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On Best Proximity Points in Metric and Banach Spaces

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Abstract

Notice that best proximity point results have been studied to find necessary conditions such that the minimization problem $\min_{x\in A\bigcup B} d(x,Tx)$ has at least one solution, where T is a cyclic mapping defined on $A\bigcup B$. A point $p\in A\bigcup B$ is a best proximity point for T if and only if that is a solution of the minimization problem (2.1). Let (A,B) be a nonempty pair in a normed linear space X and $S,T : A\bigcup B\to A\bigcup B$ be two cyclic mappings. Let (A,B) be a nonempty pair in a normed linear space X and $S,T : A\bigcup B\to A\bigcup B$ be two cyclic mappings. A point $p\in A\bigcup B$ is called a common best proximity point for the cyclic pair (T,S) provided that $\|p-Tp\| = d(A,B) = \|p-Sp\|$ In this paper, we survey the existence, uniqueness and convergence of a common best proximity point for a cyclic weak $ST-\phi$-contraction map, which is equivalent to study of a solution for a nonlinear programming problem in the setting of uniformly convex Banach spaces. Moreover, we provide some examples to illustrate and support the results.

1 Introduction

Existence and convergence of best proximity points are interesting topics in optimization theory. The first result in this area was introduced in 2003, by Kirk [8]. Later, investigation in this area was continued by many researchers and obtained many result([1], [2], [5], [6], [7], [11], [13], [14]). In 2006, Eldered and Veeramani [3] continued investigation about best proximity points of cyclic contraction maps and proved some results in this field.

In 1969, Boyd and Wong [6] gave the definition of $\phi-$contraction:

A self mapping T on a metric space X is called $\phi-$contraction if there exists an upper semi-continuous function $\phi:[0,\infty)\to[0,\infty)$ such that

$$d(Tx,Ty)\leq\phi(d(x,y))$$

for all $x,y\in X$.

Later, in 1997, Alber and Guerre-Delabriere [4], introduced the definition of weak $\phi-$contraction:

A self-mapping T on a metric space X is called weak $\phi-$contraction if for each $x,y\in X$, there exists a function $\phi:[0,\infty)\to[0,\infty)$ such that

$$d(Tx,Ty)\leq d(x,y)-\phi(d(x,y)).$$

In 2009, Al-Thagafi and Shahzad [1] introduced cyclic $\phi-$contraction maps in metric spaces. In 2012, Karapinar [7] introduced generalized cyclic contractions and obtained existence of best proximity points for this maps.

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Inspired by these results, we introduce a weak ϕ-contraction map of first kind and cyclic weak ϕ-contraction map of second kind in metric spaces and prove existence and convergence theorems of best proximity points in metric spaces.

The main purpose of this paper is to discuss about existence of best proximity points for cyclic weak $ST-\phi-$contraction map in Banach space H. This section reviews basic definitions, facts, and notation from Banach spaces that will be used throughout the paper.

Definition 1.1. *([1]) Let A and B be non-empty closed subsets of a metric space (X, d) and $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. A map $T : A \bigcup B \to A \bigcup B$ is called a cyclic weak $\phi-$contraction if*

$$T(A) \subset B, T(B) \subset A \; and \; d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) + \phi(d(A, B)), \tag{1.1}$$

for all $x \in A$ and $y \in B$, where $d(A, B) = inf\{d(a, b) : a \in A, b \in B\}$.

Definition 1.2. *([7]) Let A and B be non-empty subsets of a metric space (X, d) and $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. A map $T : A \bigcup B \to A \bigcup B$ is said to be Kannan type cyclic weak $\phi-$contraction if*

$$T(A) \subset B, T(B) \subset A \; and \; d(Tx, Ty) \leq u(x, y) - \phi(u(x, y)) + \phi(d(A, B))$$

for all $x \in A$ and $y \in B$, in which $u(x, y) = 1/2[d(x, Tx) + d(y, Ty)]$.

Definition 1.3. *([1]) Let A and B be nonempty subsets of a normed linear space X, $T : A \bigcup B \to A \bigcup B$, $T(A) \subset B$ and $T(B) \subset A$. We say that T satisfies the proximal property*
if $x_n \overset{w}{\to} x \in A \bigcup B$,$\|x_n - Tx_n\| \to d(A, B)$ then $\|x - Tx\| = d(A, B)$.
for $\{x_n\}_{n\geq 0} \in A \bigcup B$.

Definition 1.4. *A Banach space X is said to be (a) uniformly convex if there exists a strictly increasing function $\delta : (0, 2] \to [0, 1]$ such that the following implication holds for all $x_1, x_2, p \in X, R > 0$ and $r \in [0, 2R]$:*
$\|x_i - p\| \leq R, i = 1, 2$ and $x_1 - x_2 \geq r \to \|(x_1 + x_2)/2 - p\| \leq (1 - \delta(r/R))R$
(b) strictly convex if the following impilication holds for all $x_1, x_2, p \in X, R > 0$
$\|x_i - p\| \leq R, i = 1, 2$ and $x_1 \neq x_2 \to \|(x_1 + x_2)/2 - p\| \leq R$.

Definition 1.5. *[16] A sequence $\{x_n\}$ in a Banach space X is weakly Cauchy if $lim_{n\to\infty} x^*(x_n)$ exists for every $x^* \in X^*$.*

Definition 1.6. *[16] A Banach space X is said to be weakly sequentially complete (wsc) if every weakly Cauchy sequence in X convergence weakly.*

Theorem 1.1. *[16] A reflexive Banach space X is weakly sequentially complete (wsc).*

Theorem 1.2. *[16](Rosenthals l_1 Theorem). Let $\{x_n\}$ be a bounded sequence in an infinite-dimensional Banach space X. Then either:*
(a) $\{x_n\}$ has a subsequence which is weakly Cauchy, or
(b) $\{x_n\}$ has a subsequence which is basic and equivalent to the canonical basic of l_1.

2 Main Results

Notice that best proximity point results have been studied to find necessary conditions such that the minimization problem

$$min_{x\in A\bigcup B}d(x,Tx) \qquad (2.1)$$

has at least one solution, where T is a cyclic mapping defined on $A\bigcup B$.
A point $p\in A\bigcup B$ is a best proximity point for T if and only if that is a solution of the minimization problem (2.1).
Let (A,B) be a nonempty pair in a normed linear space X and $S,T : A\bigcup B\to A\bigcup B$ be two cyclic mappings. A point $p\in A\bigcup B$ is called a common best proximity point for the cyclic pair (T,S) provided that

$$\|p-Tp\| = d(A,B) = \|p-Sp\|$$

In view of the fact that

$$min\{\|x-Tx\|,\|x-Sx\|\} \ \ \geq \ \ d(A,B), x\in A\bigcup B \qquad (2.2)$$

the optimal solution to the problem of

$$min_{x\in A\bigcup B}\{\|x-Tx\|,\|x-Sx\|\}$$

will be the one for which the value $d(A,B)$ is attained. There by, a point $p\in A\bigcup B$ is a common best proximity point for the cyclic pair (T,S) if and only if that is a solution of the minimization problem (2.2). In this section, we provide some sufficient conditions in order to study the existence of a solution for (2.1) and (2.2). We begin with the following definition.

Definition 2.1. *Let A and B be non-empty subsets of a metric space (X,d) and $\phi:[0,\infty)\to[0,\infty)$ be a strictly increasing map and $S,T : A\bigcup B\to A\bigcup B$ be two cyclic mappings. Cyclic pair (S,T) is said to be cyclic weak $ST-\phi$-contraction map if $S(A)\subset T(A)\subset B$ and $S(B)\subset T(B)\subset A$ and*
$d(Sx,Sy)\leq d(Tx,Ty)-\phi(d(Tx,Ty))+\phi(d(A,B))$ *(2.3) for all $(x,y)\in A\times B$.*

Example 2.1. *Suppose $X=l^2$ and let $A=\{te_1+e_2 : 0\leq t\leq 1/4\}$ and $A=\{se_3+e_2 : 0\leq s\leq 1/4\}$. Define the cyclic pair. (S,T) as below*
$S(te_1+e_2)=e_2+t^2e_3$ *and* $S(e_2+se_3)=s^2e_1+e_2, t,s\in[0,1/4]$.

$T(te_1+e_2)=e_2+te_3$ *and* $T(e_2+se_3)=se_1+e_2, t,s\in[0,1/4]$.
It is clear that $S(A)\subset T(A)=B$ and $S(B)\subset T(B)=A$.
Also,
$TS(te_1+e_2)=T(e_2+t^2e_3)=t^2e_1+e_2=S(e_2+te_3)=ST(te_1+e_2)$,
$TS(e_2+se_3)=T(s^2e_1+e_2)=e_2+s^2e_3=T(s^2e_1+e_2)=TS(e_2+se_3)$,

that is, T and S are commuting. Now define the function $\phi\Box$ with

$$\phi(r) = 1/4(r), if 0 \leq r < 1, \phi(r) = r/(r+1), if 1 \leq r$$

For $x := te_1 + e_2 \in A$ and $y := se_3 + e_2 \in B$ we have

$$\|Sx - Sy\| = \sqrt{s^4 + t^4} \leq \sqrt{s^2 + t^2} - 1/4(\sqrt{s}^2 + t^2) = \|Tx - Ty\| - \phi(\|Tx - Ty\|)$$

Therefore, Cyclic pair (S,T) is cyclic weak $ST - \phi$-contraction map.

Choose $x_0 \in A$. Since $S(A) \subset T(A)$, there exists $x_1 \in A$ such that $Sx_0 = Tx_1$. Again, by the fact that $S(A) \subset T(A)$, there exists $x_2 \in A$ such that $Sx_1 = Tx_2$. Continuing this process, we can find a sequence $\{x_n\}$ in A such that

$$Sx_n = Tx_{n+1} \qquad (2.4)$$

Remark 2.1. *Since $d(A,B) \leq d(Tx,Ty)$ for all $(x,y) \in A \times B$, we have $d(Sx,Sy) \leq d(Tx,Ty)$.*

Theorem 2.1. *Let (X,d) be an metric space, A and B be non-empty subsets of X. Suppose Cyclic pair (S,T) is a cyclic weak $ST - \phi$-contraction map, that is, T,S satisfies (2.3). Also T and S commute and T,S satisfies(2.4). Define $d_n := d(Tx_n, TTx_{n+1}) = d(Sx_{n-1}, SSx_{n-1})$ for all $n \in N$. Then $d_n \to d(A,B)$.*

Proof. by Remark 1.1, we have

$$\begin{aligned} d(Sx_n, SSx_n) &\leq d(Tx_n, TSx_n) \\ &= d(Sx_{n-1}, STx_n) \\ &= d(Sx_{n-1}, SSx_{n-1}) \end{aligned}$$

hence

$$d_{n+1} \leq d_n$$

Hence the sequence d_n is non-increasing and bounded below. So $lim_{n\to\infty} d_n = t_0$. Since Cyclic pair (S,T) is a cyclic weak $ST - \phi$-contraction map, we obtain

$$\begin{aligned} d(Sx_n, SSx_n) &\leq d(Tx_n, TSx_n) - \phi(d(Tx_n, TSx_n)) \\ &+ \phi(d(A,B)) \end{aligned}$$

Therefore

$$\begin{aligned}\phi(d(A,B)) &\leq \phi(d(Tx_n, TSx_n)) \\ &\leq d_n - d_{n+1} + \phi(d(A,B))\end{aligned}$$

Thus

$$lim_{n\to\infty} d(Tx_n, TSx_n) = d(A,B)$$

Since

$$d(Tx_n, TSx_n) \geq d_{n+1} \geq t_0 \geq d(A,B)$$

we have

$$t_0 = d(A,B)$$

□

Theorem 2.2. *Let* $\phi : [0,\infty) \to [0,\infty)$ *be a strictly increasing unbounded map. Also, let A and B be nonempty subsets of a metric space* (X,d)*, Cyclic pair* (S,T) *is a cyclic weak* $ST-\phi$*-contraction map,* $x_0 \in A$ *and* T,S *satisfies(2.4) and T and S commute. Then, the sequences* $\{Sx_{n-1}\} = \{Tx_n\}$ *and* $\{SSx_{n-1}\} = \{TTx_{n+1}\}$ *are bounded.*

Proof. By [Theorem 2.1], $d(Sx_{n-1}, SSx_{n-1}) \to d(A,B)$. Hence, it is sufficient to prove that $\{SSx_{n-1}\}$ is bounded. If not, then for each $M > 0$ there exists natural number N such that $d(Sx_2, SSx_{N+1}) > M$ and $d(Sx_2, SSx_{N-1}) \leq M$
by Remark 1.1 we have

$$\begin{aligned}M &< d(Sx_2, SSx_{N+1}) \\ &\leq d(Tx_2, TSx_{N+1}) - \phi(d(Tx_2, TSx_{N+1})) + \phi(d(A,B)) \\ &= d(Sx_1, STx_{N+1}) - \phi(d(Tx_2, TSx_{N+1})) + \phi(d(A,B)) \\ &= d(Sx_1, SSx_N) - \phi(d(Sx_1, SSx_N)) + \phi(d(A,B)) \\ &\leq d(Tx_1, TSx_N) - \phi(d(Tx_1, TSx_N)) + \phi(d(A,B)) \\ &= d(Sx_0, SSx_{N-1}) - \phi(d(Sx_0, SSx_{N-1})) + \phi(d(A,B)) \\ &\leq d(Sx_0, Sx_2) + d(Sx_2, SSx_{N-1}) - \phi(d(Sx_0, SSx_{N-1})) + \phi(d(A,B)) \\ &\leq d(Sx_0, Sx_2) + M - \phi(d(Sx_0, SSx_{N-1})) + \phi(d(A,B))\end{aligned}$$

Therefore

$$\phi(d(Sx_0, SSx_{N-1})) \quad \leq \quad d(Sx_0, Sx_2) + \phi(d(A, B))$$

Since, ϕ is unbounded function, we can choose M such that

$$\phi(M) > d(Sx_0, Sx_2) + \phi(d(A, B))$$

Now

$$\begin{aligned} \phi(M) &< \phi(d(Sx_2, SSx_{N+1})) \leq \phi(d(Tx_2, TSx_{N+1})) \\ &= \phi(d(Sx_1, SSx_N)) \\ &\leq \phi(d(Sx_0, SSx_{N-1})) \\ &\leq d(Sx_0, Sx_2) + \phi(d(A, B)) \end{aligned}$$

a contradiction. □

Theorem 2.3. *let A and B be nonempty weakly closed subsets of a reflexive Banach space and* $A \bigcap B = \emptyset$ *and cyclic pair* (S, T) *is a cyclic weak* $ST - \phi$*-contraction map,* $x_0 \in A$ *and* T, S *satisfies(2.4) and T and S commute. Then there exists* $(q, p) \in A \times B$ *such that*

$$\|p - q\| = d(A, B).$$

Proof. Assume that $d(A, B) > 0$. For $x_0 \in A$, define $Sx_n = Tx_{n+1}$ for all $n \geq 1$. By Theorem 2.2, the sequences $\{Sx_{n-1}\}$ and $\{SSx_{n-1}\}$ are bounded. Since X is reflexive and B is weakly closed, the sequence $\{Sx_{n-1}\}$ has a subsequence $\{Sx_{n_k-1}\}$ such that $Sx_{n_k-1} \stackrel{w}{\to} p \in B$ as $k \to \infty$. Also A is weakly closed, hence the sequence $\{SSx_{n-1}\}$ has a subsequence $\{SSx_{n_k-1}\}$ such that $SSx_{n_k-1} \stackrel{w}{\to} q \in A$ as $k \to \infty$. Since $Sx_{n_k-1} - SSx_{n_k-1} \stackrel{w}{\to} p - q \neq 0$ as $k \to \infty$, there exists a bounded linear functional $f : X \to [0, \infty)$ such that $\|f\| = 1$ and $f(p - q) = \|p - q\|$.
For all $k \geq 1$, we have

$$|f(Sx_{n_k-1} - SSx_{n_k-1})| \quad \leq \quad \|f\| \, \|Sx_{n_k-1} - SSx_{n_k-1}\| = \|Sx_{n_k-1} - SSx_{n_k-1}\| .$$

Since $f(Sx_{n_k-1} - SSx_{n_k-1}) \to \|p - q\|$ as $k \to \infty$. by using Theorem 2.1, we get

$$\begin{aligned} \|p-q\| &= lim_{k\to\infty} f(Sx_{n_k-1} - SSx_{n_k-1}) \\ &\leq lim_{k\to\infty} \|Sx_{n_k-1} - SSx_{n_k-1}\| \\ &= d(A,B). \end{aligned}$$

So $\|p-q\| = d(A,B)$. □

Corollary 2.1. *let A and B be nonempty subsets of a reflexive infinite-dimensional Banach space X contains no copy of* l_1 *and* $A \bigcap B = \emptyset$ *and cyclic pair* (S,T) *is a cyclic weak* $ST-\phi$*-contraction map,* $x_0 \in A$ *and* T,S *satisfies(2.4) and T and S commute. Then there exists* $(q,p) \in A \times B$ *such that*

$$\|p-q\| = d(A,B).$$

Proof. Since the sequences $\{Sx_{n-1}\}$ and $\{SSx_{n-1}\}$ are bounded, there exists weakly Cauchy subsequence $\{Sx_{n_k-1}\}$ and $\{SSx_{n-1}\}$ by Theorem 1.2. Since X is reflexive therefore $\{Sx_{n_k-1}\}$ is weakly convergence to $p \in B$ and $\{SSx_{n_k-1}\}$ is weakly convergence to $q \in A$. Now the proof continues similar to that of Theorem 2.3. □

Theorem 2.4. *Let A and B be nonempty weakly closed subsets of a reflexive Banach space X and* $A \bigcap B = \emptyset$ *and cyclic pair* (S,T) *is a cyclic weak* $ST-\phi$*-contraction map. Then there exists* $p \in B$ *such that*

$$\|p-Sp\| = d(A,B).$$

provided that one of the following conditions is satisfied
(a) S is weakly continuous on B.
(b) S satisfy the proximal property.

Proof. Assume that $d(A,B) > 0$. By Theorem 2.2, the sequences $\{Sx_{n-1}\}$ and $\{SSx_{n-1}\}$ are bounded. Since X is reflexive and B is weakly closed, the sequence $\{Sx_{n-1}\}$ has a subsequence $\{Sx_{n_k-1}\}$ such that $Sx_{n_k-1} \stackrel{w}{\to} p \in B$ as $k \to \infty$. From (a),$SSx_{n_k-1} \stackrel{w}{\to} Sp \in A$ as $k \to \infty$. So $Sx_{n_k-1} - SSx_{n_k-1} \stackrel{w}{\to} p - Sp \neq 0$ as $k \to \infty$. Now the proof continues similar to that of Theorem 2.3. From (b) and Theorem 2.1, $\|Sx_{n_k-1} - SSx_{n_k-1}\| \to d(A,B)$ as $k \to \infty$. Thus $\|p-Sp\| = d(A,B)$. □

Corollary 2.2. *Let A and B be nonempty subsets of a reflexive infinite-dimensional Banach space X contains no copy of* l_1 *and* $A \bigcap B = \emptyset$ *and cyclic pair* (S,T) *is a cyclic weak* $ST-\phi$*-contraction map. Then there exists* $p \in B$ *such that*

$$\|p-Sp\| = d(A,B).$$

provided that one of the following conditions is satisfied
(a) S is weakly continuous on A.
(b) S satisfy the proximal property.

Proof. the proof of theorem follows by similar proof of Corollary 2.2 and Theorem 2.4. □

Theorem 2.5. *Let A and B be nonempty weakly closed subsets of a reflexive Banach space X and $A \bigcap B = \emptyset$ and cyclic pair (S, T) is a cyclic weak $ST - \phi$-contraction map. Then there exists $p \in B$ such that*

$$\|p - Sp\| = d(A, B) = \|p - Tp\| .$$

provided that one of the following conditions is satisfied
(a) S,T is weakly continuous on B.
(b) S,T satisfy the proximal property.

Proof. Assume that $d(A, B) > 0$. By Theorem 2.2, the sequences $\{Sx_{n-1}\} = \{Tx_n\}$ and $\{SSx_{n-1}\} = \{TTx_{n+1}\}$ are bounded. Since X is reflexive and B is weakly closed, the sequence $\{Sx_{n-1}\} = \{Tx_n\}$ has a subsequence $\{y_{n_k-1}\}$ such that $y_{n_k-1} \stackrel{w}{\to} p \in B$ as $k \to \infty$. From (a),$Sy_{n_k-1} \stackrel{w}{\to} Sp \in A$ as $k \to \infty$ and $Ty_{n_k-1} \stackrel{w}{\to} Tp \in A$ as $k \to \infty$. So $Sx_{n_k-1} - Sy_{n_k-1} \stackrel{w}{\to} p - Sp \neq 0$ as $k \to \infty$ and $Tx_{n_k} - Ty_{n_k-1} \stackrel{w}{\to} p - Tp \neq 0$ as $k \to \infty$. Now the proof continues similar to that of Theorem 2.3. From (b) and Theorem 2.1, $\|y_{n_k-1} - Sy_{n_k-1}\| \to d(A, B)$ as $k \to \infty$ and $\|y_{n_k-1} - Ty_{n_k-1}\| \to d(A, B)$ as $k \to \infty$. Thus $\|p - Sp\| = d(A, B) = \|p - Tp\|$. □

Corollary 2.3. *Let A and B be nonempty subsets of a reflexive infinite-dimensional Banach space X contains no copy of l_1 and $A \bigcap B = \emptyset$ and cyclic pair (S, T) is a cyclic weak $ST - \phi$-contraction map. Then there exists $p \in B$ such that*

$$\|p - Sp\| = d(A, B) = \|p - Tp\| .$$

provided that one of the following conditions is satisfied
(a) S,T is weakly continuous on A.
(b) S,T satisfy the proximal property.

Proof. the proof of theorem follows by similar proof of Corollary 2.2 and Theorem 2.5. □

Definition 2.2. *Let A and B be nonempty subsets of a normed linear space X, $T : A \bigcup B \to A \bigcup B$, $T(A) \subset B$ and $T(B) \subset A$. We say that T satisfies the $w-$proximal property if $\{x_n\}_{n\geq 0} \in A \bigcup B$ be weakly Cauchy sequence and $\|x_n - Tx_n\| \to d(A, B)$ then there exist $x \in A \bigcup B$ such that $x_n \stackrel{w}{\to} x$ and $\|x - Tx\| = d(A, B)$.*

Remark 2.2. *Let X be Banach space and A and B be nonempty subsets of a normed linear space X, $T : A \bigcup B \to A \bigcup B$, $T(A) \subset B$ and $T(B) \subset A$. if T satisfies the $w-$proximal property then T also satisfies proximal property.*

Remark 2.3. *Let X be reflexive Banach space and A and B be nonempty subsets of a normed linear space X, $T : A \bigcup B \to A \bigcup B$, $T(A) \subset B$ and $T(B) \subset A$ and $A \bigcap B = \emptyset$. T satisfies the proximal property equivalent to T satisfies $w-$proximal property.*

Corollary 2.4. *Let A and B be nonempty weakly closed subsets of a reflexive infinite-dimensional Banach space X contains no copy of l_1 and cyclic pair (S, T) is a cyclic weak $ST - \phi$-contraction map. Then there exists $p \in B$*

such that

$$\|p - Sp\| = d(A, B).$$

provided that one of the following conditions is satisfied
(a) S is weakly continuous on A.
(b) S satisfy the $w-$proximal property.

Proof. Assume that $d(A, B) > 0$. By Theorem 2.2, the sequences $\{Sx_{n-1}\}$ and $\{SSx_{n-1}\}$ are bounded. Since the sequences $\{Sx_{n-1}\}$ and $\{SSx_{n-1}\}$ are bounded, there exists weakly Cauchy subsequence $\{Sx_{n_k-1}\}$ and $\{SSx_{n_k-1}\}$ by Theorem 1.2. Since X is reflexive therefore $\{Sx_{n_k-1}\}$ is weakly convergence to $p \in B$ by Theorem 1.1. From (a),$SSx_{n_k-1} \xrightarrow{w} Tp \in A$ as $k \to \infty$. So $Sx_{n_k-1} - SSx_{n_k-1} \xrightarrow{w} p - Tp \neq 0$ as $k \to \infty$. Now the proof continues similar to that of Theorem 2.3. From (b), by Theorem 2.1, $\|Sx_{n_k-1} - SSx_{n_k-1}\| \to d(A, B)$ as $k \to \infty$. Thus $\|p - Tp\| = d(A, B)$. □

Theorem 2.6. *Let A and B be nonempty closed and convex subsets of a reflexive and strictly convex Banach space X and $A \bigcap B = \emptyset$ and cyclic pair (S, T) is a cyclic weak $ST - \phi$-contraction map. Then there exist unique $p \in B$ such that*

$$\|p - Tp\| = d(A, B).$$

provided that one of the following conditions is satisfied
(a) S is weakly continuous on A.
(b) S satisfy the proximal property.

Proof. Assume that $d(A, B) > 0$. Since A and B are closed and convex, they are weakly closed. By Theorem 2.4, there exists $p \in B$ such that

$$\|p - Tp\| = d(A, B).$$

For the uniqueness of x, suppose that there exists $a \in B$ such that $\|a - Ta\| = d(A, B)$. By the strict convexity of X, and convexity of A and B, we have

$$\|(p + a)/2 - (Tp + Ta)/2\| = \|(p - Tp)/2 + (a - Ta)/2\| < d(A, B),$$

which is a contraction. □

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