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Free Vibration Analysis of Non-Uniform Circular Nanoplate

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ABSTRACT

In this paper, axisymmetric free vibration analysis of a circular Nano-plate having variable thickness was studied. The variation in thickness of plate was considered as a linearly in radial direction. Nonlocal elasticity theory was utilized to take into account size-dependent effects. Ritz functions was utilized to obtain the frequency equations for simply supported and clamped boundary. To verify accuracy of Ritz method, differential transform method (DTM) also used to drive the size dependent natural frequencies of circular nano-plates. The validity of solutions was performed by comparing present results with those of the literature for both classical plate and nano plate. Effect of nonlocal parameter, mode number and taper parameter on the natural frequency are investigated. Results showed that taper parameter has significant effect on the non-dimensional frequency and its effects on the clamped boundary condition is more than simply support.

Keywords: Nonlocal theory; Axisymmetric vibration; Variable thickness plate; Ritz method; Differential transform method.

1 INTRODUCTION

RECENTLY, usage of micro and nano scale structures have widely spread in modern technology fields such as aerospace, electronics, MEMS and NEMS [1], [2]. Because of size effect, these structures have excellent mechanical, thermal and electrical properties compared to ordinary scale structures [3]. For example, studies shows elastic modulus and strength of single layer graphene sheets (GSs) are more than 1 *TPa* and 130 *GPa*, respectively [4]. Therefore, they can be properly utilized in sensitive devices and high performance application such as gas detection grapheme sensor, ultra capacitors and ultra-strength composite material [3], [5]. Modeling using continuum mechanic approach offers the advantages of less computational expense in compared to atomistic modeling and continuum-atomistic modeling approaches [3]. However, applicability of classical continuum mechanic models at micro/nano scale is questionable. Therefore, several non-classic continuum theories has proposed such as strain gradient theory [6]–[10], couple stress theory [11]–[14], and nonlocal elasticity theory [10], [15], [16] to take into account size effects through continuum mechanics models. Among these theories, nonlocal theory was initiated by Eringen [15] which assumes strain of a reference point as a function of every neighborhood particles. Peddieson indicated this theory could be employed in analysis of nano structures [16], [17]. Based on the nonlocal elasticity theory, size-dependent classic beam [18], first order shear beam[19], third order shear beam[18], [20] and that of plate [20]–[22] were developed for vibration problems. Furthermore, Wang et al considered small

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scale effects on the longitudinal wave propagation in nano-plates using the nonlocal elastic theory [23]. Newly, this theory has incorporated in Functionally graded material (FGM) beams[24], [25] and plate [26], [27] models. Rahmani and Pedram obtained close-form solution for vibration behaviour of the functionally graded (FG) Timoshenko beam based on nonlocal elasticity theory [24]. Simsek studied nonlinear free vibration of functionally graded nano-beams using Euler-Bernoulli beam theory. Afterward, influence of characteristic length, vibration amplitude and material composition was investigated [25]. Hashemi-hosseini et al conducted free vibration analysis of thick circular disks for various boundary conditions [26].

In other hand, it can be observed beams and plates of variable thickness are employed in micro and nano-scale applications. In fact, tapered beam and plates are utilized due to more weight reduction, structural performance and vibrational efficiency. Variable thickness is used to optimize design, redistribute stress and reduce weight [28]–[30]. Various works related to variable thickness structures are found in several references ([31]–[34]). Danesh et al explored axial vibration of taped nano-rod using nonlocal elasticity theory. They employed Differential Quadrature Method (DQM) to solve governing equations for various boundary conditions [33]. Efraim and Eisenberger obtained exact solution for vibration analysis of variable thick annular FG plates employing first order shear deformation [35]. The Differential Transform Method (DTM) is a semi-analytic method based on Taylor series to solve differential equations proposed by Zou[36] in 1986. The merit of this method is that can be applied to coefficient variable partial differential equations without linearization, discretization, or perturbation. This method provide high accurate or exact solution for initial value problems and also boundary value problems [37]. Furthermore, Plates resting on elastic foundation are used in versatile engineering applications such as in flexible supports modeling. Mohammadi [38] et al studied free vibration of rectangular graphene sheet resting in elastic medium subjected to in-plane shear load. The effects of boundary conditions and elastic medium parameters were investigated using DQM and Galerkin methods. Pradhan et al investigated nonlocal parameter's influence on vibration of graphene sheet layer and the result showed nonlocal effect is significant for graphene sheet [39]. Behfar et al[40], Mohammadimehr et al[41] and Pradhan et al [39] researched nano-scale vibration of multilayered graphene on elastic foundation. Furturmore, Ghorbanour Arani et al carried out vibration analysis of double-layered graphene sheets on Visco-Pasternak foundation[42].Despite of extensive use of variable thickness circular plate in all fields, free vibration of nonlocal variable thickness plates has not published up to our knowledge.

In this paper, free vibration analysis of variable thickness plate is performed based on nonlocal elasticity theory to investigate size effect phenomena. The appropriate governing equations are derived using energy methods incorporated nonlocal Eringen constitutive equation. Those are solved by applying Ritz method and Differential transform method (DTM) to obtain natural frequencies and to study influence of some parameters.

2 THEORETICAL FORMULATIONS

2.1 Nonlocal elasticity theory

Nonlocal theory was proposed by Eringen [15], [43] which assumes strain of a reference point as a function of every neighborhood particles. Therefore, size effect and atomic forces are included via a characteristic length $e_0 a / L$ [43]. Based on this theory, constitutive equation is a differential equation as follows.

$$(1-\left(e_0 a / L\right)^2 \nabla^2)\sigma^{nl} = \sigma \tag{1}$$

where e_0 is a material constant, which is experimentally obtained; a is an internal characteristic length depending on granular size or molecular diameter; parameter L defines an external characteristic length. Also, σ^{nl} denotes stress induced in medium due to nonlocal effects.

2.2 Ritz method procedure

A nano-plate defined in polar coordinate are studied as shown in Fig. 1. The parameters R, h_o, m are the radius of nano-plate, the thickness at $r = 0$, and thickness variation slope, respectively. Ritz method makes reliable prediction in vibration analysis[44]. The appropriate linear elastic strain energy U_b for analyzing an orthotropic circular nano-plates using Ritz method may be given by [48]

$$U_b = \frac{1}{2}\int_0^a\int_0^{2\pi}\left[D_r\left\{\left(\frac{\partial^2 w}{\partial r^2}\right)^2 + 2\nu_\theta \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r^2}\right)\right\} + D_\theta\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)^2\right] r dr d\theta \tag{2}$$

where D_r, D_θ are flexural stiffness of plate in radial and circumferential directions respectively. Furthermore, μ denotes nonlocal parameter. By assuming $w(x,y,t) = w(x,y)e^{i\omega t}$, where ω is vibration frequency and $w(x,y)$ is transverse deflection of nano-plate the kinetic energy T_a of the plate can be expressed as [48]:

$$T_a = \frac{1}{2}\rho\omega^2\left[\int_0^a\int_0^{2\pi} h w^2 r dr d\theta + \mu^2 \int_0^a\int_0^{2\pi} h\left(\frac{\partial w}{\partial r}\right)^2 r dr d\theta\right] \tag{3}$$

The total potential energy can be expressed as:

$$\Pi = U_a - T_a \tag{4}$$

So an explicit equation for natural frequency via Rayleigh quotient that can be expressed as:

$$\omega^{*2} = \frac{\gamma a^4 \omega^2 h_0}{D_0} = \frac{\frac{1}{2}\int_0^1 f^3(\rho)\left\{\left(\frac{\partial^2 W}{\partial \rho^2}\right)^2 + 2\nu_\theta \frac{\partial^2 W}{\partial \rho^2}\left(\frac{1}{\rho}\frac{\partial W}{\partial \rho}\right) + p^2\left(\frac{1}{\rho}\frac{\partial W}{\partial \rho}\right)^2\right\}\rho d\rho d\theta}{\frac{1}{2}\left[\int_0^1\int_0^{2\pi} f(\rho) w^2 \rho d\rho d\theta + \frac{\mu^2}{a^2}\int_0^1\int_0^{2\pi} f(\rho)\left(\frac{\partial w}{\partial \rho}\right)^2 \rho d\rho d\theta\right]} \tag{5}$$

where

$$w = \frac{W}{a} \quad , \quad \rho = \frac{r}{a} \quad , \quad H = \frac{h}{a} = h_0 f(\rho) \quad , \quad D_0 = \frac{E_r h^3}{12(1-\nu_\theta \nu_r)} \quad , \quad \Omega^2 = \frac{\rho a^4 \omega^2 h_0}{D_0} \quad , \quad p^2 = \frac{E_\theta}{E_r} \tag{6}$$

As for local plate, Eq. (5) is employed for frequency analysis of nonlocal nano-plates varying thickness. According to Ritz method, approximate solution is assumed a linear combination of N known basis functions

$$W = \sum_{k=1}^{N} C_k \phi_k(\rho) \tag{7}$$

where the C_k are unknown coefficient and the ϕ_k are arbitary basis functions satisfying essential boundary conditions [45]. Here, the basis functions are chosen as:

$$\phi_k = u^{s+k-1} \qquad , \qquad k = 1, 2, \ldots, \tag{8}$$

where $u = (1-\rho^2)$, and

$$s = \begin{cases} 1 & \textit{simply supported} \\ 2 & \textit{clamped} \end{cases} \tag{9}$$

Inserting Eq. (7) in Eq. (5) and minimizing ω as a function of coefficients C_k, following algebraic equations system is obtained

$$\sum_{k=1}^{N}\left(a_{mn}-\Omega^2 b_{mn}\right)C_k=0 \tag{10}$$

where

$$a_{mn}=\int_0^1\left[f_k^{\,3}(\xi)\left\{\phi_i''\phi_j''+\frac{\nu_\theta}{\rho}\left(\phi_i''\phi_j'+\phi_i'\phi_j''\right)+p^2\frac{\phi_i'}{\rho}\frac{\phi_j'}{\rho}\right\}\right]\rho d\rho \tag{11}$$

$$b_{mn}=\int_0^1 f_k(\rho)\left(\phi_i\phi_j+\frac{\mu^2}{a^2}\phi_i'\phi_j'\right)\rho d\rho \tag{12}$$

The thickness slope parameter can be expressed as:

$$f(\rho)=1+\lambda\rho \tag{13}$$

where

$$\lambda=\tan\theta \tag{14}$$

In Eq. (10), a_{mn} and b_{mn} are stiffness and mass matrix, respectively. The Eigen value problem of Eq. (10) is solved for natural frequency of non-uniform nano-plate.

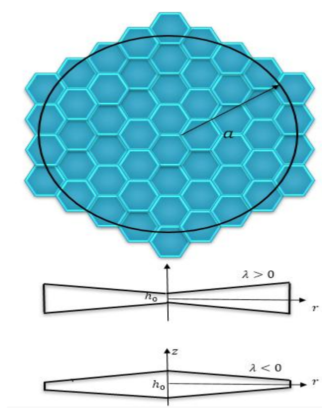

Fig.1
Non-uniform nano plate.

2.3 Equilibrium equations based on classical plate theory for DTM
2.3.1 Deriving governing equations

According to classical plate theory, displacement field can be assumed as:

$$u(x,y,z,t)=u_0(x,y,t)-z\frac{\partial w}{\partial x}\;,v(x,y,z,t)=v_0(x,y,t)-z\frac{\partial w}{\partial y}\;,w(x,y,z,t)=w_0(x,y,t) \tag{15}$$

where u_0, v_0 are displacement along coordinate line on *x-y* plane. The strain-displacement relations neglecting nonlinear terms are as follow:

$$\varepsilon_{xx}=\varepsilon_{xx}^0-z\frac{\partial^2 w}{\partial x^2}\qquad \varepsilon_{yy}=\varepsilon_{yy}^0-z\frac{\partial^2 w}{\partial y^2}\qquad \varepsilon_{xy}=\varepsilon_{xy}^0-z\frac{\partial^2 w}{\partial x\,\partial y} \tag{16}$$

where

$$\varepsilon_{xx}^0 = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy}^0 = \frac{\partial v}{\partial y} \qquad \varepsilon_{xy}^0 = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right) \tag{17}$$

Using Eq. (1), for a single layer nano-plate the strain -stress relationships are obtained as:

$$\begin{Bmatrix} \sigma_{xx}^{nl} \\ \sigma_{yy}^{nl} \\ \sigma_{xy}^{nl} \end{Bmatrix} - \mu^2\nabla^2 \begin{Bmatrix} \sigma_{xx}^{nl} \\ \sigma_{yy}^{nl} \\ \sigma_{xy}^{nl} \end{Bmatrix} = \begin{bmatrix} E_{11}/\left(1-\nu_{12}\nu_{21}\right) & \nu_{12}E_{22}/\left(1-\nu_{12}\nu_{21}\right) & 0 \\ \nu_{12}E_{22}/\left(1-\nu_{12}\nu_{21}\right) & E_{22}/\left(1-\nu_{12}\nu_{21}\right) & 0 \\ 0 & 0 & E_{11}/2\left(1+\nu_{12}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} \tag{18}$$

where $E_{11}, E_{22}, \nu_{12}, \nu_{21}$ are longitudinal elastic modulus, transverse elastic modulus, and in-plane Poisson's ratio in 1 and 2 directions, and $\mu = e_0 l_i$ is nonlocal parameter respectively. The plane stress condition is considered due to high radius to thickness ratio, therefore nonlocal stresses of $\sigma_{xx}^{nl}, \sigma_{yy}^{nl}$ and σ_{xy}^{nl} are only stresses induced in nano-plate. Stress resultant are defined as:

$$N_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}^{nl} dz\,,\; N_{yy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy}^{nl} dz\,,\; N_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy}^{nl} dz\,, \tag{19}$$

$$M_{xx} = \int_{-h/2}^{h/2} z\,\sigma_{xx}^{nl} dz\,,\; M_{yy} = \int_{-h/2}^{h/2} z\,\sigma_{yy}^{nl} dz\,,\; M_{xy} = \int_{-h/2}^{h/2} z\,\sigma_{xy}^{nl} dz\,, \tag{20}$$

Integrating Eq. (20) from $z = -h/2$ to $z = h/2$, we get nonlocal moments as:

$$\begin{Bmatrix} M_{xx}^{nl} \\ M_{yy}^{nl} \\ M_{xy}^{nl} \end{Bmatrix} - \mu^2\nabla^2 \begin{Bmatrix} M_{xx}^{nl} \\ M_{yy}^{nl} \\ M_{xy}^{nl} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \tag{21}$$

And

$$\begin{Bmatrix} N_{xx}^{nl} \\ N_{yy}^{nl} \\ N_{xy}^{nl} \end{Bmatrix} - \mu^2\nabla^2 \begin{Bmatrix} N_{xx}^{nl} \\ N_{yy}^{nl} \\ N_{xy}^{nl} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \varepsilon_{xy}^0 \end{Bmatrix} \tag{22}$$

In order to derive the governing equations using Hamilton principle, the variation of the internal energy is considered as:

$$\delta U = \int_{-h/2}^{h/2} \sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xy}\delta\gamma_{xy} + \sigma_{yy}\delta\varepsilon_{yy}\, dA\, dz \tag{23}$$

Also, the variation of work done by external forces is given as:

$$\delta V = \int_A \left(-q\,\delta w + \frac{\partial \delta w}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial \delta w}{\partial y}\left(N_{yy}\frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right)\right) dA \tag{24}$$

Considering only the transverse motion, the variation of the kinetic energy of the plate can be expressed by:

$$\delta T = \int_A \rho h \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t} dA \tag{25}$$

Hamilton principle for elastic bodies is given as:

$$\Pi = \delta \int_{t_1}^{t_2} \left(U + V - T\right) dt = 0 \tag{26}$$

Inserting Eqs.(9), (10) into (12) and integrating the resulting expressions by parts and collecting similar terms turn out following equilibrium equations.

$$\delta u :\ \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{27}$$

$$\delta v :\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0 \tag{28}$$

$$\delta w :\ \frac{\partial^2 M_{xx}}{\partial x^2} + \frac{2\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{yy}\frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right) = \rho h \frac{\partial^2 w}{\partial t^2} \tag{29}$$

Substituting moment resultants in Eq. (28) and neglecting in-plane resultant, the governing equation is obtained as:

$$\begin{aligned} & D\frac{\partial^4 w}{\partial x^4} + 2D\frac{\partial^4 w}{\partial x^2 \partial y^2} + D\frac{\partial^4 w}{\partial y^4} + 2\frac{\partial D}{\partial x}\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\,\partial y^2}\right) + 2\frac{\partial D}{\partial y}\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}\right) \\ & + \left(\frac{\partial^2 D}{\partial x^2} + \nu\frac{\partial^2 D}{\partial y^2}\right)\frac{\partial^2 w}{\partial x^2} + \left(\nu\frac{\partial^2 D}{\partial x^2} + \frac{\partial^2 D}{\partial y^2}\right)\frac{\partial^2 w}{\partial y^2} + 2\left(1-\nu\right)\frac{\partial^2 D}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y} \\ & -\left(1-\mu^2\nabla^2\right)\left(f + N_{xx}\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y} + N_{yy}\frac{\partial^2 w}{\partial y^2}\right) + \left(1-\mu^2\nabla^2\right)\left(\rho h\frac{\partial^2 w}{\partial t^2}\right) = 0 \end{aligned} \tag{30}$$

Changing Cartesian coordinate to polar system result in following equations, for the sake of convenience:

$$\begin{aligned} & D\frac{\partial^4 w}{\partial R^4} + \frac{2}{R}\left(D + R\frac{\partial D}{\partial R}\right)\frac{\partial^3 w}{\partial R^3} + \frac{1}{R^2}\left(-D - 2n^2 D + R\left(2+\nu\right)\frac{\partial D}{\partial R} + R^2\frac{\partial^2 D}{\partial R^2}\right)\frac{\partial^2 w}{\partial R^2} \\ & + \frac{1}{R^3}\left(D - R\frac{\partial D}{\partial R} + R^2\frac{\partial^2 D}{\partial R^2} + 2n^2 D - \frac{2R\,\partial D}{\partial R}\right)\frac{\partial w}{\partial R} + \left(-\frac{4D}{R^4}n^2 - 3\frac{\partial D}{R^3\partial R}n^2 + \frac{\nu}{R^2}\frac{\partial^2 D}{\partial R^2}n^2 + \frac{D}{R^4}n^4\right) \\ & -\rho\omega^2\left(1-\mu^2\nabla^2\right)\left(hw\right) = 0 \end{aligned} \tag{31}$$

where

$$N_{xx} = N_{yy} = N_{xy} = 0 \ \ ,\ f = 0 \tag{32}$$

Eq. (31) can be expressed as following by defining some non-dimensional parameter.

$$r=\frac{R}{a} \quad , \quad W=\frac{w}{a} \quad , \quad H=\frac{h}{a}$$

$$D\frac{\partial^4 W}{\partial r^4}+\frac{2}{r}\left(D+r\frac{\partial D}{\partial r}\right)\frac{\partial^3 W}{\partial r^3}+\frac{1}{r^2}\left(-D+r\left(2+\nu\right)\frac{\partial D}{\partial r}+r^2\frac{\partial^2 D}{\partial r^2}\right)\frac{\partial^2 W}{\partial r^2}$$
$$+\frac{1}{r^3}\left(D-r\frac{\partial D}{\partial r}+r^2\frac{\partial^2 D}{\partial r^2}\right)\frac{\partial W}{\partial r}-\rho\omega^2 ha^4\left(W-\frac{\mu^2}{a^2}\left(\frac{\partial^2 (W)}{\partial r^2}+\frac{1}{r}\frac{\partial (W)}{\partial r}\right)\right)=0 \tag{33}$$

Eq. (33) can be expressed as following by defining linear variation in the thickness.

$$H=h_0\left(1+\lambda r\right) \tag{34}$$

$$r^3\left(1+\lambda r\right)^3\frac{\partial^4 W}{\partial r^4}+2r^2\left(\left(1+\lambda r\right)^3-3\lambda r\left(1+\lambda r\right)^2\right)\frac{\partial^3 W}{\partial r^3}$$
$$+r\left(-\left(1+\lambda r\right)^3+3\lambda r\left(2+\nu\right)\left(1+\lambda r\right)^2+6r^2\lambda^2\left(1+\lambda r\right)\right)\frac{\partial^2 W}{\partial r^2}$$
$$+\left(\left(1+\lambda r\right)^3-3r\lambda\left(1+\lambda r\right)^2-6\nu\lambda r^2\left(1+\lambda r\right)\right)\frac{\partial W}{\partial r}$$
$$=r^3\Omega^2\left(1+\lambda r\right)W+r^3\Omega^2\frac{\mu^2}{a^2}\left[2\lambda\frac{\partial W}{\partial r}+\left(1+\lambda r\right)\frac{\partial^2 W}{\partial r^2}+\frac{\lambda W}{r}+\frac{\left(1+\lambda r\right)}{r}\frac{\partial W}{\partial r}\right] \tag{35}$$

where

$$D_0=\frac{Eh_0^3}{12\left(1-\nu^2\right)} \quad , \quad \Omega^2=\frac{\rho h_0 a^4}{D_0} \tag{36}$$

Boundary conditions equations needed to solve governing equation are given as:
Simply-supported

$$f\left(1\right)=0 \ . \ M_r\,|_{r=1}=\left\{-D\left(\frac{\partial^2 f}{\partial r^2}+\nu\frac{1}{r}\frac{\partial f}{\partial r}\right)\right\}_{r=1}=0 \tag{37}$$

Clamped

$$f\left(1\right)=0 \quad , \quad \frac{\partial f}{\partial r}\,|_{r=1}=0 \tag{38}$$

Also, it required to satisfy following equation to avoid infinite value at $r=0$

$$\frac{\partial f}{\partial r}\,|_{r=1}=0 \quad , \quad Q_r\,|_{r=0}=\left\{-D\left(\frac{\partial^3 f}{\partial r^3}+\frac{1}{r}\frac{\partial^2 f}{\partial r^2}-\frac{1}{r^2}\frac{\partial f}{\partial r}\right)+\frac{\partial D}{\partial r}\left(\frac{\partial^2 f}{\partial r^2}+\nu\frac{1}{r}\frac{\partial f}{\partial r}\right)\right\}_{r=0}=0 \tag{39}$$

2.3.2 Differential Transform Method (DTM) procedure

This method is a semi-analytic method based on Taylor series to solve differential equations by transform to algebraic equations. The Taylor series of a real valued function $f(r)$ that is infinitely differentiable at $r=r_0$ are shown as:

$$f\left(r\right)=\sum_{0}^{n}\left(r-r_0\right)^k\frac{1}{k!}\left[\frac{d^kf\left(r\right)}{dr^k}\right]_{r=r_0} \tag{40}$$

where r_0, $n!$, k denote domain center, factorial of n and k *th* derivative of $f(r)$ obtained at $r=r_0$, respectively. From above Taylor series, Eq. (40), differential transform of the *K th* derivative is defined as:

$$F_k=\frac{1}{k!}\left[\frac{d^kf\left(r\right)}{dr^k}\right]_{r=r_0} \tag{41}$$

The differential inverse transform of the F_k is given as:

$$f\left(r\right)=\sum_{0}^{n}\left(r-r_0\right)^k F_k \tag{42}$$

The number of terms i.e. value of n depends on the convergence of natural frequencies. Some basic theorems, which are frequently used in the practical problems, are given in Table 1.

Using basic properties of DTM (Table 1), we get transformed form of the Eq. (35) as:

$$\begin{aligned}&\left(k^2-1\right)^2F_{k+1}=-3\lambda k\left(k-1\right)\left(k^2-k-1+\nu\right)F_k\\&+\left\{\begin{aligned}&-3\lambda k\left(k-4\right)\left(k-3\right)\left(k-2\right)\left(k-1\right)+6\nu\lambda^2\left(k-2\right)\left(k-1\right)\\&-3\lambda^2\left(k-1\right)\left(6k^2-25k+25-2\nu\right)-\frac{\mu^2}{a^2}\left(k-1\right)^2\end{aligned}\right\}F_{k-1}\\&-\lambda^3\left(k-2\right)\left\{k^3-4k^2+\left(2+3\nu\right)k-3\nu+1-\frac{\lambda\mu^2}{a^2}\left(k-1\right)^2\right\}F_{k-2}+\Omega^2F_{k-3}+\lambda\Omega^2F_{k-4}\end{aligned} \tag{43}$$

Repetitive using of the recursive relation of Eq. (43) gives us all F_i coefficients excepting F_0 and F_2. In fact all coefficients are obtained in term of F_0 and F_2 after simplifying.

Using above mentioned rules, following equations is obtained for boundary conditions

Simply supported

$$\sum_{0}^{N}F_k=0\ .\quad\sum_{0}^{N}\left[k\left(k-1\right)+\nu k\right]F_k=0 \tag{44}$$

Clamped

$$\sum_{0}^{N}F_k=0\ .\quad\sum_{0}^{N}kF_k=0 \tag{45}$$

Conditions at $r=0$:

$$F_1=0\ .\ F_3=-\frac{2}{3}\lambda\left(1+\nu\right)F_2 \tag{46}$$

Appling boundary conditions above, Eqs. (44-46), results in a homogenous system of linear equations as following:

$$\begin{aligned}
&\phi_{11}^{(m)}(\Omega)F_0+\phi_{12}^{(m)}(\Omega)F_2=0\\
&\phi_{21}^{(m)}(\Omega)F_0+\phi_{22}^{(m)}(\Omega)F_2=0
\end{aligned} \tag{47}$$

For non-trivial solution, determinant of coefficient matrix must vanish and thus

$$\begin{vmatrix}\phi_{11}^{(m)}(\Omega) & \phi_{12}^{(m)}(\Omega)\\ \phi_{21}^{(m)}(\Omega) & \phi_{22}^{(m)}(\Omega)\end{vmatrix}=0 \tag{48}$$

It is noticeable that $\phi_{ij}^{(m)}$ coefficient are function of frequency Ω, therefore solving above nonlinear algebraic equations, Eq. (48), yields natural frequencies for nano-plate varying thickness.

Table 1
Some basic theorems frequently used in the practical problems.

Original functions	Transformed functions
$f(r)=g(r)\pm h(r)$	$F_k=G_k+H_k$
$f(r)=\lambda g(r)$	$F_k=\lambda G_k$
$f(r)=g(r)h(r)$	$F_k=\sum_{l=0}^{k}G_l H_{k-l}$
$f(r)=\dfrac{d^n g(r)}{dr^n}$	$F_k=\dfrac{(k+n)!}{k!}G_{k+n}$
$f(r)=r^n$	$F_k=\delta(k-n)=\begin{cases}1 & k=n\\ 0 & k\neq n\end{cases}$

3 NUMERICAL RESULTS AND DISCUSSIONS

The formulation derived in previous sections is employed to investigate free vibration analysis of axisymmetric circular nano-plate. Firstly, a comparison study is perfumed to show accuracy of formulation and solution. Secondly, analyzing impact of various parameters were carried out in sake of parametric study.

3.1 Comparison studies

For validation purpose a comparison is performed with a classic nano-plate. A nano-plate with elastic module 1060 *GPa* and Poisson's ratio $V=0.34$, and initial thickness 0.34 *nm* is considered. Table 2. demonstrates the result obtained for simply supported and clamped boundary condition. The result obtained are in good agreement with literature data and show accuracy of formulation and solution. Another comparison also is done with nano-plate varying thickness. Table 3. compares the dimensionless natural frequency for three boundary condition obtained using DTM, Ritz methods and another study [31]. The good agreement observed in the table depicts precision of frequency obtained. Table 4. depicts natural frequency obtained from two different solution method for nano-plate having various thickness variation slope. Furthermore, Table 5. compares values of frequencies of higher vibration modes calculated employing DTM and Ritz method for simply supported and clamped boundary conditions. Finally, the convergence analysis is shown in Figs. 2, 3. The figures demonstrate 9, 50 terms required to attain convergence for Ritz and DTM, respectively. However, the higher frequencies need more term due to their complicated mode shape.

Table 2
Value of natural frequency of simply supported nano-plate.

Boundary conditions	$e_0a(nm)$	Mode number							
		1	1	1	2	2	2	3	3
		[46]	Ritz	DTM	Ref1	Ritz	DTM	Ritz	DTM
Simple supported	0	4.9345	4.9352	4.9351	29.7198	29.7200	29.7200	74.1561	74.1561
	.5	4.8997	4.8998	4.8998	28.6485	28.6488	28.6488	68.0584	68.0584
	1	4.7979	4.7980	4.7980	26.0189	26.0191	26.0191	56.0753	56.0753
	1.5	4.6409	4.6416	4.6416	22.8917	22.8918	22.8918	45.2576	45.2576
	2	4.4455	4.4462	4.4462	19.9529	19.9532	19.9532	37.1012	37.1012
Clamped	0	10.2158	10.2158	10.2158	39.7706	39.7711	39.7711	89.1041	89.1041
	.5	10.1283	10.1285	10.1285	38.2059	38.2061	38.2061	81.3694	81.3694
	1	9.8784	9.8789	9.8789	34.4275	34.4275	34.4275	66.4873	66.4873
	1.5	9.4999	9.5002	9.5002	30.0446	30.0446	30.0445	53.3723	53.3723
	2	9.0348	9.0351	9.0351	26.0253	26.0253	26.0253	43.6484	43.6484

Table 3
Dimensionless natural frequency for simply support boundary condition obtained using DTM and Ritz methods.

	mode	1	2	3
Boundary condition	method			
		$\lambda = -.5$		
	Ritz	6.2929	37.7455	93.0764
Simply support	DTM	6.2927	37.7422	93.042
	[47]	6.2928	37.743	93.042
		$\lambda = -.3$		
	Ritz	5.7483	34.5633	85.6318
	DTM	5.7483	34.5625	85.6206
	[43]	5.7483	34.5625	85.6206
		$\lambda = -.1$		
	Ritz	5.2061	31.3465	78.0332
	DTM	5.2061	31.3465	78.0323
	[47]	5.2061	31.3465	78.0323
		$\lambda = 0$		
	Ritz	4.9351	29.7200	74.1561
	DTM	4.9351	29.7200	74.1561
	[47]	4.9351	29.7200	74.1561
		$\lambda = .1$		
	Ritz	4.6637	28.0775	70.2132
	DTM	4.6637	28.0774	70.2127
	[47]	4.6637	28.0774	70.2127
		$\lambda = .3$		
	Ritz	4.1158	24.7268	62.0726
	DTM	4.1158	24.7265	62.0704
	[47]	4.1158	24.7265	62.0704
		$\lambda = .5$		
	Ritz	3.5499	21.2391	53.4435
	DTM	3.5498	21.2386	53.4402
	[47]	3.5498	21.2386	53.4402

Table 4
Natural frequency obtained from two different solution method for nano-plate having various thickness variation slope and $\mu = 1$.

	mode	1	2	3
Boundary conditions	method			
		$\lambda = -0.5$		
	Ritz	6.0974	32.9109	70.0513
Simply support	DTM	6.0637	32.8659	69.8449
	Ritz	13.7565	44.0732	83.2819
clamp support	DTM	13.6506	44.0954	83.0025
		$\lambda = -0.3$		
	Ritz	5.5764	30.1859	64.5762
Simply support	DTM	5.5569	30.1699	64.4861
	Ritz	12.2037	40.2832	76.7279
clamp support	DTM	12.1460	40.3053	76.5920
		$\lambda = -0.1$		
	Ritz	5.0574	27.4216	58.9632
Simply support	DTM	5.0512	27.4195	58.9369
	Ritz	10.6535	36.4071	69.9725
clamp support	DTM	10.6362	36.4167	69.9315
		$\lambda = 0.1$		
	Ritz	4.5380	24.5993	53.1217
Simply support	DTM	4.5439	24.5981	53.1443
	Ritz	9.1044	32.4143	62.9163
clamp support	DTM	9.1195	32.4024	62.9520
		$\lambda = 0.3$		
	Ritz	4.0130	21.6918	46.9665
Simply support	DTM	4.0297	21.6783	47.0232
	Ritz	7.5535	28.2598	55.4510
clamp support	DTM	7.5922	28.2181	55.5428
		$\lambda = 0.5$		
	Ritz	3.4701	18.6477	40.3437
Simply support	DTM	3.4959	18.6083	40.4187
	Ritz	5.9934	23.8617	47.3794
clamp support	DTM	6.0460	23.7833	47.5045

Table 5
Values of frequencies of higher vibration modes calculated employing DTM and Ritz method.

Boundary conditions	e_0a	Mode number					
	mode	1		2		3	
		Ritz	DTM	Ritz	DTM	Ritz	DTM
			$\lambda = .5$				
Simple supported	0	6.2927	6.2927	37.7455	37.7455	93.0368	93.0368
	.5	6.2421	6.2330	36.3375	36.3242	85.2380	85.1579
	1	6.0973	6.0637	32.9083	32.8659	70.0126	69.8449
	1.5	5.8768	5.8097	28.8644	28.7925	56.3730	56.1878
	2	5.6048	5.5025	25.0947	25.0004	46.1469	45.9783
clamped	0	14.3022	14.3021	51.3485	51.3480	112.6385	112.6361
	.5	14.1598	14.1309	49.1988	49.2046	102.4920	102.3765
	1	13.7563	13.6506	44.0703	44.0954	83.2490	83.0025
	1.5	13.1520	12.9461	38.2252	38.2788	66.5699	66.2803
	2	12.4228	12.1176	32.9554	33.0369	54.3415	54.0490
			$\lambda = -.5$				
Simple supported	0	3.5498	3.5498	21.2386	21.2386	53.4408	53.4408
	.5	3.5294	3.5361	20.4917	20.4794	49.0219	49.0538
	1	3.4701	3.4959	18.6477	18.6083	40.3435	40.4187
	1.5	3.3776	3.4319	16.4381	16.3713	32.5305	32.6246
	2	3.2597	3.3480	14.3485	14.2599	26.6574	26.7566
clamped	0	6.1504	6.1505	27.3003	27.3002	63.0615	63.0611
	.5	6.1100	6.1238	26.3034	26.2802	57.7431	57.7919
	1	5.9934	6.0460	23.8617	23.7833	47.3791	47.5045
	1.5	5.8130	5.9229	20.9690	20.8261	38.1231	38.3043
	2	5.5853	5.7631	18.2625	18.0584	31.2040	31.4286

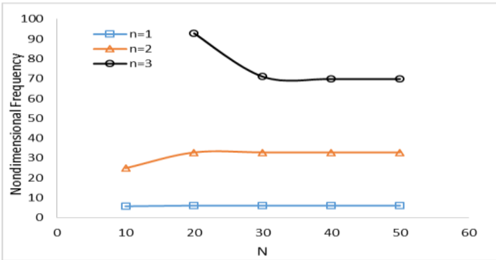

Fig.2
Variation of three first frequencies as the number of basis functions increases for Ritz method.

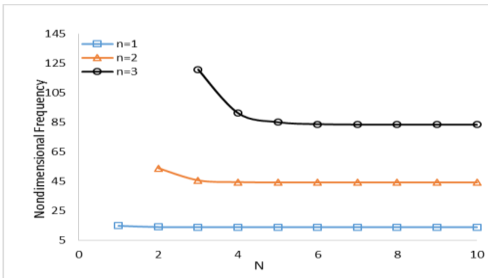

Fig.3
Variation of three first frequencies as the number of series terms increases for DTM.

3.2 Parametric studies

In this section, result obtained using validated solution above are given to study influence of various parameters.

3.2.1 Effect of characteristic length

Fig. 4 presents variations of first three non-dimensional natural frequencies of circular nano-plate versus characteristic length with two thickness slope parameter for clamped boundary condition. The figure shows as the nonlocal characteristic length increases the non-dimensional frequency decreases especially in higher frequency modes. Furthermore, positive thickness slope parameter results in higher frequencies. Fig. 5 shows variations of first three non-dimensional natural frequencies of circular nano-plate versus nonlocal characteristic length with two thickness slope parameter for simply supported boundary condition. Comparison of Figs. 4, 5 suggests corresponding frequencies of clamped boundary condition are more than those of simply supported boundary condition.

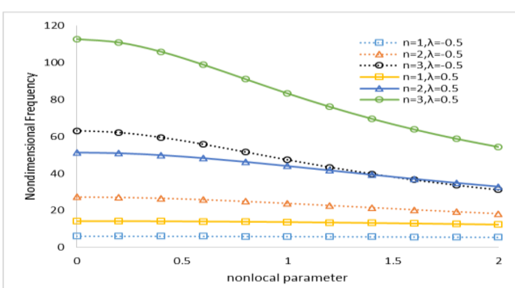

Fig.4
Variations of first three non-dimensional natural frequencies of clamped circular nano-plate versus characteristic length with two thickness parameter.

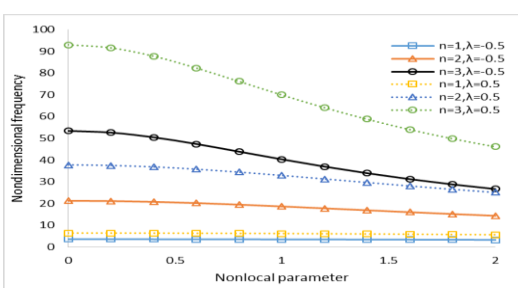

Fig.5
Variations of first three non-dimensional natural frequencies of simply supported circular nano-plate versus characteristic length with two thickness parameter.

3.2.2 Effect of thickness parameter

Fig. 6 depicts first three non-dimensional natural frequency of circular nano-plate vs. thickness slope parameter with two nonlocal characteristic length. It is observed that increasing the thickness slope parameter causes rise of non-dimensional frequency. However, the increasing rate of non-dimensional frequency is lower in higher value of thickness parameter. Fig. 7 illustrates first three non-dimensional natural frequency of circular nano-plate vs. thickness slope parameter with two nonlocal characteristic length and two plate radius. The figure shows as nano-plate radius increases, the non-dimensional frequencies decline. This decrease of frequencies is more for higher vibrational modes and larger thickness slope parameters.

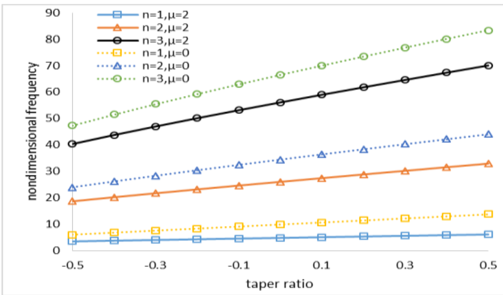

Fig.6
Variation of first three non-dimensional natural frequency of circular nano-plate *vs*. thickness slope parameter with two nonlocal characteristic length.

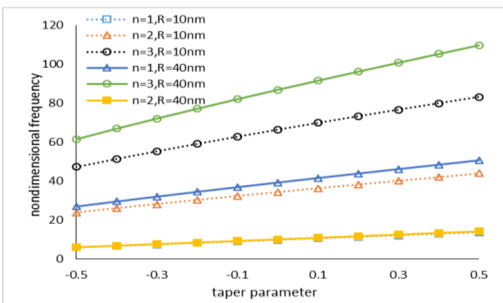

Fig.7
Variation of first three non-dimensional natural frequency of circular nano-plate *vs*. thickness slope parameter with two nonlocal characteristic length and two plate radius.

3.2.2 Effect of nano-plate radius

Effect of plate radius on first natural frequency of nano-plate is shown for various thickness slope parameter in Fig. 8. It is observed as plate radius increases plate non-dimensional frequencies increases and finally converges to classical plate result. The figure shows the nano-plates having lower thickness slopes parameters converges rapidly. Fig. 9 presents first three non-dimensional frequencies *vs* radius plate with two different thickness slope parameter. It is concluded the first natural frequency approximately remained constant, whereas higher mode frequencies rapidly rise as nano-plate radius increases. Furthermore, non-dimensional frequencies of nano-plates having positive thick slope parameter go up rapidly compared to the negative parameter.

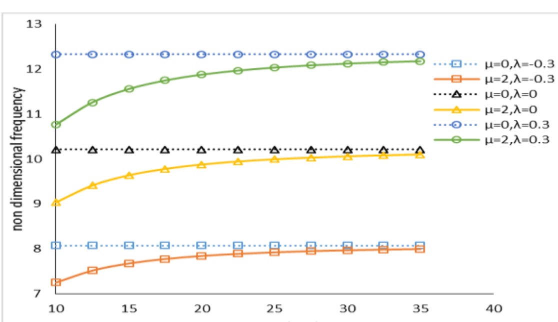

Fig.8
Variation of first non-dimensional frequency *vs* radius plate with two different thickness slope parameter and characteristic lengths.

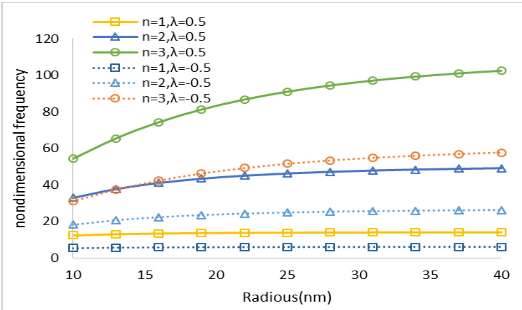

Fig.9
Variation of first three non-dimensional frequencies *vs* radius plate with two different thickness slope parameter.

3.2.4 Effect of orthotropic properties

Fig. 10 illustrates variations of non-dimensional natural frequency of nano-plate *vs*. orthotropic parameter with two different thickness slope parameters and characteristic lengths. As can be seen in the figure when orthotropic parameter increases, non-dimensional frequency rises. Also, a decrease in characteristic length causes a significant rise in non-dimensional frequency value. However, frequency rise is negligible for $\lambda = -0.5$ compared to that of $\lambda = 0.5$.

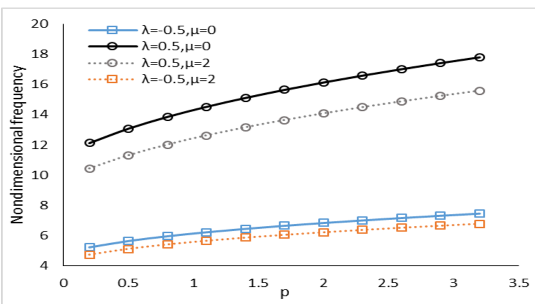

Fig.10
Variations of non-dimensional natural frequency of nano-plate *vs*. Orthotropy parameter $p = \sqrt{\dfrac{E_\theta}{E_r}}$ with two different thickness slopes parameter and characteristic length.

4 CONCLUSIONS

In present study, free vibration analysis of non-uniform thin nano-plate was investigated. Eringen nonlocal elasticity theory was used to take into account size effects. Afterwars, approperiate govening equations were derived for Ritz method and DTM. Convergense study was performed to find enough terms for accurate results. Results for local/classical plates and constant thickness plate was obtained as a special case and was compared to the available literature. The influense of nonlocal characteristic length, thickness slope parameter, nano-plate radius and boundary condition on natural frequency was studied. It was observed nonlocal characteristic length has a profound effect on higher frequencies modes. Also, the results showed corresponding frequencies of clamped boundary condition are more than those of simply supported boundary condition. It was demonstrated increasing in the thickness slope parameter causes rise of non-dimensional frequency. However, increasing rate of non-dimensional frequency was lower in higher value of thickness parameter. It was shown the free vibration response of nano-plates having lower thickness slopes parameters converges rapidly to that of classical plates as plate radius increases. Higher natural frequencies rapidly rose compared to lower natural frequencies when nano-plate radius increased. A decrease in nonlocal parameter resulted in a substantial rise in natural frequencies of nano-plate having positive thickness slope parameter (i.e. $\lambda = 0.5$) in comparison with nano-plate having negative thickness slope parameter (i.e. $\lambda = -0.5$).

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