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Coupled Vibration of Partially Fluid-Filled Laminated Composite Cylindrical Shells

M.R. Saviz *

Mechanical Engineering Department, Azarbaijan Shahid Madani University, Tabriz, Iran

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ABSTRACT

In this study, the free vibration of partially fluid-filled laminated composite circular cylindrical shell with arbitrary boundary conditions has been investigated by using Rayleigh-Ritz method. The analysis has been carried out with strain-displacement relations based on Love's thin shell theory and the contained fluid is assumed irrotational, incompressible and inviscid. After determining the kinetic and potential energies of fluid filled laminated composite shell, the eigenvalue problem has been obtained by means of Rayleigh-Ritz method. To demonstrate the validity and accuracy of the results, comparison has been made with the results of similar works for the empty and partially fluid-filled shells. Finally, an extensive parameter study on a typical composite tank is accomplished and some conclusions are drawn.

Keywords : Laminated composite; Rayleigh-Ritz; Partially fluid-filled; Cylindrical shell; Vibration analysis.

1 INTRODUCTION

FLUID-FILLED circular cylindrical shells made of composite materials have been widely used in industries, e.g. aerospace, civil engineering, marine, petrochemical engineering, oil containers and tankers due to their high strength-to-weight ratio and better corrosion resistance as well as the advantages of composite materials. Since most of failures of these structures are caused by dynamic loading, the dynamics of coupled fluid shells is a problem that has been focused recently. Free vibration of the partially fluid filled laminated shell is one of the most difficult problems in acoustics and structural dynamics.

Many studies have been conducted on the vibration of the composite cylindrical shells without fluid and different approaches have been applied in these studies, e.g. Analytical method by Wang and Lin [18], Rayleigh-Ritz method (Zhao et al.,[20]; Kim and Lee[11]; Zhang[21]), finite element method (Balamurugan and Narayanan [4]; Liu and To [15]) and experimental studies by Jafari and Bagheri [8]. On the other hand, various methods have been used to investigate the vibration of the partially fluid-filled isotropic and composite tanks, e.g. the expansion method for isotropic tanks by Jeong and Kim [9]; Kondo [10], Rayleigh's quotient method by Gupta and Hutchinson [6], Rayleigh-Ritz method (Amabili [1]; Amabili [2]; Amabili et al. [3]; Kim et al.,[12]), finite element method by Lee and Lu [14]; Goncalves and Ramos[7], harmonic balance method by Chiba and Abe [5] and experimental studies (Mazuch et al.,[16]) can be found in the literature and some papers have focused on composite tanks, for example finite element method by Ramasamy and Ganesan [17] and experimental studies by Yu et al. [19]. The Rayleigh-Ritz method has been proved to be very efficient in studying vibration of empty cylindrical shells or

*Corresponding author.
E-mail address: saviz@azaruniv.ac.ir (M.R.Saviz).

partially fluid-filled isotropic tanks from economic point of view in time and amount of calculation, but in order to obtain correct results, the trial functions must satisfy all the geometrical boundary conditions.

In this paper, free vibration analysis of the partially fluid-filled laminated composite circular cylindrical shells with various boundary conditions is investigated by using the Raleigh-Ritz method, which is based on the energy parameters. The kinetic and potential energies of different ingredients of the shell filled with fluid are obtained. The analysis is carried out with Love's thin shell theory for strain-displacement relations and the contained fluid is assumed incompressible and inviscid. For modeling the vibration of the fluid, the velocity potential is supposed as sum of two sets of velocity potentials: that of the fluid associated with the flexible shell (cause to bulging type frequencies where structure of tank plays main role) and the other one is due to the free surface of the fluid in the rigid shell (cause to sloshing type frequencies where free surface of tank plays main role). Finally, numerical comparison is made with available results and the effects of some related parameters on natural frequencies have been discussed.

2 GOVERNING EQUATIONS

The Rayleigh-Ritz method is based on the energy parameters. Therefore, the kinetic and potential energies of cylindrical shell, fluid and fluid-shell interaction have been obtained. Eventually, the eigenvalue problem for the partially fluid-filled laminated composite circular cylindrical shell has been obtained.

2.1 Kinetic and potential energies of composite shell

A cylindrical shell with mid-plane radius R, length L and thickness h is considered. The deformations are defined with reference to cylindrical coordinate system (r, θ and x) given in Fig.1. The constitutive relation for a thin generally orthotropic cylindrical composite layer under plane stress condition is given by

$$\{\sigma\} = [\bar{Q}]\{e\}, \tag{1}$$

where $\{\sigma\}$ and $\{e\}$ represent the stress and strain vectors and $[\bar{Q}]$ is the reduced transformed stiffness matrix. The stress vector is defined as follow

$$\{\sigma\}^T = \{\sigma_x, \sigma_\theta, \sigma_{x\theta}\}, \tag{2}$$

where σ_x and σ_θ are the normal stresses in x and θ directions and $\sigma_{x\theta}$ is the shear stress in the $x\theta$ plane. Similarly, the strain vector is defined by the following vector

$$\{e\}^T = \{e_x, e_\theta, e_{x\theta}\}, \tag{3}$$

where e_x and e_θ are the normal strains in x and θ directions and $e_{x\theta}$ is the shear strains in the $x\theta$ plane. The transformed reduced stiffness matrix $[\bar{Q}]$ is defined as:

$$[\bar{Q}] = [T]^{-1}[Q][T]^{-T}, \tag{4}$$

where $[T]$ is the transformation matrix between principal material coordinates and the shell coordinates defined by the following matrix

$$[T] = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix}, \tag{5}$$

where α is the angle of fibers orientation with cylinder axis (the counter-clockwise direction is taken to be positive). The reduced stiffness matrix $[Q]$ is defined as follow:

$$[Q]=\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}, \tag{6}$$

where $Q_{ij}\,(i,j=1,2$ and 6) are defined for an orthotropic material as below:

$$Q_{11}=\frac{E_{11}}{1-\nu_{12}\nu_{21}},Q_{12}=\frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}},Q_{22}=\frac{E_{22}}{1-\nu_{12}\nu_{21}},Q_{66}=G_{12}, \tag{7}$$

where E_{11} and E_{22} are Young's moduli, G_{12} is the shear modulus and finally ν_{12} and ν_{21} are Poisson's ratios. Using Love's shell theory (Leissa [13]), the strain components are obtained

$$e_x=e_1-zk_1,\ e_\theta=e_2-zk_2,\ e_{x\theta}=\gamma-2z\tau, \tag{8}$$

In the above equations e_1,e_2 and γ are the reference surface strains and k_1,k_2 and τ are the surface curvatures. These surface strains and curvatures are given as below:

$$\{e_1,e_2,\gamma,k_1,k_2,\tau\}=\left\{\frac{\partial u}{\partial x},\frac{1}{R}\left(\frac{\partial v}{\partial\theta}+w\right),\frac{\partial v}{\partial x}+\frac{1}{R}\frac{\partial u}{\partial\theta},\quad -\frac{\partial^2 w}{\partial x^2},-\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial\theta^2}-\frac{\partial v}{\partial\theta}\right),-\frac{1}{R}\left(\frac{\partial^2 w}{\partial x\partial\theta}-\frac{\partial v}{\partial x}\right)\right\}, \tag{9}$$

For a thin cylindrical shell the force and the moment resultants are defined as follows:

$$\{N_x,N_\theta,N_{x\theta}\}=\int_{-h/2}^{h/2}\{\sigma_x,\sigma_\theta,\sigma_{x\theta}\}\,dz, \tag{10}$$

$$\{M_x,M_\theta,M_{x\theta}\}=\int_{-h/2}^{h/2}\{\sigma_x,\sigma_\theta,\sigma_{x\theta}\}\,zdz, \tag{11}$$

After substituting Eqs.(9) and (2) into Eqs. (10) and (11), the following constitutive vector equation is obtained

$$\{N\}=[S]\{\varepsilon\}, \tag{12}$$

where $\{N\},\{\varepsilon\}$ and $[S]$ are

$$\{N\}^T=\{N_x,N_\theta,N_{x\theta},M_x,M_\theta,M_{x\theta}\}, \tag{13}$$

$$\{\varepsilon\}^T=\{e_1,e_2,\gamma,k_1,k_2,2\tau\}, \tag{14}$$

$$[S]=\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}, \tag{15}$$

A_{ij}, B_{ij} and D_{ij} represent the extensional, coupling and bending stiffness matrices, respectively, which are given as follow:

$$\left\{A_{ij}, B_{ij}, D_{ij}\right\} = \int_{-h/2}^{h/2} \bar{Q}_{ij} \left\{1, z, z^2\right\} dz, \tag{16}$$

For an arbitrarily laminated shell, the above matrices can be rewritten

$$A_{ij} = \sum_{k=1}^{N_l} \bar{Q}_{ij}^k \left(h_k - h_{k-1}\right), \quad B_{ij} = \frac{1}{2}\sum_{k=1}^{N_l} \bar{Q}_{ij}^k \left(h_k^2 - h_{k-1}^2\right), \quad D_{ij} = \frac{1}{3}\sum_{k=1}^{N_l} \bar{Q}_{ij}^k \left(h_k^3 - h_{k-1}^3\right), \tag{17}$$

where N_l denotes the number of layers in the shell, h_k and h_{k-1} define the distances from the shell reference surface to the outer and inner surfaces of the k_{th} lamina as indicated in Fig. 1. The strain or potential energy of the laminated composite shell is expressed as follow:

$$U_s = \frac{1}{2}\int_0^L \int_0^{2\pi} \{\varepsilon\}^T [S] \{\varepsilon\} R d\theta dx, \tag{18}$$

Further, the kinetic energy of the laminated composite shell is expressed by the following relation

$$T_s = \frac{1}{2}\int_0^L \int_0^{2\pi} \rho_T \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] R d\theta dx, \tag{19}$$

where ρ_T is the mass density per unit length and is defined as:

$$\rho_T = \int_{-h/2}^{h/2} \rho dz, \tag{20}$$

The following spatial displacement field can be used for free vibration of a cylindrical shell with different boundary conditions

$$u_n(x,\theta) = \sum_{m=1}^{M} U_{mn} \frac{\partial \psi_m(x)}{\alpha_m \partial x} \cos(n\theta), \tag{21}$$

$$v_n(x,\theta) = \sum_{m=1}^{M} V_{mn} \psi_m(x) \sin(n\theta), \tag{22}$$

$$w_n(x,\theta) = \sum_{m=1}^{M} W_{mn} \psi_m(x) \cos(n\theta), \tag{23}$$

where U_{mn}, V_{mn} and W_{mn} are constants denoting the amplitude of the displacement in the axial, circumferential and radial directions. Also, m and n denote the number of axial and circumferential wave numbers in mode shape, respectively. The axial mode function $\psi_m(x)$ is

$$\psi_m(x) = c_1 \sin\left(\frac{\eta x}{L}\right) + c_2 \cos\left(\frac{\eta x}{L}\right) + c_3 \sinh\left(\frac{\eta x}{L}\right) + c_4 \cosh\left(\frac{\eta x}{L}\right), \tag{24}$$

where c_1, c_2, c_3 and c_4 are determined for each boundary conditions. The parameter η is a real number depending on the number of axial waves. For simply supported boundary condition: $\eta = m\pi$.

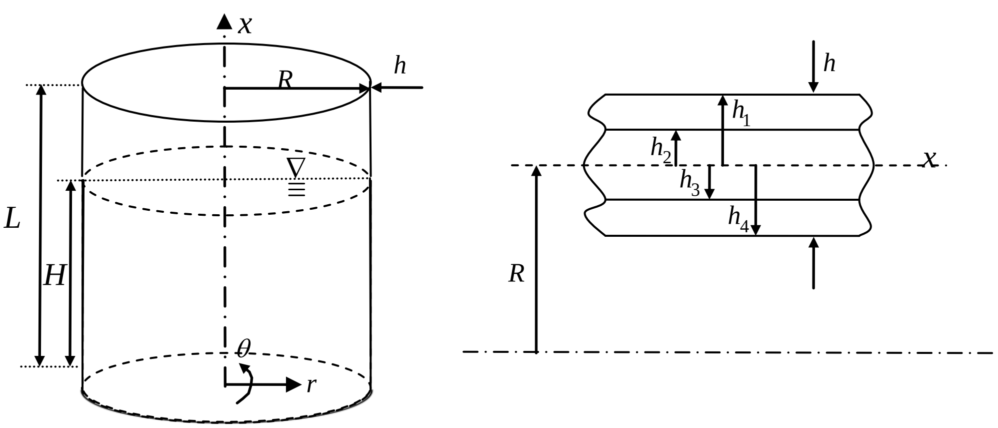


Fig.1
Geometry of a partially fluid-filled laminated composite circular cylindrical shell.

2.2 Kinetic and potential energies of the fluid

A numerous papers on the vibrations of partially fluid-filled shells have been published. This formulated part is an elaboration of the theory and approach developed by Amabili [2]; Amabili et al.[3]. The incompressible and inviscid fluid can be described by a velocity potential $\bar{\varphi}(r,\theta,x,t)$ which must satisfy the Laplace equation:

$$\nabla^2\bar{\varphi} = \frac{\partial^2\bar{\varphi}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\varphi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\bar{\varphi}}{\partial\theta^2} + \frac{\partial^2\bar{\varphi}}{\partial x^2} = 0, \tag{25}$$

The velocity potential is written in terms of deformation potential $\varphi(r,\theta,x)$ by the following form:

$$\bar{\varphi}(r,\theta,x,t) = -i\omega\varphi(r,\theta,x)e^{i\omega t}, \tag{26}$$

which is assumed to be harmonic; *i* is the imaginary unit and ω is the natural frequency of the vibration. By using the principle of superposition, the deformation potential of fluid can be divided into two parts (Kim et al., [12])

$$\varphi = \varphi^{(1)} + \varphi^{(S)}, \tag{27}$$

where the function $\varphi^{(1)}$ describes the deformation potential of the fluid associated with flexible shell considering the bottom plate to be rigid and neglecting free surface waves and $\varphi^{(S)}$ is the deformation potential due to sloshing in the presence of a rigid structure. The boundary conditions imposed to the liquid for the two complementary boundary conditions are as follows:

$$\left(\partial\varphi^{(1)}/\partial r\right)_{r=R} = w(x,\theta), \tag{28}$$

$$\left(\partial\varphi^{(1)}/\partial x\right)_{x=0} = 0, \tag{29}$$

$$\left(\varphi^{(1)}\right)_{x=H} = 0, \tag{30}$$

$$\left(\partial\varphi^{(S)}/\partial r\right)_{r=R} = 0, \tag{31}$$

$$\left(\partial\varphi^{(S)}/\partial x\right)_{x=0}=0, \tag{32}$$

$$\left(\partial\varphi/\partial x\right)_{x=H}=\left(\omega^2/g\right)\left(\varphi^{(S)}\right)_{x=H}, \tag{33}$$

g is the gravitational acceleration. By combining Eqs.(28)-(30), the linearized sloshing condition (33) can be rewritten as follow:

$$\left(\partial\varphi^{(1)}/\partial x\right)_{x=H}+\left(\partial\varphi^{(S)}/\partial x\right)_{x=H}=\frac{\omega^2}{g}\left(\varphi^{(S)}\right)_{x=H} \quad , \tag{34}$$

Finally, the natural frequencies of tank vibrations are obtained

$$\omega^2=\frac{U_s+U_L}{T_s^*+\tilde{T}_L^*}, \tag{35}$$

The total kinetic energy of fluid can be expressed

$$\tilde{T}_L^*=\frac{1}{2}\rho_L\int_S\varphi\frac{\partial\varphi}{\partial n}dS=\frac{1}{2}\rho_L\int_{S_F}\varphi\frac{\partial\varphi}{\partial x}dS+T_L^*, \tag{36}$$

where

$$T_L^*=\frac{1}{2}\rho_L\int_{S_1}\varphi\frac{\partial\varphi}{\partial n}dS=\frac{1}{2}\rho_L\int_{S_1}\left(\varphi^{(1)}+\varphi^{(S)}\right)\frac{\partial\left(\varphi^{(1)}+\varphi^{(S)}\right)}{\partial n}dS, \tag{37}$$

In the above equations, n denotes the normal unit vector at any point on the boundary surface S, which is the sum of S_1, the shell lateral surface and S_F, the free surface of fluid. The maximum potential energy U_L of the free surface waves of the liquid is

$$U_L=\frac{1}{2}\rho_L g\int_{S_F}\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial x}dS=\frac{1}{2}\rho_L\omega^2\int_{S_F}\varphi\frac{\partial\varphi}{\partial x}dS, \tag{38}$$

By using Eqs. (36) and (38), the natural frequencies of tank is written as below:

$$\omega^2=\frac{U_s}{T_s^*+T_L^*}, \tag{39}$$

Also, by using Eqs. (28) and (31), the kinetic energy of fluid Eqs. (37) is written as below:

$$T_L^*=\frac{1}{2}\rho_L\int_{S_1}\left(\varphi^{(1)}+\varphi^{(S)}\right)w\,dS=T_L^{*(1)}+T_L^{*(1-S)}, \tag{40}$$

$T_L^{\bullet(1)}$ is the kinetic energy of fluid associated with flexible shell and $T_L^{\bullet(1-S)}$ is the kinetic energy of fluid due to the sloshing. After applying separation of variables method on Eqs. (25), (29) and (30), the fluid deformation potential associated with the flexible shell and the rigid bottom plate, is obtained as follow:

$$\varphi^{(1)}=\sum_{m=1}^{M}W_{mn}\sum_{s=1}^{\infty}A_{mns}I_n\left(\frac{2s-1}{2}\pi\frac{r}{H}\right)\times\cos\left(n\theta\right)\cos\left(\frac{2s-1}{2}\pi\frac{x}{H}\right), \tag{41}$$

where A_{mns} are coefficients depending on the integers *m*, *n* and *s*. Eq. (28) is used to compute the coefficients A_{mns}

$$\sum_{s=1}^{\infty} A_{mns}\left(\frac{2s-1}{2}\pi\frac{1}{H}\right)I'_n\left(\frac{2s-1}{2}\pi\frac{R}{H}\right)\times\cos\left(\frac{2s-1}{2}\pi\frac{x}{H}\right)=\psi_m\left(x\right), \tag{42}$$

Multiplication of Eq. (42) by $\cos\left(\left(2s-1/2\right)\pi\left(x/H\right)\right)$ and then integration between zero and *H* results in the following relation:

$$A_{mns}=\frac{4\gamma_{ms}}{\left(2s-1\right)\pi I'_n\left(\frac{2s-1}{2}\pi\frac{R}{H}\right)}, \tag{43}$$

where

$$\gamma_{ms}=\int_0^H \psi_m\left(x\right)\cos\left(\frac{2s-1}{2}\pi\frac{x}{H}\right)dx, \tag{44}$$

Therefore, the term $T_L^{*(1)}$ in Eq. (40) can be obtained as:

$$T_L^{*(1)}=\frac{1}{2}\rho_L\int_0^{2\pi}\int_0^H\left(\varphi^{(1)}\right)_{r=R}wRd\theta dx=\frac{\rho_L RC_n}{2}\sum_{m=1}^{M}\sum_{j=1}^{M}W_{mn}W_{jn}\sum_{s=1}^{\infty}\frac{4\gamma_{ms}\gamma_{js}I_n\left(\frac{2s-1}{2}\pi\frac{R}{H}\right)}{\left(2s-1\right)\pi I'_n\left(\frac{2s-1}{2}\pi\frac{R}{H}\right)}, \tag{45}$$

The deformation potential of fluid due to the sloshing in the presence of rigid structure for the asymmetric modes $\left(n>0\right)$ is assumed as follow:

$$\varphi^{(S)}=\sum_{k=1}^{K}F_{nk}J_n\left(\varepsilon_{nk}\frac{r}{R}\right)\cosh\left(\varepsilon_{nk}\frac{x}{R}\right)\cos\left(n\theta\right), \tag{46}$$

where J_n is the Bessel function of first kind of order *n*. The above assumption has been obtained by using separation of variables method and satisfying the boundary conditions. By applying Eqs. (31) and (32) on Laplace Eq. (25), the following equation is derived for ε_{nk}

$$J'_n\left(\varepsilon_{nk}\right)=0, \tag{47}$$

By substituting Eqs. (41) and (46) into the sloshing condition (33), and then multiplying obtained equation by $J_n\left(\varepsilon_{nk}r/R\right)rdr$ and integrating between zero and *R*, noting the orthogonality of Bessel functions, the following sloshing equation is obtained:

$$F_{nk}\left(\frac{\varepsilon_{nk}}{R}\right)\sinh\left(\varepsilon_{nk}\frac{H}{R}\right)\alpha_{nk}-\sum_{m=1}^{M}W_{mn}\sum_{s=1}^{\infty}A_{mns}\left(\frac{2s-1}{2}\pi\frac{1}{H}\right)\sin\left(\frac{2s-1}{2}\pi\right)\beta_{nsk}=\frac{\omega^2}{g}F_{nk}\cosh\left(\varepsilon_{nk}\frac{H}{R}\right)\alpha_{nk}, \tag{48}$$

where

$$\alpha_{nk}=\frac{1}{R^2}\int_0^R J_n^2\left(\varepsilon_{nk}\frac{r}{R}\right)rdr=\frac{1}{2}\left[1-\left(\frac{n}{\varepsilon_{nk}}\right)^2\right]J_n^2\left(\varepsilon_{nk}\right), \tag{49}$$

$$\beta_{nsk} = \frac{1}{R^2}\int_0^R I_n\left(\frac{2s-1}{2}\pi\frac{r}{H}\right)J_n\left(\varepsilon_{nk}\frac{r}{R}\right)rdr = \frac{\frac{2s-1}{2}\pi\frac{R}{H}}{\left(\frac{2s-1}{2}\pi\frac{R}{H}\right)^2+\varepsilon_{nk}^2}J_n\left(\varepsilon_{nk}\right)I_n'\left(\frac{2s-1}{2}\pi\frac{R}{H}\right), \tag{50}$$

where I_n is the modified first kind Bessel function of order *n*. For the axisymmetric modes $(n=0), \varphi^{(S)}$ can be written in the following form:

$$\varphi^{(S)} = F_{00} + \sum_{k=1}^{K} F_{0k} J_0\left(\varepsilon_{0k}\frac{r}{R}\right)\cosh\left(\varepsilon_{0k}\frac{x}{R}\right), \tag{51}$$

Substituting Eqs. (41) and (51) into the sloshing condition (33), and then multiplying the obtained equation by rdr and integrating between zero and *R*, the following sloshing equation will be obtained

$$-\sum_{m=1}^{M} W_{m0}\sum_{s=1}^{\infty} A_{m0s}\left(\frac{2s-1}{2}\pi\frac{1}{H}\right)\times\sin\left(\frac{2s-1}{2}\pi\right)\eta_{s0} = \frac{1}{2}\frac{(R\omega)^2}{g}F_{00}, \tag{52}$$

where

$$\eta_{s0} = \frac{1}{R^2}\int_0^R I_0\left(\frac{2s-1}{2}\pi\frac{r}{H}\right)rdr = \frac{2}{2s-1}\pi\frac{H}{R}I_1\left(\frac{2s-1}{2}\pi\frac{R}{H}\right), \tag{53}$$

For the asymmetric modes $(n>0)$, the term $T_L^{*(1-S)}$ in Eq. (40) can be calculated

$$T_L^{*(1-S)} = \frac{1}{2}\rho_L\int_0^{2\pi}\int_0^H\left(\varphi^{(S)}w\right)_{r=R}Rdxd\theta = \frac{1}{2}\rho_L C_n R\sum_{k=1}^{K}\sum_{m=1}^{M}F_{nk}W_{mn}J_n\left(\varepsilon_{nk}\right)\zeta_{mkn}, \tag{54}$$

where

$$\zeta_{mkn} = \int_0^H \psi_m(x)\cosh\left(\varepsilon_{nk}\frac{x}{R}\right)dx, \tag{55}$$

For the axisymmetric modes $(n=0)$, the term $T_L^{*(1-S)}$ in Eq. (40) is obtained as follow:

$$T_L^{*(1-S)} = \frac{1}{2}\rho_L C_0 R\left\{\sum_{m=1}^{M}F_{00}W_{m0}\tau_m + \sum_{k=1}^{K}\sum_{m=1}^{M}F_{0k}W_{m0}J_0\left(\varepsilon_{0k}\right)\zeta_{mk0}\right\}, \tag{56}$$

where

$$\tau_m = \int_0^H \psi_m(x)dx, \tag{57}$$

3 SOLUTION METHOD OF EIGENVALUE PROBLEM

Eventually, the eigenvalue problem for the partially fluid-filled laminated composite circular cylindrical shell has been obtained. The vector of the Rayleigh-Ritz parameters *q* expansions (generalized coordinates) and generalized force vector are defined as follows:

$$\{q\} = \{\{\bar{u}\} \quad \{\bar{v}\} \quad \{\bar{w}\}\}^T, \qquad \{F\} = \{F_{n1} \quad F_{n2} \quad \cdots \quad F_{nK}\}^T, \tag{58}$$

where

$$\{\bar{u}\} = \{U_{1n} \quad U_{2n} \quad \cdots \quad U_{Mn}\}, \quad \{\bar{v}\} = \{V_{1n} \quad V_{2n} \quad \cdots \quad V_{Mn}\}, \quad \{\bar{w}\} = \{W_{1n} \quad W_{2n} \quad \cdots \quad W_{Mn}\}, \tag{59}$$

For the axisymmetric modes, the coefficient F_{00} must be included in the vector $\{F\}$. The maximum potential energy of the shell, Eq.(18) becomes

$$U_s = \frac{1}{2}\{q\}^T [K_s]\{q\}, \tag{60}$$

where

$$[K_s]_{3M\times 3M} = \begin{bmatrix} [K_s^{11}] & [K_s^{12}] & [K_s^{13}] \\ [K_s^{12}] & [K_s^{22}] & [K_s^{23}] \\ [K_s^{13}] & [K_s^{23}] & [K_s^{33}] \end{bmatrix}, \tag{61}$$

where submatrices $[K_s^{ij}]_{M\times M}$ have given in Appendix A **.** The reference kinetic energy of the shell, Eq.(19) becomes

$$T_s^* = \frac{1}{2}\{q\}^T [M_s]\{q\}, \tag{62}$$

where

$$[M_s]_{3M\times 3M} = \begin{bmatrix} [M_s^{11}] & 0 & 0 \\ 0 & [M_s^{22}] & 0 \\ 0 & 0 & [M_s^{33}] \end{bmatrix}, \tag{63}$$

and submatrices $[M_s^{ii}]_{M\times M}$ are given in Appendix **A**. The reference kinetic energy of fluid associated with flexible shell (Eq. (45)) becomes

$$T_L^{*(1)} = \frac{1}{2}\{\bar{w}\}^T \left[M_L^{(1)}\right]\{\bar{w}\}, \tag{64}$$

where

$$\left[M_L^1\right]_{mj} = \rho_L R C_n \sum_{s=1}^{\infty} \frac{4\gamma_{ms}\gamma_{js}}{(2s-1)\pi} \frac{I_n\left(\dfrac{2s-1}{2}\pi\dfrac{R}{H}\right)}{I_n'\left(\dfrac{2s-1}{2}\pi\dfrac{R}{H}\right)}, \tag{65}$$

The reference kinetic energy of fluid due to the sloshing Eq.(54) for the asymmetric modes $(n > 0)$ becomes

$$T_L^{*(1-S)} = \frac{1}{2}\{\bar{w}\}^T \left[M_L^{(1-S)}\right]\{F\}, \tag{66}$$

where

$$\left[M_L^{(1-S)}\right]_{mk} = \rho_L C_n R J_n(\varepsilon_{nk})\zeta_{mkn}, \tag{67}$$

and for axisymmetric modes $(n=0)$, $\left[M_L^{(1-S)}\right]$ is modified by adding the following row to Eq. (67):

$$\left[M_L^{(1-S)}\right]_{m0} = \rho_L C_0 R \tau_m, \tag{68}$$

From Eqs. (39), (60), (62), (64) and (66), the following matrix equation is obtained

$$\begin{bmatrix} \left[K_s^{11}\right] & \left[K_s^{12}\right] & \left[K_s^{13}\right] & 0 \\ \left[K_s^{12}\right] & \left[K_s^{22}\right] & \left[K_s^{23}\right] & 0 \\ \left[K_s^{13}\right] & \left[K_s^{23}\right] & \left[K_s^{33}\right] & 0 \end{bmatrix} \begin{Bmatrix} \{q\} \\ \{F\} \end{Bmatrix} - \omega^2 \begin{bmatrix} \left[M_s^{11}\right] & 0 & 0 & 0 \\ 0 & \left[M_s^{22}\right] & 0 & 0 \\ 0 & 0 & \left[M_s^{33}\right]+\left[M_L^{(1)}\right] & \left[M_L^{(1-S)}\right] \end{bmatrix} \begin{Bmatrix} \{q\} \\ \{F\} \end{Bmatrix} = 0, \tag{69}$$

Matrix form of sloshing conditions, Eqs.(48) for the asymmetric modes $(n>0)$ becomes

$$\begin{bmatrix} 0 & 0 & \left[K_{ssl}\right] & \left[K_{sl}\right] \end{bmatrix} \begin{Bmatrix} \{q\} \\ \{F\} \end{Bmatrix} - \omega^2 \begin{bmatrix} 0 & 0 & 0 & \left[M_{sl}\right] \end{bmatrix} \begin{Bmatrix} \{q\} \\ \{F\} \end{Bmatrix} = 0, \tag{70}$$

where

$$\left[M_{sl}\right]_{ik} = \delta_{ik} \frac{\alpha_{nk}}{g} \cosh\left(\varepsilon_{nk} \frac{H}{R}\right), \tag{71}$$

$$\left[K_{sl}\right]_{ik} = \delta_{ik} \frac{\varepsilon_{nk}\alpha_{nk}}{R} \sinh\left(\varepsilon_{nk} \frac{H}{R}\right), \tag{72}$$

$$\left[K_{ssl}\right]_{km} = -\sum_{s=1}^{\infty} A_{mns} \left(\frac{2s-1}{2}\pi \frac{1}{H}\right) \sin\left(\frac{2s-1}{2}\pi\right) \beta_{nsk}, \tag{73}$$

For axisymmetric modes $(n=0)$, $\left[M_{sl}\right]$, $\left[K_{sl}\right]$ and $\left[K_{ssl}\right]$ are modified by adding the following rows to Eqs. (71), (72) and (73):

$$\left[M_{sl}\right]_{00} = \frac{1}{2g}, \tag{74}$$

$$\left[K_{sl}\right]_{0k} = 0, \tag{75}$$

$$\left[K_{ssl}\right]_{0m} = -\sum_{s=1}^{\infty} A_{m0s} \left(\frac{2s-1}{2}\pi \frac{1}{H}\right) \sin\left(\frac{2s-1}{2}\pi\right) \eta_{s0}, \tag{76}$$

By combining Eqs. (69) and (70), the eigenvalue problem of fluid structure interaction of shell can be written in the following form:

$$\begin{bmatrix} \left[K_s^{11}\right] & \left[K_s^{12}\right] & \left[K_s^{13}\right] & 0 \\ \left[K_s^{12}\right] & \left[K_s^{22}\right] & \left[K_s^{23}\right] & 0 \\ \left[K_s^{13}\right] & \left[K_s^{23}\right] & \left[K_s^{33}\right] & 0 \\ 0 & 0 & \left[K_{ssl}\right] & \left[K_{sl}\right] \end{bmatrix} \begin{Bmatrix} \{q\} \\ \{F\} \end{Bmatrix} - \omega^2 \begin{bmatrix} \left[M_s^{11}\right] & 0 & 0 & 0 \\ 0 & \left[M_s^{22}\right] & 0 & 0 \\ 0 & 0 & \left[M_s^{33}\right]+\left[M_L^{(1)}\right] & \left[M_L^{(1-S)}\right] \\ 0 & 0 & 0 & \left[M_{sl}\right] \end{bmatrix} \begin{Bmatrix} \{q\} \\ \{F\} \end{Bmatrix} = 0, \tag{77}$$

4 RESULTS AND DISCUSSION

Firstly, the validity and accuracy of the results obtained from the present formulation has been checked. Case study #1 has been presented to compare the bulging and sloshing frequencies of the simply supported isotropic cylindrical shell from the present method with corresponding results in Kondo [10]; Gupta and Hutchinson [6]; Amabili [2].

In case study #2, the bulging frequencies of laminated composite cylindrical shell are compared with those obtained by finite element method from Chiba and Abe [5]. Subsequent case studies investigate the effects of different parameters of the fluid-filled laminated composite cylindrical shell on the natural frequencies.

Case study #1: Fluid-filled isotropic shell with simply supported boundary conditions

In order to verify the theoretical results and comparison with the corresponding results published in previous studies, the following dimensions and material properties of the simply supported steel shell are taken: $R = 25(m), L = 30(m)$, $H = 21.6(m), h_s = 0.03(m), E = 206(GPa), \rho = 8750(Kg/m^3), v = 0.3$ and the density of water is $\rho_L = 1000(Kg/m^3)$. In Tables1. and 2, the bulging and sloshing frequencies, obtained from present work and previous works (Kondo [10]; Gupta and Hutchinson [6]; Amabili [2]), have been presented. Table 1. compares the five natural frequencies of axisymmetric vibrations. As it is observed the greatest difference in bulging mode results belongs to fifth mode with respect to Kondo[10] and the greatest difference in sloshing mode results belongs to third mode with respect to results of Kondo [10]. It is observed that generally, there exists a good agreement between present results and those of Amabili [2].

In Table 2. the five lowest natural frequencies of asymmetric vibrations are compared with the results of Amabili [2]. The greatest difference which belongs to fifth mode of bulging mode is about 1.2%.

Table 1
Comparison of bulging and sloshing frequencies (*rad/s*) for the simply supported fluid-filled isotropic cylindrical shell for ($n = 0, R = 25(m), L = 30(m), H = 21.6(m), h_s = 0.03(m), \rho_L = 1000(Kg/m^3), E = 206(GPa), v = 0.3, \rho_s = 7850(Kg/m^3)$).

Mode	Bulging modes				Sloshing modes			
	Present study	Kondo	Gupta and Hutchinson	Amabili [2]	Present study	Kondo	Gupta and Hutchinson	Amabili [2]
1	22.224	22.096	22.349	22.244	1.2278	1.2238	1.2244	1.2244
2	44.877	43.762	44.170	44.010	1.6604	1.6582	1.6591	1.6591
3	57.163	56.829	58.244	57.191	2.0071	1.9969	1.9979	1.9980
4	68.600	66.888	69.513	67.291	2.2873	2.2853	2.2864	2.2865
5	77.392	75.347	79.189	75.850	2.5432	2.5409	2.5422	2.5422

Table 2
Comparison of bulging and sloshing frequencies (*rad/s*) for the simply supported fluid-filled isotropic cylindrical shell for ($n = 4, R = 25(m), L = 30(m), H = 21.6(m), h_s = 0.03(m), \rho_L = 1000(Kg/m^3), E_s = 206(GPa), v = 0.3, \rho_s = 7850(Kg/m^3)$).

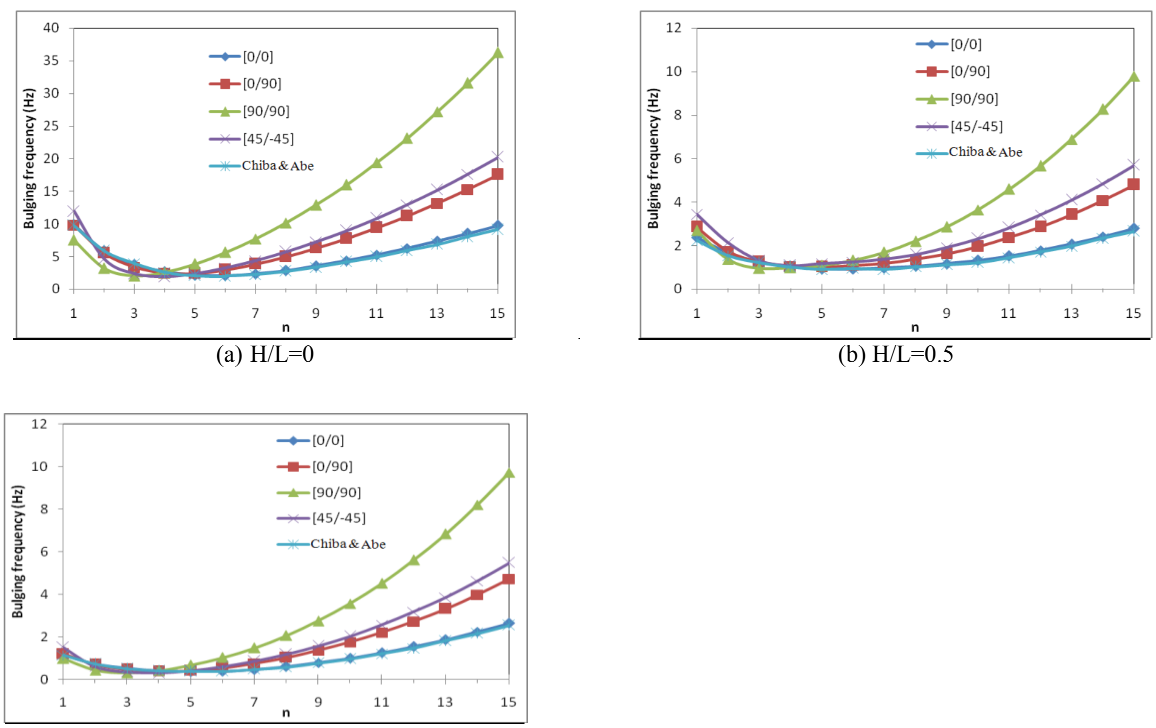
Mode	Bulging modes		Sloshing modes	
	Present study	Amabili [2]	Present study	Amabili [2]
1	13.610	13.658	1.447	1.4425
2	34.245	34.441	1.915	1.9081
3	49.269	49.692	2.239	2.2305
4	61.193	61.877	2.514	2.5027
5	70.951	71.804	2.758	2.7444

Case study #2: Fluid-filled laminated composite shell with different boundary conditions

The second example is a two layer laminated composite cylindrical shell with clamped-free, simply supported and clamped-clamped boundary conditions and partially filled with water. To comply with the reported solutions of Chiba and Abe [5], the following dimensions and material properties of the composite shell are taken: $R = 7.22(m)$ $L = 21.96(m), h = 0.0127(m)$ for each layer, $E_1 = 76(GPa), E_2 = 5.5(GPa), G_{12} = 2.31(GPa), \rho_s = 1460(Kg/m^3), v_{12} = 0.34,$ $\rho_L = 1000(Kg/m^3)$.

Figs. 2-4 illustrate the variation of bulging natural frequencies of empty, half and full filled cylindrical shell with different boundary conditions and fiber angles. In Figs. 2 the first fifteen bulging natural frequencies of (a) empty, (b) half filled and (c) full filled cylindrical shell with clamped-free boundary condition for [0°/0°], [0°/90°], [90°/90°] and [+45°/-45°] laminations are presented. The present results for fiber angle [0°/0°] are compared with the finite element results provided by Chiba and Abe [5] in this figure. There is a good agreement between the results and finite element results. Figs. 3 illustrate the bulging natural frequencies of (a) empty, (b) half filled and (c) full filled simply supported cylindrical shell. In Figs. 4 the bulging natural frequencies of (a) empty, (b) half filled and (c) full filled cylindrical shell with clamped-clamped boundary conditions are shown.

From these figures it is easily inferred that the general shapes of the natural frequency curves of the partially fluid-filled shells are similar to those for the corresponding empty shells. However, for a given circumferential wave number, the bulging natural frequencies of the partially fluid-filled shells are lower than those of the corresponding empty shells. This is because of the fact that the fluid filled in the shell has increased the total mass of the tank. Besides, for *n*≤ *6*, the bulging natural frequencies of shell with different fiber angles are almost identical except fiber angle [+45°/-45°] in simply supported and clamped-clamped shells. However for *n*>*6*, difference among bulging natural frequencies of shell for different fiber angles increases. In case of fiber angle [+45°/-45°], the bulging natural frequencies of shell are larger than that for the corresponding other fiber angles, but the bulging natural frequencies of shell is smaller than that of the corresponding fiber angle [90/90] regardless the height of fluid and boundary condition.

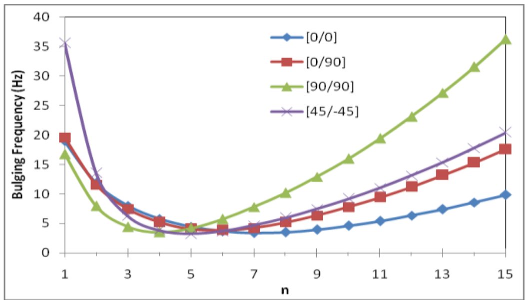


(a) H/L=0

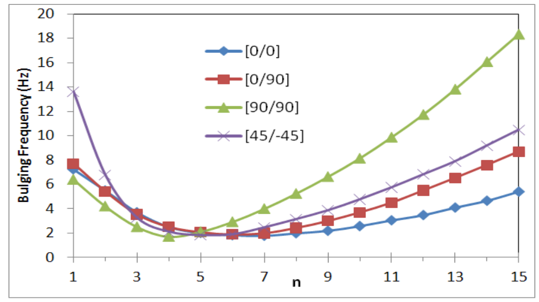
(b) H/L=0.5

(c) H/L=1

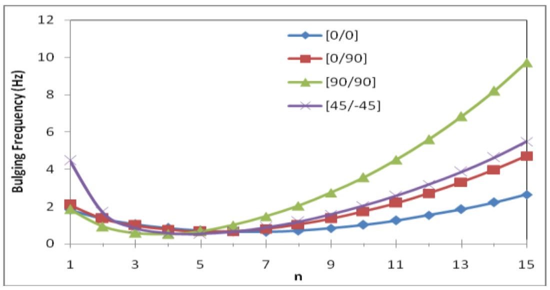
Fig.2
The Bulging frequencies (*Hz*) for the clamped-free laminated composite cylindrical shell; ($R = 7.22(m), L = 21.96(m), h = 0.0127(m)$ $\rho_L = 1000(Kg/m^3), E_1 = 76(GPa), E_2 = 5.5(GPa), G_{12} = 2.31(GPa), v_{12} = 0.34, \rho_s = 1460(Kg/m^3)$).



(a) H/L=0

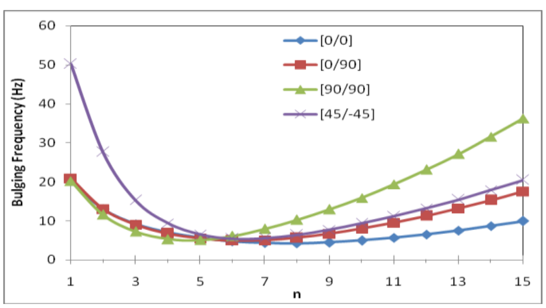


(b) H/L=0.5

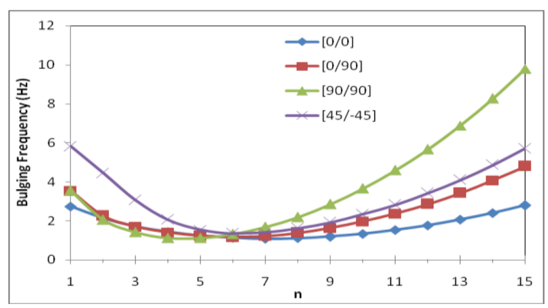


(c) H/L=1

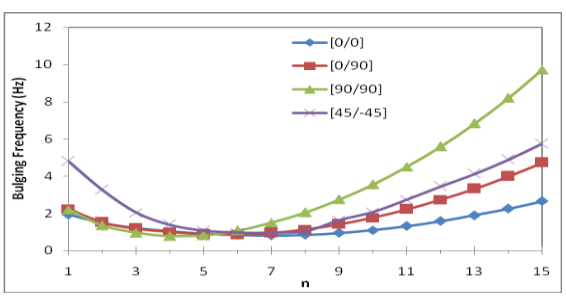
Fig.3
The Bulging frequencies (*Hz*) for the simply supported fluid-filled laminated composite cylindrical shell; $R=7.22(m), L=21.96(m)$, ($h=2\times0.0127(m), \rho_L=1000(Kg/m^3), E_1=76(GPa), E_2=5.5(GPa), G_{12}=2.31(GPa), v_{12}=0.34, \rho_s=1460(Kg/m^3)$).



(a) H/L=0



(b) H/L=0.5



(c) H/L=1

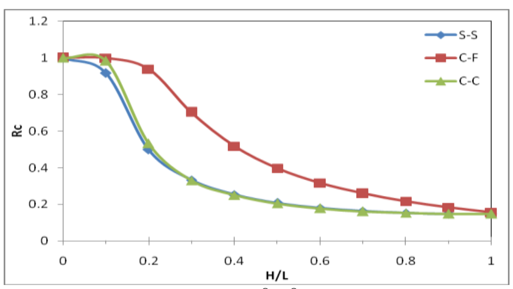
Fig.4
The Bulging frequencies (*Hz*) for the clamped-clamped fluid-filled laminated composite cylindrical shell; $R=7.22(m), L=21.96(m)$, ($h=2\times0.0127(m), \rho_L=1000(Kg/m^3), E_1=76(GPa), E_2=5.5(GPa), G_{12}=2.31(GPa), v_{12}=0.34, \rho_s=1460(Kg/m^3)$).

Case study #3: Effect of the fluid level on the bulging frequencies of the fluid filled laminated composite cylindrical shell

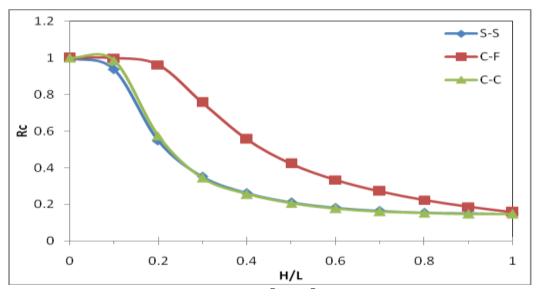
Consider the free vibration of fluid filled laminated composite cylindrical shell with the same material and geometry as case study #2 and with different fluid levels. Figs.5 show the first fundamental bulging frequency ratios (*Rc*) of fluid filled composite cylindrical shell with different boundary conditions versus height ratio (H/L) of contained fluid for n=4 and [0°/0°], [0°/90°], [90°/90°] and [+45°/-45°] laminations, respectively.

The fundamental bulging frequency ratio is defined as the ratio of the bulging frequency of partially fluid-filled shell to the natural frequency of the empty shell. From Figs.5, it can be inferred that for all laminations, by increasing the height of the fluid the fundamental bulging frequency ratio decreases. The effect of the fluid level on the natural frequency of a shell mostly depends on boundary conditions, whereas different fiber angles have insignificant role on trend of reduction of fundamental bulging frequency ratio. The changes in fundamental frequency ratio of simply supported and clamped-clamped fluid filled shells approximately have the same trends of reduction.

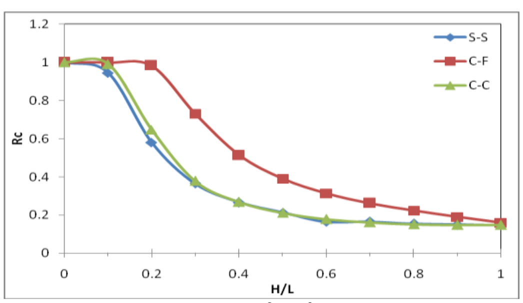
For small *H/L* ratios, by increasing height of the fluid, the reduction of the fundamental frequency ratio for a clamped-free shell has a slower trend than that for the corresponding simply supported and clamped-clamped shells and is almost constant, whereas the trend of reduction in natural frequency for a simply supported shell is faster than that of the clamped-free shell while the curve corresponding to clamped-clamped shell has an overshoot and then a sudden decrease. For *H/L≤ 0.1*, the values and variations of the fundamental frequency ratio for all boundary conditions are almost identical. However, for *0.1<H/L<0.3*, the bulging natural frequencies of clamped-free shell have both a larger value and smoother variations than the other boundary conditions. Finally, for *0.3<H/L<1*, the rate of reduction for the fundamental frequency ratio of the clamped-free shell is larger than simply supported and clamped-clamped shells.



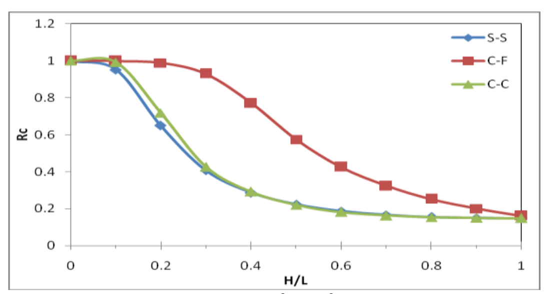
(a) [0°/0°]



(b) [0°/90°]



(c) [90°/90°]



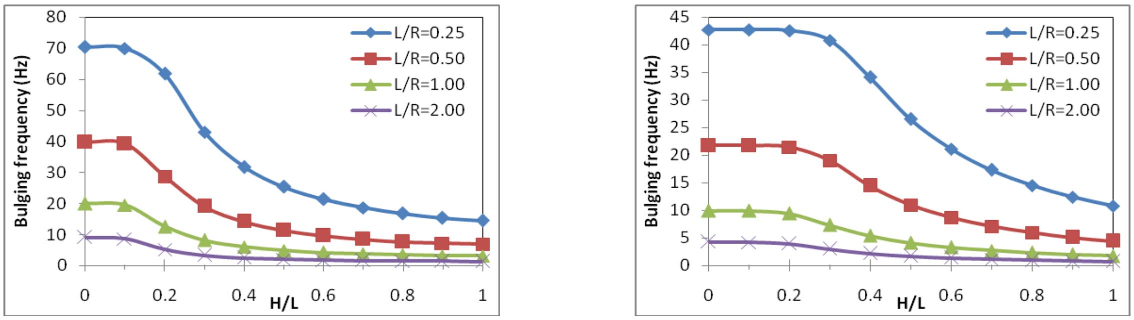
(d)[+45°/-45°]

Fig.5
The fundamental bulging frequency ratio for the fluid-filled laminated composite cylindrical shell versus height of fluid;
($R=7.22(m), L=21.96(m), h_s=0.0254(m), \rho_L=1000(Kg/m^3), E_1=76(GPa), E_2=5.5(GPa), G_{12}=2.31(GPa), v_{12}=0.34, \rho=1460(Kg/m^3)$): (a) [0/0], (c)[0/90], (b)[90/90], (d)[+45/-45].

Case study #4: Effect of the shell geometry on bulging frequencies of fluid filled shell

To investigate the effect of the length to radius ratio on the free vibration of fluid filled laminated composite cylindrical shell, again the shell of case study #2 with simply supported and clamped-free boundary conditions are considered. The fiber angle is taken [0/90] lamination. Figs. 6 show the first bulging frequencies for n=4 versus height of fluid for different length to radius ratios of shell. From these figures, it can be seen that at constant fluid level in shell, by increasing the shell length the fundamental natural frequency decreases. For shorter shells, the effect of length on fundamental natural frequency is more sensible, especially for simply supported boundary condition.

As expected, for a longer shell, there is more remarkable effect of the filled fluid on its bulging natural frequency. This is because of the fact that a longer shell becomes more flexible, and it is thus susceptible to the filled fluid.



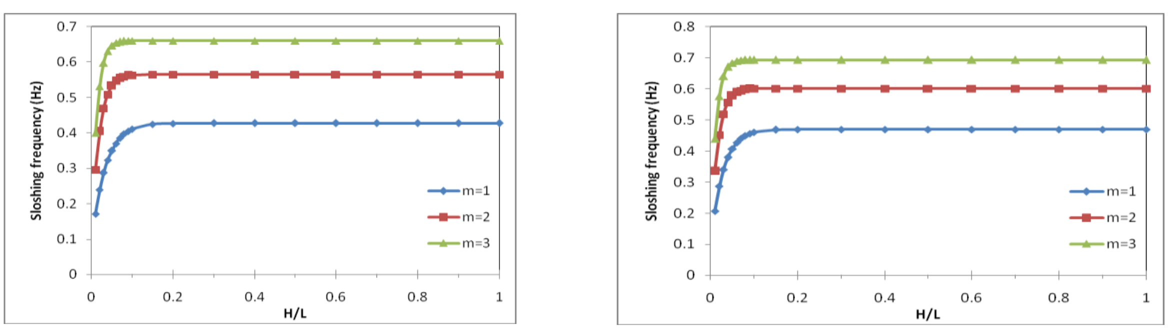
(a) simply supported (b) clamped-free

Fig.6
The bulging frequency for the fluid-filled laminated composite cylindrical shell versus height of fluid with different ratio of length to radius of shell;($n=4, R=7.22(m), h_s=0.0254(m), \rho_L=1000(Kg/m^3), E_1=76(GPa), E_2=5.5(GPa), G_{12}=2.31(GPa), v_{12}=0.34, \rho=1460(Kg/m^3)$.

Case study #5: Effect of height of fluid on sloshing frequencies of fluid filled laminated composite cylindrical shell

In this part, the effect of fluid height on sloshing natural frequencies of fluid filled laminated composite cylindrical shell vibration is studied. Consider the composite shell of case study #2 with simply supported boundary condition and the fiber angle is taken [0/90], because according to study #1 the sloshing frequencies of the shells with the same radius and length are approximately equal regardless of boundary conditions and fiber angle. This phenomenon can be explained so that the deformation of shell does not occur in the sloshing vibration mode and the shell is stiff enough for sloshing mode but an extremely flexible shell will show a larger displacement.

Figs. 7 show the first three sloshing frequencies (*m*=1,2, 3) for (a) *n*= 4 and (b) *n*= 5 versus height of fluid. From these figures, it can be easily seen that sloshing natural frequencies grow quickly by increasing contained fluid up to 0.1<*H/L*<0.2, and then frequencies remain at a constant value till the shell is filled completely. This is because of the fact that sloshing frequency is only related to free surface of fluid; when depth of fluid becomes inasmuch as surface of fluid could move easily, the height of fluid has no significant effect on sloshing frequency.



(a) *n*=4 (b) *n*=5

Fig.7
The sloshing frequency for the simply supported fluid-filled laminated composite cylindrical shell versus height of fluid;
$R=7.22(m), L=21.96(m), h_s=0.0254(m), \rho_L=1000(Kg/m^3), E_1=76(GPa), E_2=5.5(GPa), G_{12}=2.31(GPa), v_{12}=0.34, \rho=1460(Kg/m^3)$.

5 CONCLUSIONS

A semi-analytical method for the coupled vibration of fluid filled laminated composite cylindrical shell with various boundary conditions has been presented by using Rayleigh-Ritz method. The results are validated by comparing with the previous works on the vibration of fluid filled isotropic shell and the results obtained from finite element method. Numerical case studies are given for investigating some related parameters on the bulging and sloshing frequency of fluid-filled shell and the following conclusions may be drawn:

- The variation of the bulging frequency of a partially fluid-filled laminated composite circular cylindrical shell is similar to that for the corresponding empty shell, as circumferential wave number increases.
- The effect of the filled fluid on the bulging frequency of a laminated composite circular cylindrical shell increases with the length to radius ratio.
- The effect of the filled fluid on the sloshing frequency of a laminated composite circular cylindrical shell decreases with height of fluid.

APPENDIX A

$$\left[K_s^{11}\right]_{ij}=\int_0^L\left\{C_n\frac{A_{11}}{\alpha_i\alpha_j}\frac{\partial^2\psi_i}{\partial x^2}\frac{\partial^2\psi_j}{\partial x^2}+S_n\left(\frac{n}{R}\right)^2\frac{A_{66}}{\alpha_i\alpha_j}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}\right\}Rdx, \tag{A.1}$$

$$\left[K_s^{12}\right]_{ij}=\int_0^L\left\{\left[C_n\left(\frac{n}{R}\right)\frac{A_{12}}{\alpha_i}+C_n\left(\frac{n}{R^2}\right)\frac{B_{12}}{\alpha_i}\right]\frac{\partial^2\psi_i}{\partial x^2}\psi_j-\left[S_n\left(\frac{n}{R}\right)\frac{A_{66}}{\alpha_i}+2S_n\left(\frac{n}{R^2}\right)\frac{B_{66}}{\alpha_i}\right]\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}\right\}Rdx, \tag{A.2}$$

$$\left[K_s^{13}\right]_{ij}=\int_0^L\left\{\left[C_n\left(\frac{1}{R}\right)\frac{A_{12}}{\alpha_i}+C_n\left(\frac{n}{R}\right)^2\frac{B_{12}}{\alpha_i}\right]\frac{\partial^2\psi_i}{\partial x^2}\psi_j-C_n\frac{B_{11}}{\alpha_i}\frac{\partial^2\psi_i}{\partial x^2}\frac{\partial^2\psi_j}{\partial x^2}-2S_n\left(\frac{n}{R}\right)^2\frac{B_{66}}{\alpha_i}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}\right\}Rdx, \tag{A.3}$$

$$\left[K_s^{22}\right]_{ij}=\int_0^L\left\{\left[C_nA_{22}\left(\frac{n}{R}\right)^2+2C_nB_{22}\left(\frac{n^2}{R^3}\right)+C_nD_{22}\left(\frac{n^2}{R^4}\right)\right]\psi_i\psi_j+\left[S_nA_{66}+4S_nB_{66}\left(\frac{1}{R}\right)+4S_nD_{66}\left(\frac{1}{R^2}\right)\right]\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}\right\}Rdx, \tag{A.4}$$

$$\begin{aligned}\left[K_s^{23}\right]_{ij}=\int_0^L\Bigg\{&\left[C_nA_{22}\left(\frac{n}{R^2}\right)+C_nB_{22}\left(\frac{n}{R}\right)^3+C_nB_{22}\left(\frac{n}{R^3}\right)+C_nD_{22}\left(\frac{n^3}{R^4}\right)\right]\psi_i\psi_j-\left[C_nB_{12}\left(\frac{n}{R}\right)+C_nD_{12}\left(\frac{n}{R^2}\right)\right]\psi_i\frac{\partial^2\psi_j}{\partial x^2}\\&+\left[2S_nB_{66}\left(\frac{n}{R}\right)+4S_nD_{66}\left(\frac{n}{R^2}\right)\right]\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}\Bigg\}Rdx,\end{aligned} \tag{A.5}$$

$$\begin{aligned}\left[K_s^{33}\right]_{ij}=\int_0^L\Bigg\{&\left[C_nA_{22}\left(\frac{1}{R^2}\right)+2C_nB_{22}\left(\frac{n^2}{R^3}\right)+C_nD_{22}\left(\frac{n}{R}\right)^4\right]\psi_i\psi_j-\left[C_nB_{12}\left(\frac{1}{R}\right)+C_nD_{12}\left(\frac{n}{R}\right)^2\right]\psi_i\frac{\partial^2\psi_j}{\partial x^2}\\&-\left[C_nB_{12}\left(\frac{1}{R}\right)+C_nD_{12}\left(\frac{n}{R}\right)^2\right]\frac{\partial^2\psi_i}{\partial x^2}\psi_j+C_nD_{11}\frac{\partial^2\psi_i}{\partial x^2}\frac{\partial^2\psi_j}{\partial x^2}+4S_nD_{66}\left(\frac{n}{R}\right)^2\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}\Bigg\}Rdx,\end{aligned} \tag{A.6}$$

$$\left[M_s^{11}\right]_{ij}=\rho_TC_n\int_0^L\left\{\frac{1}{\alpha_i\alpha_j}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}\right\}Rdx, \tag{A.7}$$

$$\left[M_s^{22}\right]_{ij}=\rho_TS_n\int_0^L\left\{\psi_i\psi_j\right\}Rdx, \tag{A.8}$$

$$\left[M_s^{33}\right]_{ij} = \rho_T C_n \int_0^L \left\{\psi_i \psi_j\right\} Rdx, \tag{A.9}$$

where

$$C_n = \begin{cases} 2\pi; & n = 0 \\ \pi; & n > 1 \end{cases}, \qquad S_n = \begin{cases} 0; & n = 0 \\ \pi; & n > 1 \end{cases} \tag{A.10}$$

REFERENCES

[1] Amabili M., 1996, Free vibration of partially filled, horizontal cylindrical shells, *Journal of Sound and Vibration* **191**(5): 757-780.

[2] Amabili M., 1997, Shell-plate interaction in the free vibrations of circular cylindrical tanks partially filled with a liquid: The artificial spring method, *Journal of Sound and Vibration* **199**(3): 431-452.

[3] Amabili M., Paidoussis M. P., Lakis A. A., 1998,Vibrations of partially filled cylindrical tanks with ring-stiffeners and flexible bottom, *Journal of Sound and Vibration* **213**(2): 259-299.

[4] Balamurugan V., Narayanan S., 2001, Shell finite element for smart piezoelectric composite plate/shell structures and its application to the study of active vibration control, *Finite Elements in Analysis and Design* **37**: 713-738.

[5] Chiba M., Abe K., 1999, Nonlinear hydroelastic vibration of a cylindrical tank with an elastic bottom containing liquid-analysis using harmonic balance method, *Thin-Walled Structure* **34**: 233-260.

[6] Gupta R. K., Hutchinson G. L., 1988, Free vibration analysis of liquid storage tanks, *Journal of Sound and Vibration* **122**: 491-506.

[7] Goncalves P. B., Ramos N. R. S. S., 1996, Free vibration analysis of cylindrical tanks partially filled with liquid, *Journal of Sound and Vibration* **195**(3): 429-444.

[8] Jafari A. A., Bagheri M., 2006, Free vibration of non-uniformly ring stiffened cylindrical shells using analytical, experimental and numerical methods, *Thin-Walled Structures* **44**: 82-90.

[9] Jeong K. H., Kim K. J., 1998, Free vibration of a circular cylindrical shell filled with bounded compressible fluid, *Journal of Sound and Vibration* **217**(2): 197-221.

[10] Kondo H., 1981, Axisymmetric vibration analysis of a circular cylindrical tank, *Bulletin of the Japan Society of Mechanical Engineers* **24**: 215-221.

[11] Kim Y. W., Lee Y. S., 2002,Transient analysis of ring-stiffened composite cylindrical shells with both edges clamped, *Journal of Sound and Vibration* **252**(1): 1-17.

[12] Kim Y. W., Lee Y. S., Ko S. H., 2004, Coupled vibration of partially fluid-filled cylindrical shells with ring-stiffeners, *Journal of Sound and Vibration* **276**:869-897.

[13] Leissa A. W., 1973, *Vibration of Shells*, NASA SP-288, Washington, Government Printing Office.

[14] Lee L. T., Lu J. C., 1995, Free vibration of cylindrical shells filled with liquid, *Computers & Structures* **54**(5): 997-1001.

[15] Liu M. L., To C. W. S., 2003, Free vibration analysis of laminated composite shell structures using hybrid strain based layerwise finite elements, *Finite Elements in Analysis and Design* **40**(1): 83-120.

[16] Mazuch T., Horacek J., Trnka J., Vesely J., 1996, Natural modes and frequencies of thin clamped-free steel cylindrical storage tank, *Journal of Sound and Vibration* **193**:669-690.

[17] Ramasamy R., Ganesan N., 1999,Vibration and damping analysis of fluid filled orthotropic cylindrical shells with constrained viscoelastic damping, *Computers and Structures* **70**: 363-376.

[18] Wang R. T., Lin Z. X., 2006, Vibration analysis of ring-stiffened cross-ply laminated cylindrical shells, *Journal of Sound and Vibration* **295**: 964-987.

[19] Yu L., Cheng L., Yam L. H., Yan Y. J., Jiang J. S., 2007, Experimental validation of vibration-based damage detection for static laminated composite shells partially filled with fluid, *Computers and Structures* **79**: 288-299.

[20] Zhao X., Liew K. M., Ng T. Y., 2002,Vibrations of rotating cross-ply laminated circular cylindrical shells with stringer and ring stiffeners, *International Journal of Solids and Structures* **39**:529-545.