Theory of Approximation and Applications

Vol. 12, No.2, (2018), 29-42

λ-Symmetry Method and the Prelle-Singer Method for Third-Order Differential Equations

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Received 3 March 2018; accepted 13 December 2018

Abstract

In this paper, we will obtain first integral, integrating factor, λ-symmetry of third-order ODEs $\dddot{u} = F(x, u, \dot{u}, \ddot{u})$. Also comparing Prelle-Singer (PS) method and λ-symmetry for third-order differential equations.

Key words: Symmetry, λ-Symmetry, Integrating factor, First integral, Order reduction.
2010 AMS Mathematics Subject Classification : 33E30

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1 Introduction

There are many examples of ODEs that have trivial Lie symmetries. In 2001, Muriel and Romero introduced λ-symmetry to find general solutions for such examples. Recently, they [8] presented techniques to obtain first integral, integrating factor, λ-symmetry of second-order ODEs $\ddot{u} = F(x, u, \dot{u})$ and the relationship between them.

In addition, the study of a λ-symmetry method of the ODEs permit us the determination of an integrating factor and reduce the order of the ODEs and explain the reduction process of many ODEs that lack Lie symmetries.

In this paper, first we will recall some of the foundational results about symmetry and λ-symmetry rather briefly. we present some theorems about an integrating factor, first integral and reduce the order of the ODEs. second, we will calculate an integrating factor, first integral and reduce of third-order ODEs $\dddot{u} = F(x, u, \dot{u}, \ddot{u})$, through λ-symmetry method. Finally, we comparing Prelle-Singer (PS) method and λ-symmetry method for third-order differential equations.

2 Symmetries and λ-Symmetries on ODEs

In this section we recall some of the foundational results about symmetry and λ-symmetry rather briefly [5,11].
Let $\mathbf{v}$ be a vector field defined on an open subset $M \subset X \times U$. We denote by $M^{(n)}$ the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$, for $n \in N$. their elements are $(x, u^{(n))}) = (x, u, u_1, ..., u_n)$, where, for $i = 1, 2, ..., n$, u_i denotes the derivative of order i of u with respect to x. Suppose

$$\Delta(x, u^{(n)}) = 0, \tag{2.1}$$

be an ODE defined over the total space M. The latter characterizes a Lie symmetry of an ODE as a vector filed $\mathbf{v} = \xi(x, u)\partial/\partial x + \eta(x, u)\partial/\partial u$, that satisfies

$$\mathbf{v}^{(n)}\Big[\Delta(x,u^{(n)})\Big]=0, \qquad \text{if} \qquad \Delta(x,u^{(n)})=0,$$

where $\mathbf{v}^{(n)}$ that called $n-th$ prolongation of $\mathbf{v}$ is

$$\mathbf{v}^{(n)}=\xi(x,u)\frac{\partial}{\partial x}+\eta(x,u)\frac{\partial}{\partial u}+\sum_{i=1}^{n}\eta^{(i)}(x,u^{(i)})\frac{\partial}{\partial u_i},$$

where $\eta^{(i)}(x,u^{(i)})=D_x\Big(\eta^{(i-1)}(x,u^{(i-1)})\Big)-D_x(\xi(x,u))u_i$, and $\eta^{(0)}(x,u)=\eta(x,u)$, for $i=1,...,n$, where D_x denote the total derivative operator with respect to x [11].

If an ODE does not have Lie point symmetry, then we using λ-symmetry method for reduce of order the ODE. λ-symmetry method is as follows[5]:

For every function $\lambda\in C^\infty(M^{(1)})$, we will define a new prolongation and Lie symmetry of $\mathbf{v}$ in the following way:

Let $\mathbf{v}=\xi(x,u)\partial/\partial x+\eta(x,u)\partial/\partial u$, be a vector field defined on M, and let $\lambda\in C^\infty(M^{(1)})$ be an arbitrary function. The λ-prolongation of order n of $\mathbf{v}$, denoted by $\mathbf{v}^{[\lambda,(n)]}$, is the vector field defined on M by

$$\begin{aligned}\mathbf{v}^{[\lambda,(n)]} &= \xi(x,u)\tfrac{\partial}{\partial x}+\eta(x,u)\tfrac{\partial}{\partial u}\\ &+\textstyle\sum_{i=1}^{n}\eta^{(i)}(x,u^{(i)})\tfrac{\partial}{\partial u_i}\end{aligned}$$

where $\eta^{[\lambda,(i)]}(x,u^{(i)})=(D_x+\lambda)(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)}))-\Big((D_x\lambda)(\xi(x,u))\Big)u_i$ and $\eta^{[\lambda,(0)]}(x,u)=\eta(x,u)$, for $i=1,2,3,...,n$.
A vector field $\mathbf{v}$ is a λ-symmetry of the Eq.(2.1), if there exists function $\lambda\in C^\infty(M^{(1)})$, such that

$$\mathbf{v}^{[\lambda,(n)]}\Big[\Delta(x,u^{(n)})\Big]=0, \qquad \text{if} \qquad \Delta(x,u^{(n)})=0.$$

Note. Suppose vector field $v=\partial/\partial u$ be a λ-symmetry of the Eq.(2.1), then

$$v^{[\lambda,(n-1)]} = \frac{\partial}{\partial u} + (D_x+\lambda)(1)\frac{\partial}{\partial u_1} + (D_x+\lambda)(D_x+\lambda)(1)\frac{\partial}{\partial u_2} + ...$$
$$+(D_x+\lambda)(D_x+\lambda)...(D_x+\lambda)(1)\frac{\partial}{\partial u_{n-1}},$$

or equivalent

$$v^{[\lambda,(n-1)]} = \sum_{i=0}^{n-1}(D_x+\lambda)^i(1)\frac{\partial}{\partial u_i}. \tag{2.2}$$

An integrating factor of the Eq.(2.1), is a function $\mu(x, u^{(n-1)})$ such that the equation $\mu.\Delta = 0$ is an exact equation:

$$\mu(x, u^{(n-1)}).\Delta(x, u^{(n)}) = D_x(G(x, u^{(n-1)})).$$

Function $G(x, u^{(n-1)})$, will be called a first integral of the Eq.(2.1), and

$$D_x(G(x, u^{(n-1)})) = 0,$$

is a conserved form of the Eq.(2.1) [7].

Let

$$u_n = F(x, u^{(n-1)}), \tag{2.3}$$

be a nth-order ordinary differential equation, where F is an analytic function of its arguments. We denote by $A = \partial_x + u_1\partial_u + u_2\partial_{u^{(1)}} + ... + F(x, u^{(n-1)})\partial_{u^{(n-1)}}$ the vector field associated with Eq.(2.3) [5].
Function $I(x, u^{(n-1)})$ is a first integral [8] of Eq.(2.3), such that $A(I) = 0$ and an integrating factor of Eq.(2.3), is any function $\mu(x, u^{(n-1)})$ such that

$$\mu(x, u^{(n-1)})(u^{(n)} - F(x, u^{(n-1)})) = D_x I(x, u^{(n-1)}).$$

By using (2.2), It can be checked that the vector field $v = \partial_u$ is a λ-symmetry of Eq.(2.3), if the function $\lambda(x, u^{(k)})$, for some $k < n$, is any particular solution of the equation

$$(D_x+\lambda)^n(1) = \sum_{i=0}^{n-1}(D_x+\lambda)^i(1)\frac{\partial F}{\partial u_i}. \tag{2.4}$$

3 First integral, Integrating factor and λ-symmetry for third-order ODEs

For $n = 3$, the corresponding third-order ODEs can be written in explicit form as particular

$$\dddot{u} = F(x, u, \dot{u}, \ddot{u}). \tag{3.1}$$

We denote by $A = \partial_x + \dot{u}\partial_u + \ddot{u}\partial_{\dot{u}} + F(x, u, \dot{u}, \ddot{u})\partial_{\ddot{u}}$ the vector field associated with Eq.(3.1). Function $I(x, u, \dot{u}, \ddot{u})$ is a first integral of Eq.(3.1) such that $A(I) = 0$ and an integrating factor of Eq.(3.1) is any function $\mu(x, u, \dot{u}, \ddot{u})$ such that $\mu(\dddot{u} - F(x, u, \dot{u}, \ddot{u})) = D_x(I)$. By using (2.4), if $\lambda(x, u, \dot{u}, \ddot{u})$ be any particular solution of

$$D_x^2\lambda + D_x\lambda^2 + \lambda D_x\lambda + \lambda^3 = F_u + \lambda F_{\dot{u}} + (D_x\lambda + \lambda^2)F_{\ddot{u}},$$

then the vector field $v = \partial_u$ is a λ-symmetry of Eq. (3.1).

Theorem 3.1 *If $I(x, u, \dot{u}, \ddot{u})$ is a first integral of Eq. (3.1), then $\mu(x, u, \dot{u}, \ddot{u}) = I_{\ddot{u}}(x, u, \dot{u}, \ddot{u})$ is a integrating factor of (3.1).*

Proof. If $I(x, u, \dot{u}, \ddot{u})$ be a first integral of Eq. (3.1), then

$$0 = A(I) = I_x + \dot{u}I_u + \ddot{u}I_{\dot{u}} + F(x, u, \dot{u}, \ddot{u})I_{\ddot{u}},$$

therefore

$$I_x + \dot{u}I_u + \ddot{u}I_{\dot{u}} = -F(x, u, \dot{u}, \ddot{u})I_{\ddot{u}},$$

and

$$\begin{aligned} D_xI &= I_x + \dot{u}I_u + \ddot{u}I_{\dot{u}} + \dddot{u}\,I_{\ddot{u}} \\ &= -F(x, u, \dot{u}, \ddot{u})I_{\ddot{u}} + \dddot{u}\,I_{\ddot{u}} \\ &= I_{\ddot{u}}(\dddot{u} - F(x, u, \dot{u}, \ddot{u})). \end{aligned}$$

Hence $\mu = I_{\ddot{u}}$. $\square$**Note**(see [5]). If $v^{[\lambda,(k)]}(\alpha) = v^{[\lambda,(k)]}(\beta) = 0$ where $\alpha = \alpha(x, u^{(k)}), \beta = \beta(x, u^{(k)}) \in C^\infty(M^{(k)})$ then

$$v^{[\lambda,(k+1)]}\left(\frac{D_x\alpha}{D_x\beta}\right) = 0.$$

Theorem 3.2 *If $I(x,u,\dot{u},\ddot{u})$ is a first integral of Eq. (3.1), then the vector field $v=\partial_u$ is a λ-symmetry of Eq.(3.1) such that λ is solution $I_u+\lambda I_{\dot{u}}+(D_x\lambda+\lambda^2)I_{\ddot{u}}=0$ and $v^{[\lambda,(2)]}(I)=0$.*

Proof. Since for any function $\lambda(x,u,\dot{u},\ddot{u})$, we have $v^{[\lambda,(2)]}=\partial u+\lambda\partial\dot{u}+(D_x\lambda+\lambda^2)\partial\ddot{u}$, therefore,

$$v^{[\lambda,(2)]}(I)=I_u+\lambda I_{\dot{u}}+(D_x\lambda+\lambda^2)I_{\ddot{u}}=0.$$

Since functions $g(x,u,\dot{u},\ddot{u})=x$ and $I(x,u,\dot{u},\ddot{u})$ are first integral of $v^{[\lambda,(2)]}$ then $v^{[\lambda,(3)]}(\frac{D_xI}{D_xx})=v^{[\lambda,(3)]}(D_xI)=0$, i.e., D_xI is an invariant of $v^{[\lambda,(3)]}$. By applying $v^{[\lambda,(3)]}$ to identity $\mu(\dddot{u}-F(x,u,\dot{u},\ddot{u}))=D_x(I)$, we obtain

$$\begin{aligned}
v^{[\lambda,(3)]}\Big(\mu(\dddot{u}-F)\Big) &= v^{[\lambda,(3)]}(D_x(I))\\
v^{[\lambda,(3)]}\Big(I_{\ddot{u}}(\dddot{u}-F)\Big) &= 0\\
v^{[\lambda,(3)]}(I_{\ddot{u}})\Big(\dddot{u}-F\Big)+I_{\ddot{u}}v^{[\lambda,(3)]}\Big(\dddot{u}-F\Big) &= 0.
\end{aligned}$$

Therefore $I_{\ddot{u}}v^{[\lambda,(3)]}\Big(\dddot{u}-F\Big)=0$, when $\dddot{u}=F(x,u,\dot{u},\ddot{u})$, since $I_{\ddot{u}}\neq 0$, hence the vector field $v=\partial_u$ is a λ-symmetry of Eq.(3.1). □

Theorem 3.3 *If $\mu(x,u,\dot{u},\ddot{u})$ is an integrating factor of Eq. (3.1), then there is a first integral $I(x,u,\dot{u},\ddot{u})$ of Eq. (3.1), such that $\mu(x,u,\dot{u},\ddot{u})=I_{\ddot{u}}(x,u,\dot{u},\ddot{u})$.*

Proof. If $\mu(x,u,\dot{u},\ddot{u})$ is an integrating factor of Eq. (3.1), then

$$\mu(x,u,\dot{u},\ddot{u})(\dddot{u}-F(x,u,\dot{u},\ddot{u}))=D_x(I)=I_x+\dot{u}I_u+\ddot{u}I_{\dot{u}}+\dddot{u}I_{\ddot{u}},$$

for some function $I(x,u,\dot{u},\ddot{u})$ then $\mu(x,u,\dot{u},\ddot{u})=I_{\ddot{u}}(x,u,\dot{u},\ddot{u})$ also, we have

$$-\mu F=-I_{\ddot{u}}F=I_x+\dot{u}I_u+\ddot{u}I_{\dot{u}},$$

therefore $I_x+\dot{u}I_u+\ddot{u}I_{\dot{u}}+F(x,u,\dot{u},\ddot{u})I_{\ddot{u}}=0$, i.e. $A(I)=0$. □The vector field $v=\xi(x,u)\partial_x+\eta(x,u)\partial_u$ is a λ-symmetry of Eq. (3.1) if and only if

$[v^{[\lambda,(2)]}, A] = \lambda v^{[\lambda,(2)]} + \tau A$ where $\tau = -(A+\lambda)(\xi(x,u))$. When $v = \partial_u$ is a λ-symmetry of Eq.(3.1) if and only if $[v^{[\lambda,(2)]}, A] = \lambda v^{[\lambda,(2)]}$.(see [5])

Theorem 3.4 *If $v = \partial_u$ is a λ-symmetry of Eq.(3.1) for some function $\lambda(x,u,\dot{u},\ddot{u})$, then there is a first integral $I(x,u,\dot{u},\ddot{u})$ of Eq.(3.1) such that $v^{[\lambda,(2)]}(I) = 0$.*

Proof. If $v = \partial_u$ is a λ-symmetry of Eq.(3.1) for some function $\lambda(x,u,\dot{u},\ddot{u})$, then

$$[v^{[\lambda,(2)]}, A] = \lambda v^{[\lambda,(2)]}.$$

Therefore $\{v^{[\lambda,(2)]}, A\}$ is an involutive set of vector fields in $M^{(2)}$ and there is function $I(x,u,\dot{u},\ddot{u})$ such that $v^{[\lambda,(2)]}(I) = 0$ and $A(I) = 0$. □

Theorem 3.5 *A system of the form*

$$\begin{cases} I_x = \mu\Big(\Psi\dot{u} - H\ddot{u} - F\Big) \\ I_u = -\mu\Psi \\ I_{\dot{u}} = \mu H \\ I_{\ddot{u}} = \mu \end{cases} \tag{3.2}$$

where $\Psi = \lambda H + D_x\lambda + \lambda^2$, is compatibly for some function $\lambda(x,u,\dot{u},\ddot{u})$, $\mu(x,u,\dot{u},\ddot{u})$ and $H(x,u,\dot{u},\ddot{u})$, if and only if μ is an integrating factor of Eq.(3.1) and $v = \partial_u$ is a λ-symmetry of Eq.(3.1). In this case I is a first integral of Eq.(3.1).

Proof. If I be a first integral of Eq.(3.1) then $\mu = I_{\ddot{u}}$ is an integrating factor of Eq.(3.1) and if $v = \partial_u$ be a λ-symmetry of Eq.(3.1) then $A(I) = 0$ and $v^{[\lambda,(2)]}(I) = 0$, i.e.

$$\begin{aligned} I_x &= -\dot{u}I_u - \ddot{u}I_{\dot{u}} - FI_{\ddot{u}} = -\dot{u}I_u - \ddot{u}I_{\dot{u}} - F\mu, \\ I_u &= -\lambda I_{\dot{u}} - (D_x\lambda + \lambda^2)I_{\ddot{u}} = -\lambda I_{\dot{u}} - (D_x\lambda + \lambda^2)\mu. \end{aligned}$$

If $I_{\dot{u}} = \mu H$, where $H(x,u,\dot{u},\ddot{u})$ is arbitrary function, then system (3.2) is compatible. We are going to prove that, when (3.2) is compatible necessarily $v = \partial_u$ is a λ-symmetry. Suppose (3.2) is compatible,i.e.,

$I_{xu} = I_{ux}$,$I_{x\dot{u}} = I_{\dot{u}x}$,$I_{x\ddot{u}} = I_{\ddot{u}x}$,$I_{u\dot{u}} = I_{\dot{u}u}$,$I_{u\ddot{u}} = I_{\ddot{u}u}$,$I_{\dot{u}\ddot{u}} = I_{\ddot{u}\dot{u}}$. Obviously that $I_{x\ddot{u}} = I_{\ddot{u}x}$,$I_{u\ddot{u}} = I_{\ddot{u}u}$,$I_{\dot{u}\ddot{u}} = I_{\ddot{u}\dot{u}}$, implies that

$$\begin{aligned}
\mu_x &= (I_{\ddot{u}})_x = (I_x)_{\ddot{u}} = \mu_{\ddot{u}}\Big(\Psi\dot{u} - H\ddot{u} - F\Big) + \mu\Big(\Psi\dot{u} - H\ddot{u} - F\Big)_{\ddot{u}},\\
\mu_u &= (I_{\ddot{u}})_u = (I_u)_{\ddot{u}} = -\mu_{\ddot{u}}\Psi - \mu\Psi_{\ddot{u}},\\
\mu_{\dot{u}} &= (I_{\ddot{u}})_{\dot{u}} = (I_{\dot{u}})_{\ddot{u}} = \mu_{\ddot{u}}H + \mu H_{\ddot{u}}.
\end{aligned} \tag{3.3}$$

The compatibility of system (3.2) , i.e. $I_{xu} = I_{ux}$, $I_{x\dot{u}} = I_{\dot{u}x}$, $I_{u\dot{u}} = I_{\dot{u}u}$, and by using of (3.3) implies that

$$\begin{aligned}
A(\Psi) &= +F_u + (F_{\ddot{u}} + H)\Psi,\\
A(H) &= -F_{\dot{u}} + HF_{\ddot{u}} + H^2 + \Psi,\\
A(\mu) &= -\mu F_{\ddot{u}} - \mu H.
\end{aligned} \tag{3.4}$$

By using (3.4) we have

$$\begin{aligned}
0 &= \mu\Big[A(\lambda H + D_x\lambda + \lambda^2) - F_u - (F_{\ddot{u}} + H)(\lambda H + D_x\lambda + \lambda^2)\Big]\\
&= \ldots\\
&= \mu\Big[D_x^2\lambda + D_x\lambda^2 + \lambda D_x\lambda + \lambda^3 - F_u - \lambda F_{\dot{u}} - (D_x\lambda + \lambda^2)F_{\ddot{u}}\Big].
\end{aligned}$$

Hence,

$$\mu\Big[D_x^2\lambda + D_x\lambda^2 + \lambda D_x\lambda + \lambda^3 - F_u - \lambda F_{\dot{u}} - (D_x\lambda + \lambda^2)F_{\ddot{u}}\Big] = 0, \tag{3.5}$$

when $\mu \neq 0$, (3.5) implies that $v = \partial_u$ is a λ-symmetry. □

Corollary 3.1 *A procedure to find an integrating factor $\mu(x, u, \dot{u}, \ddot{u})$ and consequently a first integral $I(x, u, \dot{u}, \ddot{u})$ of Eq.(3.1), by λ-symmetry method, is as follows:*

- *First, we find a function λ of Eq.(3.1). The function $\lambda(x, u, \dot{u}, \ddot{u})$ is any particular solution of the equation*

$$D_x^2\lambda + D_x\lambda^2 + \lambda D_x\lambda + \lambda^3 = F_u + \lambda F_{\dot{u}} + (D_x\lambda + \lambda^2)F_{\ddot{u}}, \tag{3.6}$$

- *Second, we find an integrating factor* μ *of Eq.(3.1). The integrating factor* $\mu(x, u, \dot{u}, \ddot{u})$ *and function* $H(x, u, \dot{u}, \ddot{u})$ *are particular solutions of the system*

$$\begin{cases} A(\Psi) - F_u - (F_{\ddot{u}} + H)\Psi = 0 \\ -A(H) - F_{\dot{u}} + HF_{\ddot{u}} + H^2 + \Psi = 0 \\ A(\mu) + \mu F_{\ddot{u}} + \mu H = 0 \end{cases} \tag{3.7}$$

  where $\Psi = \lambda H + D_x\lambda + \lambda^2$.
- *Finally, we find a first integral* I *of Eq.(3.1). The first integral* $I(x, u, \dot{u}, \ddot{u})$ *is any particular solution of the system*

$$\begin{cases} I_x = \mu\Big(\Psi\dot{u} - H\ddot{u} - F\Big) \\ I_u = -\mu\Psi \\ I_{\dot{u}} = \mu H \\ I_{\ddot{u}} = \mu. \end{cases} \tag{3.8}$$

  Therefor, we have $\mu(\dddot{u} - F) = D_x(I)$.

Example Consider the third-order differential equation

$$\dddot{u} - \frac{2x}{x^2+1}\ddot{u} - \frac{3x^2}{x^2+1} = 0, \tag{3.9}$$

where $F(x, u, \dot{u}, \ddot{u}) = \dfrac{2x}{x^2+1}\ddot{u} + \dfrac{3x^2}{x^2+1}$ is an analytic function of its arguments. It can be checked that $\lambda = \dfrac{1}{x}$ is a particular solution of (3.6). Substituting $F(x, u, \dot{u}, \ddot{u}) = \dfrac{2x}{x^2+1}\ddot{u} + \dfrac{3x^2}{x^2+1}$ and $\lambda = \dfrac{1}{x}$ into (3.7) and solving them, we obtain $H = 0$ and $\mu = x^2 + 1$. Therefore, by using of system (3.8), we have

$$\begin{cases} I_x = -\mu F = 2x\ddot{u} + 3x^2 \\ I_u = 0 \\ I_{\dot{u}} = 0 \\ I_{\ddot{u}} = \mu = x^2 + 1. \end{cases} \tag{3.10}$$

A solution of this system is $I(x, u, \dot{u}, \ddot{u}) = (x^2 + 1)\ddot{u} + x^3$. Therefore, by using of (2.2), i.e. $(x^2+1)\Big(\dddot{u} - \frac{2x}{x^2+1}\ddot{u} - \frac{3x^2}{x^2+1}\Big) = D_x\Big((x^2+1)\ddot{u}+x^3\Big)$, implies that, we reduce the order of equation $\dddot{u} - \frac{2x}{x^2+1}\ddot{u} - \frac{3x^2}{x^2+1} = 0$ to the equation $(x^2 + 1)\ddot{u} + x^3 = 0$.

4 The Prelle-Singer (PS) method

The PS procedure has many attractive features. For a large class of integrable systems, this procedure provides the integrals of motion/general solution in a straightforward way. In fact this is true for any order. The PS method not only gives the first integrals but also the underlying integrating factors. Further, like Lie-symmetry analysis and Noether's theorem the PS method can also used to solve linear as well as nonlinear ODEs. In addition to the above, the PS procedure is applicable to deal with both Hamiltonian and non-Hamiltonian systems.

Prelle and Singer introduced construct a method to integrating factor of first-order ODEs and higher-order ODEs (see [2],[3]). The main characteristic its as follows:

Let us consider a class of third order ODEs of the form

$$\dddot{u} = \frac{P(x, u, \dot{u}, \ddot{u})}{Q(x, u, \dot{u}, \ddot{u})}, \tag{4.1}$$

where over dot denotes differentiation with respect to x and P and Q are polynomials in $x, u, \dot{u}$ and $\ddot{u}$. Let us assume that the ODE (4.1) admits a first integral $I(x, u, \dot{u}, \ddot{u}) = c$, such that c constant on the solutions ODEs (4.1), so that the total differential gives

$$dI = I_x dx + I_u du + I_{\dot{u}} d\dot{u} + I_{\ddot{u}} d\ddot{u} = 0. \tag{4.2}$$

where subscript denotes partial differentiation with respect to that variable. Rewriting equation (4.1) of the form

$$\frac{P}{Q} dx - d\ddot{u} = 0. \tag{4.3}$$

The existence of the functions $S_i(x, u, \dot{u}, \ddot{u})$, $i = 1, 2$, such that

$$\begin{aligned} &S_1(x, u, \dot{u}, \ddot{u})\dot{u}dx - S_1(x, u, \dot{u}, \ddot{u})du = 0, \\ &S_2(x, u, \dot{u}, \ddot{u})\ddot{u}dx - S_2(x, u, \dot{u}, \ddot{u})d\dot{u} = 0. \end{aligned} \tag{4.4}$$

By adding terms (4.3) and (4.4), we obtain the 1−form

$$\Big(\frac{P}{Q} + S_1\dot{u} + S_2\ddot{u}\Big)dx - S_1 du - S_2 d\dot{u} - d\ddot{u} = 0. \tag{4.5}$$

Multiplying (4.5) by some function $R(x, u, \dot{u}, \ddot{u})$, we have that

$$dI = R\Big[\Big(F + S_1\dot{u} + S_2\ddot{u}\Big)dx - S_1 du - S_2 d\dot{u} - d\ddot{u}\Big] = 0. \tag{4.6}$$

where $F \equiv \dfrac{P}{Q}$. Comparing equation (4.2) and (4.6), we have the system

$$\begin{cases} I_x = R\Big(F + S_1\dot{u} + S_2\ddot{u}\Big) \\ I_u = -RS_1 \\ I_{\dot{u}} = -RS_2 \\ I_{\ddot{u}} = -R. \end{cases} \tag{4.7}$$

The compatibility of system (4.7), $I_{xu} = I_{ux}$, $I_{x\dot{u}} = I_{\dot{u}x}$, $I_{x\ddot{u}} = I_{\ddot{u}x}$, $I_{u\dot{u}} = I_{\dot{u}u}$, $I_{u\ddot{u}} = I_{\ddot{u}u}$, $I_{\dot{u}\ddot{u}} = I_{\ddot{u}\dot{u}}$, implies that, we obtain the conditions

$$\begin{aligned}
A(S_1) &= -F_u + S_1(F_{\ddot{u}} + S_2),\\
A(S_2) &= -F_{\dot{u}} + S_2(F_{\ddot{u}} + S_2) - S_1,\\
A(R) &= -R(F_{\ddot{u}} + S_2),\\
R_x &= -R_{\ddot{u}}\Big(F + S_1\dot{u} + S_2\ddot{u}\Big) - R\Big(F + S_1\dot{u} + S_2\ddot{u}\Big)_{\ddot{u}},\\
R_u &= R_{\ddot{u}}S_1 + R(S_1)_{\ddot{u}}\,,\\
R_{\dot{u}} &= R_{\ddot{u}}S_2 + R(S_2)_{\ddot{u}}.
\end{aligned} \tag{4.8}$$

Corollary 4.1 *A procedure to find an integrating factor $R(x,u,\dot{u},\ddot{u})$ and consequently a first integral $I(x,u,\dot{u},\ddot{u})$ of Eq.(3.1), by the PS method, is as follows:*

- *First, we find an integrating factor R of Eq.(3.1). The integrating factor $R(x,u,\dot{u},\ddot{u})$ and functions $S_1(x,u,\dot{u},\ddot{u})$ and $S_2(x,u,\dot{u},\ddot{u})$ are particular solutions of the system*

$$\begin{cases}
A(S_1) + F_u - S_1(F_{\ddot{u}} + S_2) = 0,\\
A(S_2) + F_{\dot{u}} - S_2(F_{\ddot{u}} + S_2) + S_1 = 0,\\
A(R) + R(F_{\ddot{u}} + S_2) = 0.
\end{cases} \tag{4.9}$$

- *Finally, we find a first integral I of Eq.(3.1). The first integral $I(x,u,\dot{u},\ddot{u})$ is any particular solution of the system*

$$\begin{cases}
I_x = R\Big(F + S_1\dot{u} + S_2\ddot{u}\Big)\\
I_u = -RS_1\\
I_{\dot{u}} = -RS_2\\
I_{\ddot{u}} = -R.
\end{cases} \tag{4.10}$$

Therefor, we have $R(\dddot{u} - F) = D_x(I)$.

5 Comparing λ-symmetry method and The PS method

As a consequence of section 3 and 4, by comparing λ-symmetry method and Prelle-Singer (PS) method for third-order differential equations, we have the following corollary:

Corollary 5.1 *If* $R = -\mu$, $S_1 = -\Psi$ *and* $S_2 = H$ *where* $\Psi = \lambda H + D_x\lambda + \lambda^2$ *then the systems (3.7) and (3.8) are equivalent to systems (4.9) and (4.10).*

References

[1] B. Abraham-Shrdauner, *Hidden symmetries and non-local group generators for ordinary differential equation*, IMA J. Appl. Math. 56 (1996) 235–252.

[2] V.K. Chandrasekar, M. Senthilvelan, M. Lakshmanan, *Extended Prelle–Singer method and integrability/solvability of a class of nonlinear nth-order ordinary differential equations*, J. Nonl. Math.Phys. 12 (2005) 184–201.

[3] V.K. Chandrasekar, M. Senthilvelan, M. Lakshmanan, *A note on solving third order ordinary differential equations through the extended Prelle-Singer procedure*, National conference on nonlinear systems and dynamic. (2005).

[4] G. Gaeta, P. Morando, *On the geometry of lambda-symmetries and PDEs reduction*, J. Phys. A 37 (2004) 6955–6975.

[5] C. Muriel, J. L. Romero, *New methods of reduction for ordinary differential equation*, IMA J. Appl. Math. 66 (2001) 111–125.

[6] C. Muriel, J. L. Romero, *C^∞-symmetries and reduction of equation without Lie point symmetries*, J. Lie Theory 13 (2003) 167–188.

[7] C. Muriel, J. L. Romero, *λ-symmetries and integrating factors*, J. Nonlinear Math. Phys. 15 (2008) 290–299.

[8] C. Muriel, J. L. Romero, *First integrals, integrating factors and λ-symmetries of second-order differential equations*, J. Phys. A:Math. Theor. 43 (2009) .

[9] C. Muriel, J. L. Romero, *Second-order ordinary differential equations and first integral of the form $A(t,x)\dot{x} + B(t,x)$*, J. Nonlinear Math. Phys. 16 (2009) 209–222.

[10] C. Muriel, J. L. Romero, *A λ-symmetry-based method for the linearization and determination of first-integrals of a family of second-order differential equations*, J. Phys. A:Math. Theor. 44 (2011) 245201.

[11] P.J. Olver, *Applications of Lie Groups to Differential Equations*, New York, 1986.