function over $T = [a_1, b_1] \times [a_1, b_1]$, and u is unknown on D.

Theorem 2.1. *[24] Let $k(x, t)$ be a continuous on T and $f(x)$ be a fuzzy continuous function over the D. If*

$$|\lambda| \leq \frac{1}{M(b_1 - a_1)},$$

where $M = \max_{(x,t)\in T} |k(x, t)|$, then equation (2.2) has a fuzzy unique solution.

Remark 2.1. *[13] Put $u(\alpha) = (\underline{u}(\alpha), \overline{v}(\alpha))$,$0 \leq \alpha \leq 1$ be a fuzzy number, we take*

$$\begin{cases} u^c(\alpha) = \frac{\underline{u}(\alpha)+\overline{u}(\alpha)}{2}, \\ u^d(\alpha) = \frac{\overline{u}(\alpha)-\underline{u}(\alpha)}{2}. \end{cases} \tag{2.3}$$

Obvious that $u^d(\alpha) \geq 0$ and $\underline{u}(\alpha) = u^c(\alpha) - u^d(\alpha)$, $\overline{u}(\alpha) = u^c(\alpha) + u^d(\alpha)$, also the fuzzy number $u \in E^1$ is symmetric if $u^c(\alpha)$ is independent from α for $0 \leq \alpha \leq 1$.

Remark 2.2. *[13] Put $u(\alpha) = (\underline{u}(\alpha), \overline{u}(\alpha))$, $v(\alpha) = (\underline{v}(\alpha), \overline{v}(\alpha)$ and k, s are arbitrary real numbers. If $w = ku+sv$ then*

$$\begin{cases} w^c(\alpha) = ku^c(\alpha) + sv^c(r), \\ w^d(\alpha) = |k|u^d(\alpha) + |s|v^d(\alpha). \end{cases} \tag{2.4}$$

Remark 2.3. *[13] By referring to remark 2.1 we have*

$$\begin{array}{rcl} |\bar{u}(r) - \bar{v}(r)| & = & |u^c(r) - v^c(r)| + |u^d(r) - v^d(r)|, \\ |\underline{u}(r) - \underline{v}(r)| & = & |u^c(r) - v^c(r)| + |u^d(r) - v^d(r)|, \end{array} \tag{2.5}$$

hence for all $r \in [0, 1]$

$$D(u, v) \leq \sup_{0\leq r\leq1} \left\{|u^c(r) - v^c(r)| + |u^d(r) - v^d(r)|\right\}.$$

Thus if $|u^c(r) - v^c(r)|$ and $|u^d(r) - v^d(r)|$ tend to zero then $D(u, v)$ tends to zero.

Definition 2.6. *[25] A function $f:\mathbb{R}^2 \to E^1$ is called a fuzzy function in two-dimensional space. f is said to be continuous, if for arbitrary fixed $t_0 \in \mathbb{R}^2$ and $\epsilon > 0$ a $\delta > 0$ exists such that*

$$||t-t_0|| < \delta \Rightarrow D(f(t), f(t_0)) < \epsilon, \quad t=(x,y), t_0=(x_0,y_0).$$

Definition 2.7. *Let $f:[a_1,b_1]\times[a_2,b_2] \to E^1$, for each partition $p=\{t_0,t_1,...t_m\}$ of $[a_1,b_1]$, $Q=\{s_0,s_1,...,s_n\}$ of $[a_2,b_2]$ and for arbitrary $\xi_i \in [t_{i-1},t_i]$, $2 \le i \le m$, and for arbitrary $\eta_j \in [s_{j-1},s_j]$, $2 \le j \le n$ let*

$$R_p = \sum_{i=2}^{m}\sum_{j=2}^{n} f(\xi_i,\eta_j)(t_i-t_{i-1})(s_j-s_{j-1})$$

The definite integral of $f(x,y)$ over $[a_1,b_1]\times[a_2,b_2]$ is

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y)dxdy = \lim_{(p_i,q_j)\to(0,0)} R_p,$$

such that

$$q_j = \max|s_j - s_{j-1}|, 2 \le j \le n \qquad p_i = \max|t_i, t_{i-1}|, 2 \le i \le m$$

then this limit exists in metric D.
If the function $f(x,y)$ is continuous in the metric D, it's definite integral exists [21].

$$\underline{\left(\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y;\alpha)dxdy\right)} = \int_{a_2}^{b_2}\int_{a_1}^{b_1} \underline{f}(x,y;\alpha)dxdy$$
$$\overline{\left(\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y;\alpha)dxdy\right)} = \int_{a_2}^{b_2}\int_{a_1}^{b_1} \overline{f}(x,y;\alpha)dxdy$$

3 Adomian decomposition method for solving 2D-FFIE-2

Two-dimensional fuzzy Fredholm integral equation of the second kind $(2D-FFIE-2)$ is defined as follows [16]:

$$F(x,y) = f(x,y) + \lambda \int_{a_1}^{b_1}\int_{a_2}^{b_2} K(x,y,s,t)F(s,t)dsdt, \tag{3.1}$$

where $k(x,y,s,t)$ is an arbitrary kernel function over $S=[a_1,b_1]\times[a_2,b_2]\times[a_1,b_1]\times[a_2,b_2]$ and $f(x,y)$ and $F(x,y)$ are fuzzy functions over $V=[a_1,b_1]\times[a_2,b_2]$ and $F(s,t)$ is unknown on V.
Now, we are about to introduce parametric form of 2D-FFIE-2 with respect to definition (2.2).
Let $(\underline{f}(x,y;\alpha), \overline{f}(x,y;\alpha))$ and $(\underline{F}(x,y;\alpha), \overline{F}(x,y;\alpha)), 0 \le \alpha \le 1$, be parametric form of $f(x,y)$, $F(x,y)$, respectively. Then parametric form of 2D-FFIE-2 ia as follows:

$$\begin{cases} \underline{F}(x,y;\alpha) = \underline{f}(x,y;\alpha) + \lambda \int_{a_1}^{b_1}\int_{a_2}^{b_2} \upsilon_1(x,y,s,t,\underline{F}(s,t;\alpha),\overline{F}(s,t;\alpha))dsdt, \\ \overline{F}(x,y;\alpha) = \overline{f}(x,y;\alpha) + \lambda \int_{a_1}^{b_1}\int_{a_2}^{b_2} \upsilon_2(x,y,s,t,\underline{F}(s,t;\alpha),\overline{F}(s,t;\alpha))dsdt \end{cases} \tag{3.2}$$

which

$$v_1(x,y,s,t,\underline{F}(s,t;\alpha),\overline{F}(s,t;\alpha)) = \begin{cases} k(x,y,s,t)\underline{F}(s,t;\alpha), & \text{if } k(x,y,s,t) \geq 0, \\ k(x,y,s,t)\overline{F}(s,t;\alpha), & \text{if } k(x,y,s,t) \leq 0 \end{cases}$$

and

$$v_2(x,y,s,t,\underline{F}(s,t;\alpha),\overline{F}(s,t;\alpha)) = \begin{cases} k(x,y,s,t)\overline{F}(s,t;\alpha), & k(x,y,s,t) \geq 0, \\ k(x,y,s,t)\underline{F}(s,t;\alpha), & k(x,y,s,t) \leq 0 \end{cases}$$

for $0 \leq \alpha \leq 1$, we can see that (3.1), are systems of Fredholm integral equations of the second kind with three variables in crisp case.

hence

$$F^c(x,y;\alpha) = f^c(x,y;\alpha) + \lambda \int_{a_1}^{b_1}\int_{a_2}^{b_2} K(x,y,s,t)F^c(s,t;\alpha)dsdt. \tag{3.3}$$

$$F^d(x,y;\alpha) = f^d(x,y;\alpha) + \lambda \int_{a_1}^{b_1}\int_{a_2}^{b_2} |K(x,y,s,t)|F^d(s,t;\alpha)dsdt. \tag{3.4}$$

The Adomian decomposition method consists of decomposing the unknown function $F(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$F(x) = \sum_{n=0}^{\infty} F_n(x),$$

where the components $F_n(x)$, $n \geq 0$ are to be determined in a recursive way. The decomposition method concerns itself with determining the components $F_0, F_1, F_2, \cdots$ individually. The aim of this section is using ADM for solving 2D-FFIE-2. By using that, we can write (3.3),(3.4) as follows.

$$\sum_{n=0}^{\infty} F_n^c(x,y;\alpha) = f^c(x,y,\alpha) + \lambda \int_{a_1}^{b_1}\int_{a_2}^{b_2} K(x,y,s,t)(\sum_{n=0}^{\infty} F_n^c(s,t;\alpha))dsdt \tag{3.5}$$

$$\sum_{n=0}^{\infty} F_n^d(x,y;\alpha) = f^d(x,y,\alpha) + \lambda \int_{a_1}^{b_1}\int_{a_2}^{b_2} |K(x,y,s,t)|(\sum_{n=0}^{\infty} F_n^d(s,t;\alpha))dsdt \tag{3.6}$$

we have

$$\begin{aligned} F_0^c(x,y;\alpha) &= f^c(x,y;\alpha) \\ F_1^c(x,y;\alpha) &= \lambda \int_{a_1}^{b_1}\int_{a_2}^{b_2} K(x,y,s,t)F_0^c(s,t;\alpha)dsdt \\ F_2^c(x,y;\alpha) &= \lambda \int_{a_1}^{b_1}\int_{a_2}^{b_2} K(x,y,s,t)F_1^c(s,t;\alpha)dsdt, \cdots \end{aligned}$$

and at the end

$$\begin{cases} F_0^c(x,y;\alpha) = f^c(x,y;\alpha) \\ F_{n+1}^c(x,y;\alpha) = \lambda \int_{a_1}^{b_1}\int_{a_2}^{b_2} k(x,y,s,t)F_n^c(s,t;\alpha)dsdt \end{cases} \tag{3.7}$$

and

$$\begin{cases} F_0^d(x,y;\alpha) = f^d(x,y;\alpha) \\ F_{n+1}^d(x,y;\alpha) = \lambda \int_{a_1}^{b_1} \int_{a_2}^{b_2} k(x,y,s,t) F_n^d(s,t;\alpha) dsdt \end{cases} \quad (3.8)$$

4 Convergence analysis

In this section we prove that the ADM is convergent.

Theorem 4.1. *Let $k(x,y,s,t)$ be a continuous function over S and $f(x,y)$ be a fuzzy continuous and bounded function on V. If*

$$|\lambda| < \frac{1}{M(b_1-a_1)(b_2-a_2)}, \quad (4.1)$$

where $M = \max_{(x,y,s,t)\in S} |k(x,y,s,t)|$, then the ADM in section (4), is convergence.

Proof. Without losing generality, let F^c and F^d are the exact solution of (3.3) and (3.4), and we illustrate how F_n^c and F_n^d are convergent to F^c and F^d respectively. From Eq. (3.7), we have

$$\begin{aligned} F_n^c(x,y;\alpha) &= \lambda \int_{a_1}^{b_1} \int_{a_2}^{b_2} k(x,y,s,t) F_{n-1}^c(s,t;\alpha) dsdt \\ &= \lambda \int_{a_1}^{b_1} \int_{a_2}^{b_2} k(x,y,s,t) \left(\lambda \int_{a_1}^{b_1} \int_{a_2}^{b_2} k(x,y,s,t) F_{n-2}^c(s,t;\alpha) dsdt \right) dsdt \\ &\vdots \\ &= \lambda^n \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_1}^{b_1} \int_{a_2}^{b_2} (k(x,y,s,t))^n F_0^c (dsdt)^n \end{aligned}$$

and

$$F_{n-1}^c(x,y;\alpha) = \lambda^{n-1} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_1}^{b_1} \int_{a_2}^{b_2} (k(x,y,s,t))^{n-1} F_0^c (dsdt)^{n-1},$$

therefore

$$\begin{aligned} |F_n^c - F_{n-1}^c| &= |\lambda^n \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_1}^{b_1} \int_{a_2}^{b_2} (k(x,y,s,t))^n F_0^c (dsdt)^n - \lambda^{n-1} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_1}^{b_1} \int_{a_2}^{b_2} (k(x,y,s,t))^{n-1} F_0^c (dsdt)^{n-1}| \\ &\le |\lambda^n \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_1}^{b_1} \int_{a_2}^{b_2} (k(x,y,s,t))^n F_0^c (dsdt)^n| + |\lambda^{n-1} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_1} -b_1 \int_{a_2}^{b_2} (k(x,y,s,t))^{n-1} F_0^c (dsdt)^{n-1}| \\ &\le N(\lambda M(b_1-a_1)(b_2-a_2))^n + N(\lambda M(b_1-a_1)(b_2-a_2))^{n-1} \\ &= N(\lambda M(b_1-a_1)(b_2-a_2))^{n-1} (\lambda M(b_1-a_1)(b_2-a_2)+1) \end{aligned}$$

where

$$N = \sup_{V} |F_0^c(x, y; \alpha)|.$$

So F_n^c is uniformly convergent. Consequently, we have

$$\lim_{n\to\infty} F_n^c = F^c.$$

Clearly, we can show $\lim_{n\to\infty} F_n^d = F^d$. □

5 Numerical examples

In this section, we propose some examples and compare numerical results with other methods.

Example 5.1. *[15]. Consider the following 2D-FFIE-2*

$$\underline{f}(x, y; \alpha) = \pi xy(\frac{13}{15}(\alpha^2 + \alpha) + \frac{2}{15}(4 - \alpha^3 - \alpha)),$$
$$\overline{f}(x, y; \alpha) = \pi xy(\frac{2}{15}(\alpha^2 + \alpha) + \frac{13}{15}(4 - \alpha^3 - \alpha)),$$

and

$$k(x, y, s, t) = -\frac{7}{6}\pi xy \sin(\pi x), \quad 0 \le x, y, s, t \le 1, \quad \lambda = 1.$$

and $a_1 = a_2 = 0, b_1 = b_2 = 1$*, the exact solution is*

$$\underline{F}(x, y; \alpha) = \frac{\pi}{25}xy(\frac{268}{19}\alpha^3 + \frac{568}{19}\alpha^2) + 44\alpha - \frac{1072}{19},$$
$$\overline{F}(x, y; \alpha) = -\frac{\pi}{25}xy(\frac{568}{19}\alpha^3 + \frac{268}{19}\alpha^2) + 44\alpha - \frac{2272}{19}.$$

We can see

$$f^c(x, y; \alpha) = \pi xy(4 - \alpha^3 + \alpha^2),$$
$$f^d(x, y; \alpha) = \pi xy(-\frac{11}{15}(\alpha^2 + \alpha) + \frac{11}{15}(4 - \alpha^3 - \alpha)),$$

and now

$$\begin{cases} F^c(x, y; \alpha) = f^c(x, y; \alpha) + \int_0^1 \int_0^1 (-\frac{7}{6}\pi xy \sin(\pi x))F^c(s, t, \alpha)dsdt, \\ F^d(x, y; \alpha) = f^d(x, y; \alpha) + \int_0^1 \int_0^1 | -\frac{7}{6}\pi xy \sin(\pi x)|F^c(s, t, \alpha)dsdt. \end{cases} \tag{5.1}$$

Now with doing ADM for Eq. (5.1), we have

$$\begin{cases} F_0^c(x, y; \alpha) = f^c(x, y; \alpha), \\ F_{n+1}^c(x, y; \alpha) = \lambda \int_0^1 \int_0^1 (-\frac{7}{6}\pi xy \sin(\pi x))F_n^c(s, t; \alpha)dsdt. \end{cases}$$

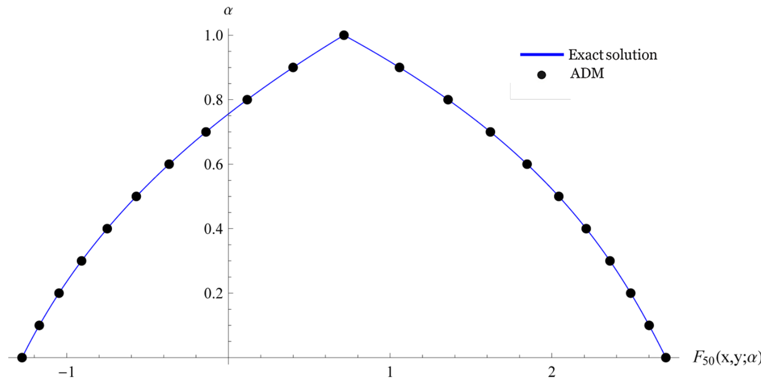

Figure 1: comparison between the exact solution and the approximate solution

$$\begin{cases} F_0^d(x,y;\alpha)=f^d(x,y;\alpha),\\ F_{n+1}^d(x,y;\alpha)=\lambda\int_0^1\int_0^1|-\frac{7}{6}\pi xy\sin(\pi x)|F_n^d(s,t;\alpha)dsdt.\end{cases}$$

and for $n\to\infty$ then

$$\underline{F}(s,t;\alpha)\simeq F_n^c(s,t;\alpha)-F_n^d(s,t;\alpha),$$
$$\overline{F}(s,t;\alpha)\simeq F_n^c(s,t;\alpha)+F_n^d(s,t;\alpha).$$

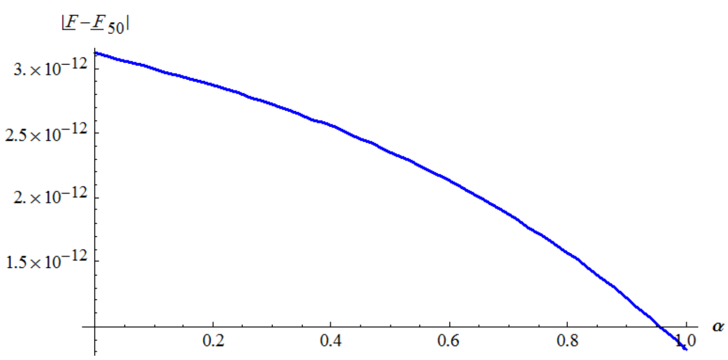

Figure 2: Absolute error $|\underline{F}-\underline{F}_{50}|$

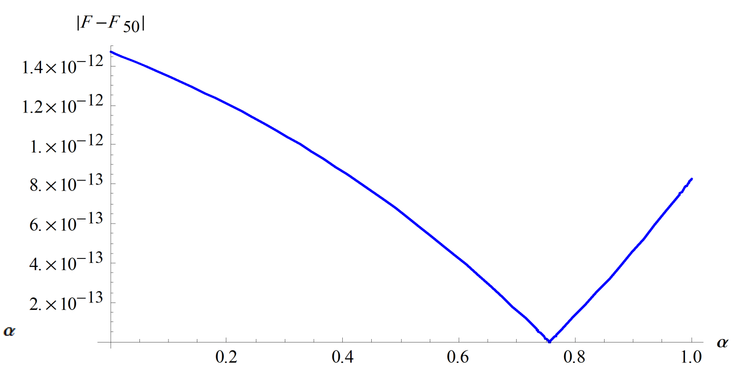

Figure 3: Absolute error $|\overline{F}-\overline{F}_{50}|$

Example 5.2. *[15]. Consider the following 2D-FFIE-2*

$$\underline{f}(x,y;\alpha)=\alpha(xy+\frac{1}{676}(x^2+y^2-2)),$$
$$\overline{f}(x,y;\alpha)=(2-\alpha)(xy+\frac{1}{676}(x^2+y^2-2)),$$

and

$$k(x,y,s,t)=\frac{1}{169}(x^2+y^2-2)(s^2+t^2-2),\quad 0\le x,y,s,t\le 1,\quad \lambda=1.$$

Table 1: Numerical results of Example 5.2

α	Exact solution $\underline{F}(x,y;\alpha),\overline{F}(x,y;\alpha)$	Approximate solution $x=0.3, y=0.6$ and $N=10$	Absolute error for AD method $x=0.3, y=0.6$ and $M=50$	Absolute error the method of Rivaz and Yousefi. [23] for $M=62$
0.0	-1.2762 , 2.7048	-1.2762 , 2.7048	3.11839e-12 , 1.47127e-12	0.0047 , 0.0009
0.1	-1.1696 , 2.6014	-1.1696 , 2.6014	2.99916e-12 , 1.34914e-12	0.0025 , 0.0016
0.2	-1.0476 , 2.4876	-1.0476 , 2.4876	2.86793e-12 , 1.20792e-12	0.0006 , 0.0046
0.3	-0.9082 , 2.3593	-0.9082 , 2.3593	2.71960e-12 , 1.04672e-12	0.0047 , 0.0084
0.4	-0.7495 , 2.2124	-0.7495 , 2.2124	2.55052e-12 , 8.64642e-13	0.0101 , 0.0134
0.5	-0.5697 , 2.0429	-0.5697 , 2.0429	2.35512e-12 , 6.56364e-13	0.0141 , 0.0094
0.6	-0.3667 , 1.8467	-0.3667 , 1.8467	2.12896e-12 , 4.22995e-13	0.0095 , 0.0054
0.7	-0.1388 , 1.6199	-0.1388 , 1.6199	1.86770e-12 , 1.60094e-13	0.0036 , 0.0003
0.8	0.1161 , 1.3582	0.1161 , 1.3582	1.56571e-12 , 1.33893e-13	0.0040 , 0.0081
0.9	0.3998 , 1.0577	0.3998 , 1.0577	1.21941e-12 , 4.61187e-13	0.0133 , 0.0183

and $a_1 = a_2 = 0, b_1 = b_2 = 1$*, the exact solution is*

$$\underline{F}(x,y;\alpha) = \alpha xy,$$
$$\overline{F}(x,y;\alpha) = (2-\alpha)xy.$$

We have

$$f^c(x,y;\alpha) = 2(xy + \frac{1}{676}(x^2+y^2-2),$$
$$f^d(x,y;\alpha) = (2-2\alpha)(xy + \frac{1}{676}(x^2+y^2-2).$$

and

$$\begin{cases} F^c(x,y;\alpha) = f^c(x,y;\alpha) + \int_0^1\int_0^1 \left(\frac{1}{169}(x^2+y^2-2)(s^2+t^2-2)\right) F^c(s,t,\alpha)dsdt, \\ F^d(x,y;\alpha) = f^d(x,y;\alpha) + \int_0^1\int_0^1 \left|\frac{1}{169}(x^2+y^2-2)(s^2+t^2-2)\right| F^c(s,t,\alpha)dsdt. \end{cases} \quad (5.2)$$

Now with doing ADM for Eq. (5.2), we have

$$\begin{cases} F_0^c(x,y;\alpha) = f^c(x,y;\alpha), \\ F_{n+1}^c(x,y;\alpha) = \lambda \int_0^1\int_0^1 \left(\frac{1}{169}(x^2+y^2-2)(s^2+t^2-2)\right) F_n^c(s,t;\alpha)dsdt. \end{cases}$$

$$\begin{cases} F_0^d(x,y;\alpha) = f^d(x,y;\alpha), \\ F_{n+1}^d(x,y;\alpha) = \lambda \int_0^1\int_0^1 \left|\frac{1}{169}(x^2+y^2-2)(s^2+t^2-2)\right| F_n^d(s,t;\alpha)dsdt. \end{cases}$$

and at the end

$$\underline{F}(s,t;\alpha) \simeq F_n^c(s,t;\alpha) - F_n^d(s,t;r),$$
$$\overline{F}(s,t;\alpha) \simeq F_n^c(s,t;\alpha) + F_n^d(s,t;\alpha).$$

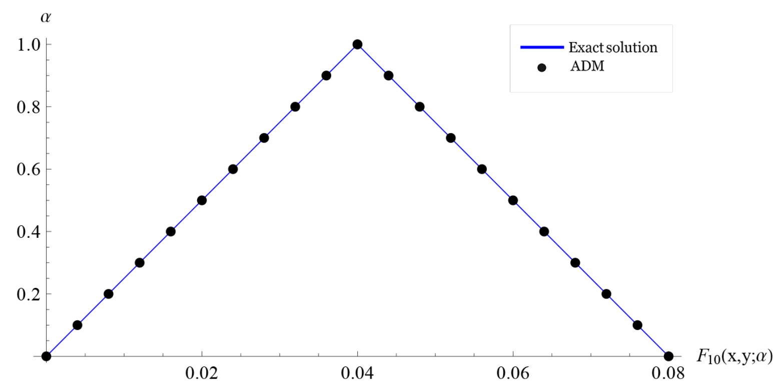

Figure 4: comparison between the exact solution and the approximate solution

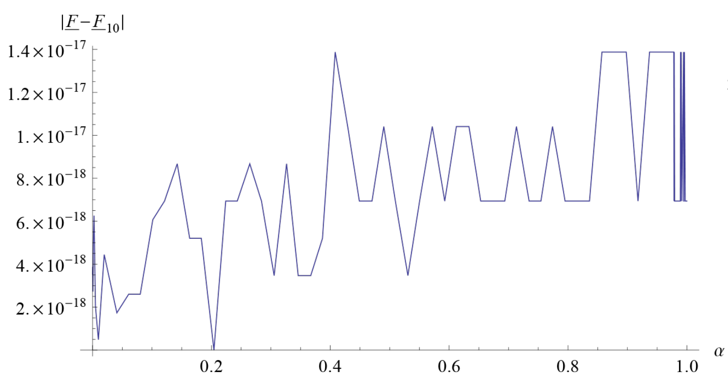

Figure 5: Absolute error $|\underline{F} - \underline{F}_{10}|$

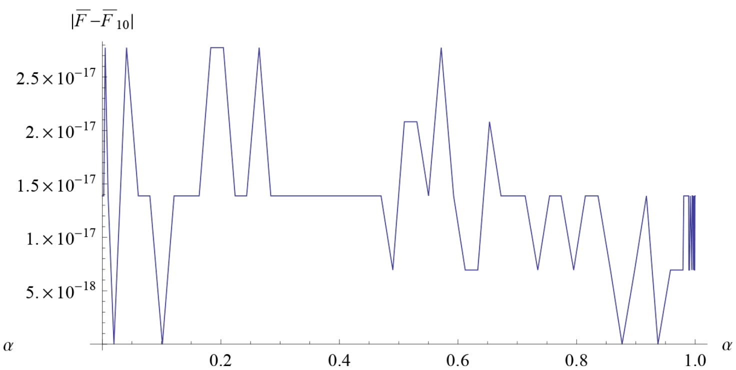

Figure 6: Absolute error $|\overline{F} - \overline{F}_{10}|$

Table 2: Numerical results of Example 5.2

α	Exact solution $\underline{F}(x,y;r),\overline{F}(x,y;r)$	Approximate solution $x=0.1,y=0.4$ and $N=10$	Absolute error for AD method $x=0.1,y=0.4$ and $M=10$	Absolute error the method of Mirzaee et al. [16] for $M=12$
0.0	0.00000 , 0.08000	0.00000 , 0.08000	0.0000000e-00 , 1.3877787e-17	0.0000000e-00 , 9.8553102e-06
0.1	0.00400 , 0.07600	0.00400 , 0.07600	2.6020852e-18 , 1.3877787e-17	4.9276551e-07 , 9.3625447e-06
0.2	0.00800 , 0.07200	0.00800 , 0.07200	1.7347234e-18 , 2.7755575e-17	9.8553102e-07 , 8.8697792e-06
0.3	0.01200 , 0.06800	0.01200 , 0.06800	6.9388939e-18 , 1.3877787e-17	1.4782965e-06 , 8.3770137e-06
0.4	0.01600 , 0.06400	0.01600 , 0.06400	6.9388939e-18 , 0.0000000e-00	1.9710620e-06 , 7.8842481e-06
0.5	0.02000 , 0.06000	0.02000 , 0.06000	3.4694469e-18 , 1.3877787e-17	2.4638275e-06 , 7.3914826e-06
0.6	0.02400 , 0.05600	0.02400 , 0.05600	6.9388939e-18 , 2.0816681e-17	2.9565930e-06 , 6.8987171e-06
0.7	0.02800 , 0.05200	0.02800 , 0.05200	1.0408340e-17 , 2.7755575e-17	3.4493585e-06 , 6.4059516e-06
0.8	0.03200 , 0.04800	0.03200 , 0.04800	6.9388939e-18 , 1.3877787e-17	3.9421240e-06 , 5.9131861e-06
0.9	0.03600 , 0.04400	0.03600 , 0.04400	1.3877787e-17 , 0.0000000e-00	4.4348896e-06 , 5.4204206e-06

6 Conclusion

In this study, we used Adomian decomposition method and new representation of the parametric form of fuzzy numbers for solving 2D-FFIE-2. By using this method a two-dimensional fuzzy Fredholm integral equation leads to two crisp. Efficiency and accuracy of the above method by providing a few examples and comparing results with other methods and also exact solution were analyzed. As we can see error in the introduced method has been decreased.

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