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# Influence of Heterogeneity on Rayleigh Wave Propagation in an Incompressible Medium Bonded Between Two Half-Spaces

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**ABSTRACT**

The present investigation deals with the propagation of Rayleigh wave in an incompressible medium bonded between two half-spaces. Variation in elastic parameters of the layer is taken linear form. The solution for layer and half-space are obtained analytically. Frequency equation for Rayleigh waves has been obtained. It is observed that the heterogeneity and width of the incompressible medium has significant effect on the phase velocity of Rayleigh waves. Some particular cases have been deduced. Results have been presented by the means of graph. Also the findings are exhibited through graphical representation and surface plot.



**Keywords:** Heterogeneity; Incompressibility; Frequency equation; Rayleigh waves.

## 1 INTRODUCTION

ELASTIC surface waves in isotropic elastic solids, discovered by Rayleigh [1] more than 120 years ago, have been studied extensively and exploited in a wide range of applications in seismology, acoustics, geophysics, telecommunications industry and material science. For the Rayleigh wave, its speed is a fundamental quantity which attracts the researchers of seismology, geophysics and in other fields of physical sciences. The formation and alteration of the oceanic lithosphere are important components of the solid earth cycles and geodynamics theme. More than two third of the earth crust is of oceanic crust type, made of different layers with varying material properties. The sedimentary layer of oceanic crust exhibits anisotropy and/or inhomogeneity. Oceanic crust is continuously being created at mid-ocean ridges. As plates diverge at these ridges, magma rises into the upper mantle and crust. As it moves away from the ridge, the lithosphere becomes cooler and denser, and sediment gradually builds on top of it. On other hand, Rayleigh waves play drastic role in damages during earthquake due to their nature of propagation. Also, they help to explain the crucial seismic observations that cannot be done by body wave theory. The above fact, demands an analytical study for propagation behavior of Rayleigh type waves near the ocean ridges. After the pioneer works of Rayleigh, many investigators have solved the problem of Rayleigh waves in a half-space and one or more superficial layers situated over it. A good amount of literature about Rayleigh waves may be found in the standard books of Love [4] and Stonely [5]. Many investigators have been studied the propagation of elastic waves in isotropic medium. Propagation of elastic waves in a system consisting of a liquid layer of finite depth overlying an isotropic half-space have been discussed by Stonely [6], Biot[7] and Tolstoy[8]. The dispersive

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properties of liquid layer overlying a semi-infinite homogeneous, transversely isotropic half-space have been studied by Abubaker and Hudson [9]. Carcione [10] has shown the possibility of propagation of two types of Rayleigh waves in isotropic viscoelastic media. The equation of motion and the constitutive relation of the isotropic linear viscoelastic solid are derived in terms of the complex Lame parameters by Carcione. Destrade [11] has derived the secular equation for surface acoustic waves propagating on an orthotropic incompressible half-space. This contribution helped others to obtain the propagation patterns of surface waves in different elastic properties. Rudzki [12] studied the propagation of an elastic surface wave in a transversely isotropic medium. Vinh and Ogden [13] have obtained an explicit formula for the speed of Rayleigh waves in orthotropic compressible elastic material by using the theory of cubic equations. Singh and Kumar [14] have analyzed the problem of propagation of Rayleigh waves due to a finite rigid barrier in a shallow ocean. Gupta [15] studied the Propagation of Rayleigh Waves in a Pre-stressed layer over a Pre-stressed half-space. He has notice that the frequency equation of Rayleigh waves are affected due to the initial stresses present in the equation. Vinh et al [16] have investigated the Rayleigh waves in an isotropic elastic half-space coated by a thin isotropic elastic layer with smooth contact. Pal et al [17] have shown the propagation of Rayleigh waves in anisotropic layer overlying a semi-infinite sandy medium. They have suggested that the sandiness of materials produces heterogeneity in the medium and has a great impact on the phase velocity. They study the behavior of Rayleigh waves when upper boundary plane is considered as free surface. The heterogeneity and anisotropy plays a key role in the seismic wave propagation. Gupta and Kumar [18] have studied the propagation of Rayleigh wave over the pre-stressed surface of a heterogeneous medium. Rayleigh waves in non-homogeneous granular medium have been investigated by Kakar and Kakar [19]. They have shown the effect of heterogeneity in granular medium. Dutta [20] explained the Rayleigh waves in two layer heterogeneous medium. Singh [21] has investigated the wave propagation in an incompressible transversely isotropic medium. Vinh and Link [22] have analyzed the Rayleigh waves in an incompressible elastic half-space overlaid with a water layer under the effect of gravity. Singh [23] discussed the Rayleigh wave in an initially stressed transversely isotropic dissipative half-space. Rayleigh Waves in a Homogeneous Magneto-Thermo Voigt-Type Viscoelastic Half-Space under Initial Surface Stresses has been discussed by Kakar [24]. Study on Rayleigh wave propagation in structures having planer boundaries is important leading to better understanding of seismic wave behavior. To be specific, Rayleigh wave propagation is significantly affected by the height of the layer.

In the present investigation, we have shown the effect of heterogeneity and width of the layer on the propagation of Rayleigh waves in an incompressible layer bonded between liquid half-space and transversely isotropic half-space. A model has been considered to represent the part of real earth where the crustal part (oceanic crust: the crust that lies at the ocean floor) appear to be sandwiched between ocean and upper mantle. Elastic conditions of the layer are taken by following the geophysical fact that the oceanic crust is thinner but denser and its different layers exhibit variations in elastic parameters. An analytical study is carried out to highlight the effect of different physical parameters on the velocity profile of considered surface wave. As the outcome of the study, it is found that wave number, wavelength, rigidity and density have their substantial effect on the phase velocity of Rayleigh waves. Numerical computation and graphical demonstration has been done to exhibit the findings. Some particular cases have been deduced.

## 2 FORMULATION OF THE PROBLEM

We consider an incompressible heterogeneous medium sandwiched between liquid half-space and transversely isotropic half-space as shown in Fig. 1. We consider a rectangular coordinate system in such a way that $x$ axis in the direction of propagation and $z$ axis pointing vertically downward. Heterogeneity in the intermediate layer has been taken in the form of $\mu_2 = \mu_1(1+\alpha z)$ and $\rho_2 = \rho_1(1+\alpha z)$ where $\mu_1$ and $\rho_1$ refers to the rigidity and density at $z = 0$, respectively.

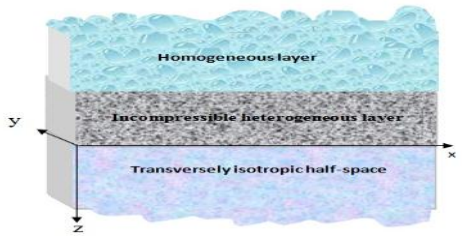


**Fig.1**
Geometry of the problem.

## 3 BASIC EQUATIONS AND SOLUTIONS

*3.1 Solution for the upper half-space*

Equation of motion for the upper half-space in terms of the displacement potential $\phi$ is given as:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=\frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2} \tag{1}$$

where $\alpha=\sqrt{\frac{\lambda_0}{\rho_0}}$ is the velocity of the dilatational wave in the liquid, $\rho_0$ is the density and $\lambda_0$ is the bulk modulus of upper layer. Displacement components $u_1, w_1$ and pressure $p$ for upper half-space are given by

$$u_1=\frac{\partial\phi}{\partial x}, w_1=\frac{\partial\phi}{\partial z} \quad \text{and} \quad p=-\sigma_{zz}=-\lambda_0\nabla^2\phi \tag{2}$$

where $\sigma_{zz}$ is the normal component of stress in the liquid. We seek wave the solution of Eq. (1) of the form

$$\phi=\phi_0 e^{ik(x-ct)} \tag{3}$$

Introducing Eq. (3) in Eq. (1), we obtain

$$\phi_0''(z)-\phi_0\left[1-\frac{c^2}{\alpha^2}\right]k^2=0 \tag{4}$$

On solving Eq. (4), we obtained

$$\phi_0=A_1e^{\beta_1 kz}+A_2e^{-\beta_1 kz} \tag{5}$$

where

$$\beta_1=\sqrt{1-\frac{c^2}{\alpha^2}} \tag{6}$$

Introducing Eq. (5) in Eq. (3), we get

$$\phi(x,z,t)=\left(A_1e^{\beta_1 kz}+A_2e^{-\beta_1 kz}\right)e^{ik(x-ct)} \tag{7}$$

Hence, displacement components for the upper half-space are

$$u_1=ik\left(A_1e^{\beta_1 kz}+A_2e^{-\beta_1 kz}\right)e^{ik(x-ct)}, \tag{8}$$

$$w_1=\beta_1 k\left(A_1e^{\beta_1 kz}-A_2e^{-\beta_1 kz}\right)e^{ik(x-ct)}. \tag{9}$$

*3.2 Solution for the intermediate layer*

Equations of motion for the incompressible layer are

$$-\frac{\partial \xi}{\partial x}+\mu_2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)u_2=\rho_2\frac{\partial^2 u_2}{\partial t^2} \tag{10}$$

$$-\frac{\partial \xi}{\partial z}+\mu_2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)w_2=\rho_2\frac{\partial^2 w_2}{\partial t^2} \tag{11}$$

where

$$u_2=\frac{\partial \phi_2}{\partial x}+\frac{\partial \psi_2}{\partial z} \tag{12}$$

$$w_2=\frac{\partial \phi_2}{\partial z}-\frac{\partial \psi_2}{\partial x} \tag{13}$$

and

$$-\xi=\rho_2\frac{\partial^2 \phi_2}{\partial t^2}. \tag{14}$$

We are assuming that $-\xi=\lim \lambda_2\Delta_2$ as $\lambda_2\to\infty$ and $\Delta_2\to 0$, and the stress components are given by

$$\left(\sigma_{zz}\right)_2=-\xi+2\mu_2\frac{\partial w_2}{\partial z} \tag{15}$$

$$\left(\sigma_{zx}\right)_2=\mu_2\left(2\frac{\partial^2 \phi_2}{\partial z\,\partial x}+\frac{\partial^2 \psi_2}{\partial z^2}-\frac{\partial^2 \psi_2}{\partial x^2}\right). \tag{16}$$

We have considered the heterogeneity in the layer in following form

$$\mu_2=\mu_1(1+az) \tag{17}$$

$$\rho_2=\rho_1(1+az) \tag{18}$$

Using Eqs. (10), (11), (16), (17) and (18) in Eqs. (14) and (15), we get the following equations

$$\frac{\partial}{\partial x}\left(\rho_1(1+az)\frac{\partial^2 \phi_2}{\partial t^2}\right)+\mu_1(1+az)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial \phi_2}{\partial x}+\frac{\partial \psi_2}{\partial z}\right)=\rho_1(1+az)\frac{\partial^2}{\partial t^2}\left(\frac{\partial \phi_2}{\partial x}+\frac{\partial \psi_2}{\partial z}\right) \tag{19}$$

and

$$\frac{\partial}{\partial z}\left(\rho_1(1+az)\frac{\partial^2 \phi_2}{\partial t^2}\right)+\mu_1(1+az)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial \phi_2}{\partial z}-\frac{\partial \psi_2}{\partial x}\right)=\rho_1(1+az)\frac{\partial^2}{\partial t^2}\left(\frac{\partial \phi_2}{\partial z}-\frac{\partial \psi_2}{\partial x}\right) \tag{20}$$

Presume the solution of Eqs. (19) and (20) as:

$$\phi_2(x,z,t)=\bar{\phi}_2(z)e^{ik(x-ct)} \tag{21}$$

$$\psi_2(x,z,t)=\bar{\psi}_2(z)e^{ik(x-ct)} \tag{22}$$

Intuducing Eqs.(21) and (22) in Eqs. (19) and (20), we obtained

$$\bar{\psi}_2'''(z)-\bar{\psi}_2'(z)\left[1-\frac{\rho_1}{\mu_1}c^2\right]k^2+\bar{\phi}_2''(z)ik-\bar{\phi}_2(z)ik^3=0 \tag{23}$$

$$\bar{\phi}_2'''(z)-\bar{\phi}_2'(z)k^2-\bar{\psi}_2''(z)ik+\bar{\psi}_2(z)\left[1-\frac{\rho_1}{\mu_1}c^2\right]ik^3=0 \tag{24}$$

Solution of Eqs. (23) and (24) may be written as:

$$\bar{\psi}_2=\left(B_1e^{\upsilon kz}+B_2e^{-\upsilon kz}\right)e^{ik(x-ct)} \tag{25}$$

$$\bar{\phi}_2=\left(C_1e^{\upsilon kz}+C_2e^{-\upsilon kz}\right)e^{ik(x-ct)} \tag{26}$$

where $\upsilon=\sqrt{1-\frac{c^2}{\alpha_1^2}}$ and $\alpha_1=\sqrt{\frac{\mu_1}{\rho_1}}$.

Mechanical displacement for incompressible layer is

$$u_2=\left[\left(B_1e^{\upsilon kz}\upsilon k-\upsilon kB_2e^{-\upsilon kz}\right)+ik\left(C_1e^{\upsilon kz}+C_2e^{-\upsilon kz}\right)\right]e^{ik(x-ct)} \tag{27}$$

$$w_2=\left[\upsilon k\left(C_1e^{\upsilon kz}-C_2e^{-\upsilon kz}\right)-ik\left(B_1e^{\upsilon kz}+B_2e^{-\upsilon kz}\right)\right]e^{ik(x-ct)} \tag{28}$$

*3.3 Solution for the lower half-space*

The strain energy volume density function for the lower half-space, Love [4]

$$2W=A\left(e_{xx}^2+e_{yy}^2\right)+Ce_{zz}^2+2F\left(e_{xx}+e_{yy}\right)e_{zz}+2(A-2N)e_{xx}e_{yy}+L\left(e_{xz}^2+e_{zy}^2\right)+Ne_{xy}^2 \tag{29}$$

where

$$e_{ij}=\begin{cases}\dfrac{\partial u_i}{\partial x_j}+\dfrac{\partial u_j}{\partial x_i}, & \text{i} \neq \text{j}\\[2ex] \dfrac{\partial u_i}{\partial x_i}, & \text{i=j}\end{cases}$$

In $(x,z)$ direction, the strain energy volume density function becomes

$$2W=Ae_{xx}^2+Ce_{zz}^2+2Fe_{xx}e_{zz}+Le_{xz}^2 \tag{30}$$

where

$$\sigma_{ij}=\frac{\partial W}{\partial e_{ij}}. \tag{31}$$

Equation of motion without body forces, are

$$A\frac{\partial^2 u}{\partial x^2}+L\frac{\partial^2 u}{\partial z^2}+(F+L)\frac{\partial^2 w}{\partial z\,\partial x}=\rho\frac{\partial^2 u}{\partial t^2}, \tag{32}$$

$$L\frac{\partial^2 w}{\partial x^2}+C\frac{\partial^2 w}{\partial z^2}+(F+L)\frac{\partial^2 u}{\partial z\,\partial x}=\rho\frac{\partial^2 w}{\partial t^2}. \tag{33}$$

Consider the solution of Eqs. (32) and (33) in the form

$$u_3(x,z,t)=\bar{u}(z)e^{ik(x-ct)} \tag{34}$$

$$w_3(x,z,t)=\bar{w}(z)e^{ik(x-ct)} \tag{35}$$

Using Eqs.(34) and (35) in Eqs. (32) and (33), the following equations reduces to

$$L\bar{u}''(z)+ik(F+L)\bar{w}'(z)-(A-\rho c^2)k^2\bar{u}(z)=0 \tag{36}$$

$$C\bar{w}''(z)+ik(F+L)\bar{u}'(z)-(L-\rho c^2)k^2\bar{w}(z)=0 \tag{37}$$

To solve the Eqs. (36) and (37), we assume the solution as:

$$\bar{u}=\chi_1 e^{-k\zeta z} \tag{38}$$

$$\bar{w}=\chi_2 e^{-k\zeta z} \tag{39}$$

Substitute Eqs. (38) and (39) in Eqs. (36) and (37), we get

$$(\zeta^2 L+b)\chi_1-(i\zeta a)\chi_2=0 \tag{40}$$

$$-(i\zeta a)\chi_1+(\zeta^2 C+d)\chi_2=0 \tag{41}$$

where $a=F+L, b=\rho c^2-A$ and $d=\rho c^2-L$.

For the non-trivial solution of Eqs. (32) and (33), we must have

$$LC\zeta^4+(bC+dL+a^2)\zeta^2+bd=0 \tag{42}$$

The roots of Eq. (42) may be given as:

$$\frac{-(bC+dL+a^2)\mp\left((bC+dL+a^2)^2-4LCad\right)^{\frac{1}{2}}}{2LC} \tag{43}$$

The ratio of the displacement components corresponding to $\zeta=\zeta_j$ is

$$\frac{\bar{w}_j}{\bar{u}_j}=\frac{\chi_{2j}}{\chi_{1j}}=\frac{L\zeta_j^2+b}{i\zeta_j a}=H_j \tag{44}$$

Thus, the solutions of Eqs. (36) and (37) can be written as:

$$\bar{u} = \left(\chi_{11}e^{-k\zeta_1 z} + \chi_{12}e^{-k\zeta_2 z} + \chi_{13}e^{k\zeta_1 z} + \chi_{14}e^{k\zeta_2 z}\right) \tag{45}$$

$$\bar{w} = H_1\left(\chi_{11}e^{-k\zeta_1 z} + \chi_{13}e^{k\zeta_1 z}\right) + H_2\left(\chi_{12}e^{-k\zeta_2 z} + \chi_{14}e^{-k\zeta_2 z}\right) \tag{46}$$

where $\chi_{11}, \chi_{12}, \chi_{13}$ and $\chi_{14}$ are arbitrary constants and

$$\zeta_j^2 = \frac{-\left(bC + dL + a^2\right) + (-1)^j\left(\left(bC + dL + a^2\right)^2 - 4LCbd\right)^{\frac{1}{2}}}{2LC} \qquad (j = 1,2) \tag{47}$$

Hence, the displacement for the lower half-space are

$$u_3 = \left(\chi_{11}e^{-k\zeta_1 z} + \chi_{12}e^{-k\zeta_2 z}\right)e^{ik(x-ct)} \tag{48}$$

$$w_3 = \left(H_1\chi_{11}e^{-k\zeta_1 z} + H_2\chi_{12}e^{-k\zeta_2 z}\right)e^{ik(x-ct)} \tag{49}$$

where $\zeta_1$ and $\zeta_2$ are real and positive.

## 4 BOUNDARY CONDITIONS

We consider the appropriate boundary conditions for the propagation of Rayleigh wave as following:
At the interface $z = -H$ , the stress and the displacement components are continuous

*a*) $u_1 = u_2$,
*b*) $w_1 = w_2$,
*c*) $(\sigma_{zz})_1 = (\sigma_{zz})_2$,
*d*) $(\sigma_{xz})_2 = 0.$

At the interface $z = 0$, the stress and the displacement components are continuous

*a*) $u_2 = u_3$,
*b*) $w_2 = w_3$,
*c*) $(\sigma_{zz})_2 = (\sigma_{zz})_3$,
*d*) $(\sigma_{xz})_2 = (\sigma_{xz})_3$.

Using all the boundary conditions we obtained the following equations

$$A_1\left(ike^{-\beta_1 kH}\right) + A_2\left(ike^{\beta_1 kH}\right) - B_1\left(\upsilon ke^{-\upsilon kH}\right) + B_2\left(\upsilon ke^{\upsilon kH}\right) - C_1\left(ike^{-\upsilon kH}\right) - C_2\left(ike^{\upsilon kH}\right) = 0 \tag{50}$$

$$A_1\left(\beta_1 ke^{-\beta_1 kH}\right) - A_2\left(\beta_1 ke^{\beta_1 kH}\right) + B_1\left(ike^{-\upsilon kH}\right) + B_2\left(ike^{\upsilon kH}\right) - C_1\left(\upsilon ke^{-\upsilon kH}\right) + C_2\left(\upsilon ke^{\upsilon kH}\right) = 0 \tag{51}$$

$$A_1Z_1 + A_2Z_2 - C_1X - C_2Y = 0 \tag{52}$$

$$B_1X_1+B_2Y_1+C_1X_2-C_2Y_2=0 \tag{53}$$

$$C_1(ik)+C_2(ik)+B_1(\upsilon k)-B_2(\upsilon k)-P_1-P_2=0 \tag{54}$$

$$C_1(\upsilon k)-C_2(\upsilon k)-B_1(ik)-B_2(ik)-P_1(m_1)-P_2(m_2)=0 \tag{55}$$

$$C_1(X_3)+C_2(Y_3)-P_1(T_2)-P_2(U_2)=0 \tag{56}$$

$$B_1\mu_1k^2(\upsilon^2+1)+B_2\mu_1k^2(\upsilon^2+1)+C_1 2i\,\mu_1k^2\upsilon-C_2 2i\,\mu_1k^2\upsilon-P_1(T_1)-P_1(U_1)=0 \tag{57}$$

where

$$Z_1=\left(-k^2\lambda_0+\lambda_0\beta_1^2k^2\right)e^{-\beta_1kH},\ Z_2=\left(-k^2\lambda_0+\lambda_0\beta_1^2k^2\right)e^{\beta_1kH}$$
$$X=\left(-\rho_1(1-aH)k^2C^2+2\mu_1(1-aH)k^2\upsilon^2-2\mu_1(1-aH)ik^2\upsilon\right)e^{-\upsilon kH}$$
$$Y=\left(-\rho_1(1-aH)k^2C^2+2\mu_1(1-aH)k^2\upsilon^2+2\mu_1(1-aH)ik^2\upsilon\right)e^{\upsilon kH}$$
$$X_1=\mu_1k^2(1-aH)(\upsilon^2+1)e^{-\upsilon kH},\ Y_1=\mu_1k^2(1-aH)(\upsilon^2+1)e^{\upsilon kH},\ X_2=2i\,\mu_1k^2\upsilon(1-aH)e^{-\upsilon kH}$$
$$Y_2=2i\,\mu_1k^2\upsilon(1-aH)e^{\upsilon kH},\ X_3=-\rho_1k^2C^2+2\mu_1\upsilon^2k^2-2i\,\mu_1\upsilon k^2,\ Y_3=-\rho_1k^2C^2+2\mu_1\upsilon^2k^2+2i\,\mu_1\upsilon k^2,$$
$$T_2=-H_1C\zeta_1k+Fik\ ,\ T_1=LikH_1-k\zeta_1\ ,\ U_2=-H_1C\zeta_2k+Fik\ ,\ U_1=LikH_2-k\zeta_2.$$

Solving Eqs. (50) to (57), we obtained

$$\begin{vmatrix} ike^{-\beta_1kH} & ike^{\beta_1kH} & -\upsilon ke^{-\upsilon kH} & \upsilon ke^{\upsilon kH} & -ike^{-\upsilon kH} & -ike^{\upsilon kH} & 0 & 0 \\ \beta_1ke^{-\beta_1kH} & -\beta_1ke^{\beta_1kH} & ike^{-\upsilon kH} & ike^{-\upsilon kH} & -\upsilon ke^{-\upsilon kH} & \upsilon ke^{\upsilon kH} & 0 & 0 \\ 0 & 0 & \upsilon k & -\upsilon k & ik & ik & -1 & -1 \\ 0 & 0 & -ik & -ik & \upsilon k & -\upsilon k & -m_1 & -m_2 \\ Z_1 & Z_2 & 0 & 0 & -X & -Y & 0 & 0 \\ 0 & 0 & X_1 & Y_1 & -X_2 & -Y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_3 & Y_3 & -T_2 & -U_2 \\ 0 & 0 & \mu_1k^2(\upsilon^2+1) & \mu_1k^2(\upsilon^2+1) & 2\mu_1ik^2\upsilon & -2\mu_1ik^2\upsilon & -T_1 & -U_1 \end{vmatrix}=0 \tag{58}$$

Eq. (58) is the frequency equation for Rayleigh waves propagating in incompressible heterogeneous layer bonded between liquid and transversally isotropic half-spaces.

## 5 SPECIAL CASES

### *5.1 Case 1*

In the absence of liquid half-space, the frequency Eq. (58) reduces to

$$\begin{vmatrix} 0 & 0 & X & Y & 0 & 0 \\ X_1 & Y_1 & X_2 & Y_2 & 0 & 0 \\ \upsilon k & \upsilon k & ik & ik & -1 & -1 \\ -ik & -ik & \upsilon k & -\upsilon k & -m_1 & -m_2 \\ 0 & 0 & X_3 & Y_3 & -T_2 & -U_2 \\ \mu_1k^2(\upsilon^2+1) & \mu_1k^2(\upsilon^2+1) & 2\mu_1ik^2\upsilon & -2\mu_1ik^2\upsilon & -T_1 & -U_1 \end{vmatrix}=0 \tag{59}$$

Eq. (59) is the frequency equation for Rayleigh wave propagation in an incompressible, heterogeneous medium lying over a transversely isotropic half-space.

*5.2 Case 2*

When the lower half-space becomes isotropic, i.e. $A = C = \lambda + 2\mu$ , $F = \lambda$ and $L = \mu$ then the frequency Eq. (58) reduces to

$$\begin{vmatrix} ike^{-\beta_1 kH} & ike^{\beta_1 kH} & -\upsilon ke^{-n_2 kH} & \upsilon ke^{n_2 kH} & -ike^{-\upsilon kH} & -ike^{\upsilon kH} & 0 & 0 \\ \beta_1 ke^{-\beta_1 kH} & -\beta_1 ke^{\beta_1 kH} & ike^{-\upsilon kH} & ike^{-\upsilon kH} & -\upsilon ke^{-\upsilon kH} & \upsilon ke^{\upsilon kH} & 0 & 0 \\ 0 & 0 & \upsilon k & -\upsilon k & ik & ik & -1 & -1 \\ 0 & 0 & -ik & -ik & \upsilon k & -\upsilon k & -m_1 & -m_2 \\ Z_1 & Z_2 & 0 & 0 & -X & -Y & 0 & 0 \\ 0 & 0 & X_1 & Y_1 & -X_2 & -Y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_3' & Y_3' & -T_2' & -U_2' \\ 0 & 0 & \mu_1 k^2(\upsilon^2+1) & \mu_1 k^2(\upsilon^2+1) & 2\mu_1 ik^2\upsilon & -2\mu_1 ik^2\upsilon & -T_1' & -U_1' \end{vmatrix} = 0 \tag{60}$$

where

$U_1' = \mu ikH_2 - k\zeta_2$ , $T_1' = \mu ikH_1 - k\zeta_1$ , $U_2' = -H_1(\lambda + 2\mu)\zeta_1 k + \lambda ik$ , $T_2' = -H_1(\lambda + 2\mu)\zeta_1 k + \lambda ik$ ,

$Y_3' = -\rho_1(\lambda + 2\mu)^2 k^2 + 2\mu_1\upsilon^2 k^2 + 2i\,\mu_1\upsilon k^2$ , $X_3' = -\rho_1(\lambda + 2\mu)^2 k^2 + 2\mu_1\upsilon^2 k^2 - 2i\,\mu_1\upsilon k^2$ ,

$\zeta_1^2 = 1 - \dfrac{c^2}{\alpha^2}$ , $\zeta_2^2 = 1 - \dfrac{c^2}{\beta^2}$ , $\chi_{11} = -\left(2 - \dfrac{c^2}{\beta^2}\right)$

Such that $\alpha^2 = \dfrac{\lambda + 2\mu}{\rho}$ and $\beta^2 = \dfrac{\mu}{\rho}$.

Eq. (60) is the frequency equation for Rayleigh waves propagating in an incompressible heterogeneous layer bonded between a uniform medium of liquid and isotropic half-space.

*5.3 Case 3*

In the absence of incompressible medium i.e. $H = 0$ , we get the dispersion relation for Rayleigh wave propagation in a liquid half-space lying over a transversely isotropic half-space.

$$\begin{vmatrix} ik & ik & -1 & -1 \\ \beta_1 k & \beta_1 k & -m_1 & -m_2 \\ \nu_1 & \nu_1 & M_1 & M_2 \\ 0 & 0 & T_1 & U_1 \end{vmatrix} = 0 \tag{61}$$

where

$\nu_1 = \left(-k^2\lambda_0 + \lambda_0\beta_1^2 k^2\right)$ , $M_1 = -H_1 C\zeta_1 k + Fik$ , $M_2 = -H_2 C\zeta_1 k + Fik$ .

## 6 NUMERICAL EXAMPLES AND DISCUSSION

To illustrate the effect of heterogeneity parameters we have done the numerical analysis of frequency equation. We have considered the following values. For the liquid layer, Ewing et al. [3],

$\lambda_0 = 0.214\times10^{11} dynes\ \text{cm}^{-2}, \rho_0 = 1g\ \text{cm}^{-3},$

For the incompressible medium we have taken the following values: Bullen [2]

$\rho_1 = 3.53g\ \text{cm}^{-3}, \quad \mu_1 = 8.1\times10^{11} dynes\ \text{cm}^{-2}.$

For the transversely isotropic half-space we have taken the following values: Love [4]

$A = 26.94\times10^{11} dynes\ \text{cm}^{-2},$
$C = 23.63\times10^{11} dynes\ \text{cm}^{-2},$
$F = 6.61\times10^{11} dynes\ \text{cm}^{-2},$
$L = 6.53\times10^{11} dynes\ \text{cm}^{-2},$
$\rho = 2.7g\ \text{cm}^{-3}.$

Numerical results are obtained by using Eq. (60) to represent the effect of heterogeneity on propagation of Rayleigh wave. In all the figures curves are plotted to exhibits the variation in wave number, wave length and Rayleigh wave velocity.

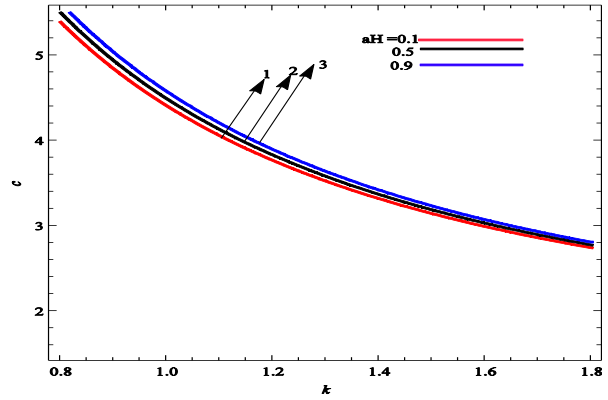


**Fig.2**
Variation of Rayleigh wave velocity $c$ with respect to wave number $k$ for different values of heterogeneity parameter $aH$.

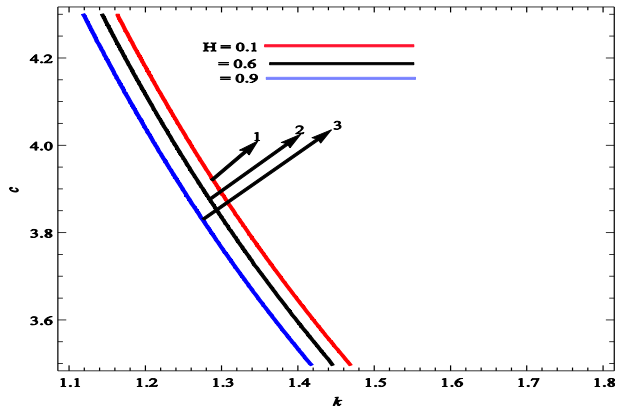


**Fig.3**
Variation of Rayleigh wave velocity $c$ with respect to wave number $k$ for different values of $H$.

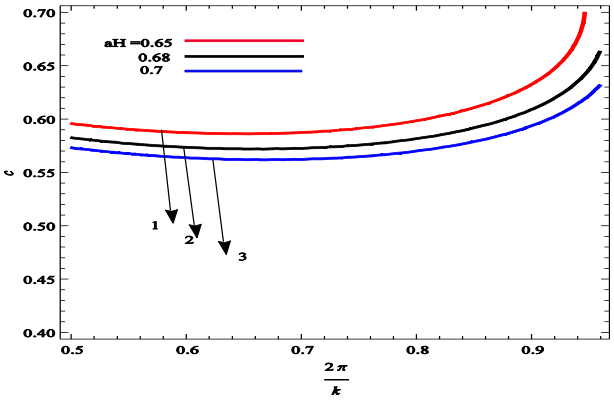


**Fig.4**
Variation of Rayleigh wave velocity $c$ with respect to wave length $\frac{2\pi}{k}$ for different values of heterogeneity parameter $aH$.

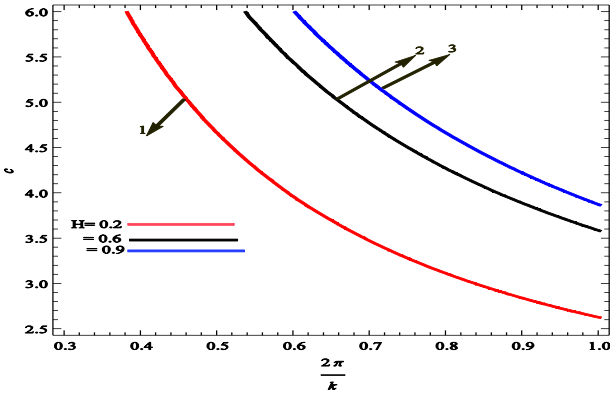


**Fig.5**
Variation of Rayleigh wave velocity $c$ with respect to wave length $\frac{2\pi}{k}$ for different values of $H$.

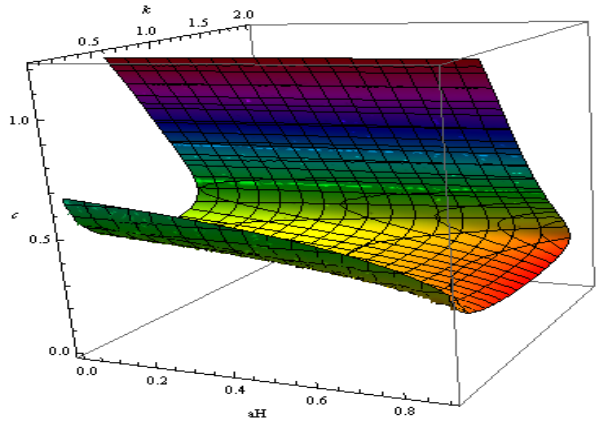


**Fig.6**
Surface plot for Rayleigh wave velocity $c$ with respect to wave number $k$ and heterogeneity parameters $aH$.

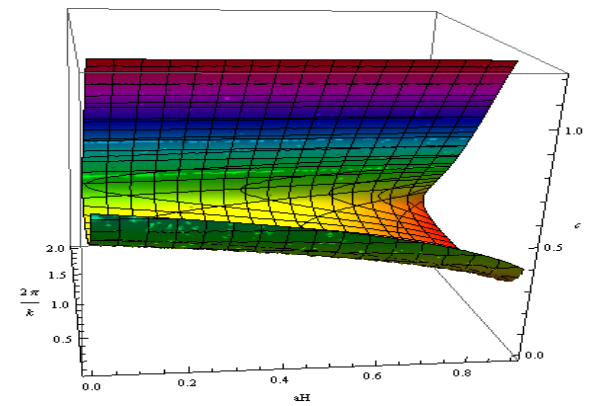


**Fig.7**
Surface plot for Rayleigh wave velocity $c$ with respect to wave length $\frac{2\pi}{k}$ and heterogeneity parameters $aH$.

Presented figures show the effect of heterogeneity on Rayleigh wave velocity $(c)$ with respect to different parameters like: wave number $(k)$ and wave length $\left(\frac{2\pi}{k}\right)$. In Fig. 2-3, the graph is plotted for the variation of wave velocity *c* against the wave number *k* for different values of *aH* and width of the layer. The figure reflects that as *aH* increases, the wave velocity *c* increases and as the wave number *k* increases the wave velocity *c* decreases. In Fig. 4, the graph is plotted for the variation of Rayleigh wave velocity *c* against the wavelength $\left(\frac{2\pi}{k}\right)$ for different values of *aH*. The figure reflects that as *aH* increases, the wave velocity *c* decreases and as the wavelength $\left(\frac{2\pi}{k}\right)$ increases the wave velocity *c* increases. Fig. 5, the graph is plotted for the variation of Rayleigh wave velocity *c* against the wavelength $\left(\frac{2\pi}{k}\right)$ for different values of *H* and observes that when we increase *H*, the wave velocity *c* increases. Figs. 6-7 display the surface plot for Rayleigh wave velocity with respect to different parameters like: wave number, heterogeneity and wavelength.

## 7 CONCLUSIONS

Effects of heterogeneity and layer width on the propagation of Rayleigh waves have been studied. Frequency equation has been obtained in determinant form. It is observed that the heterogeneity and width of the incompressible medium has great impact on the phase velocity of Rayleigh waves. In particular, Rayleigh wave velocity increases with respect to wave number as we increases the heterogeneity parameter and decreases with respect to wave length. Findings have been shown by the means of graphs. The presented model may help to understand the propagation behavior of Rayleigh type waves near the ocean ridges.

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