Journal of Solid Mechanics Vol. 1, No. 4 (2009) pp. 358-364

RESEARCH NOTE

First-Order Formulation for Functionally Graded Stiffened Cylindrical Shells Under Axial Compression

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Received 4 October 2009; accepted 18 December 2009

ABSTRACT
The buckling analysis of stiffened cylindrical shells by rings and stringers made of functionally graded materials subjected to axial compression loading is presented. It is assumed that the material properties vary as a power form of the thickness coordinate variable. The fundamental relations, the equilibrium and stability equations are derived using the first order shear deformation theory. Resulting equations are employed to obtain the critical buckling loads. The effects of the material properties and geometry of shell on the critical buckling loads are examined. Excellent agreement with the results in the literature indicates the correctness of the proposed closed form solution.

Keywords: Functionally graded material; Stiffened cylindrical shell; First-order theory; Axial compression

1 INTRODUCTION

ANALYSIS of stiffened cylindrical shells is an important topic in modern engineering, especially in aircraft and spacecraft industry. Due to the increasing demands of high structural performance requirements, the study of functionally graded structures has received considerable attention in recent years. There have been many studies on the stability of cylindrical shells but closed-form solutions are possible only for the case which all edges are simply supported. The stabilization of a functionally graded (FG) cylindrical shell under axial harmonic loading is investigated by Ng et al. [1]. Yaffe and Abramovich [2] have analyzed numerically and experimentally the dynamic buckling of cylindrical stringer-stiffened shells. Rikards et al. [3] employed a triangular finite element model to study the buckling and vibration of laminated composite stiffened shells and plates based on the first order shear deformation theory. Khazaeinejad et al. [4] developed the first order shear deformation theory (FSDT) to study the critical buckling loads of FG cylindrical shells under three types of mechanical loadings.

The aim of the present paper is to obtain the critical buckling loads of functionally graded stiffened cylindrical shells by rings and stringers under axial compression loading, and to investigate the effects of the material properties and geometry of shell on the critical buckling loads. The first order shear deformation theory is employed to derive the equilibrium and stability equations.

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2 BRIEF DESCRIPTION OF THE METHOD

A cylindrical shell of mean radius a, thickness h, and length L with the cylindrical coordinates (x,θ,z) made of functionally graded materials is considered. The Young's modulus of shell is assumed to vary as a power form of the thickness coordinate z, that is [4]

$$E(z)=E_m+(E_c-E_m)\left(\frac{2z+h}{2h}\right)^k,\quad (-h/2\le z\le h/2) \tag{1}$$

Here, k is non-negative real number called the inhomogeneity parameter and subscripts m and c refer to the metal and ceramic constituents, respectively. The first order shear deformation theory (FSDT), used in the present study, is based on the following displacement field

$$\begin{aligned} u(x,\theta,z)&=u_0(x,\theta)+zu_1(x,\theta)\\ v(x,\theta,z)&=v_0(x,\theta)+zv_1(x,\theta)\\ w(x,\theta,z)&=w_0(x,\theta)\end{aligned} \tag{2}$$

where u_0, v_0, and w_0 are the displacements of a point on the mid-surface of the shell along the $x-$, $\theta-$, and $z-$ axes, respectively and u_1 and v_1 describe the rotations about the $\theta-$ and $x-$ axes, respectively, u, v and w are the axial, circumferential, and lateral displacements of shell, respectively, ε_x, ε_θ and $\gamma_{x\theta},\gamma_{\theta z},\gamma_{xz}$ are respectively the normal and shear strains, and k_x, k_θ and $k_{x\theta}$ are the curvatures. Also, the indices x and θ refer to the axial and circumferential directions, respectively. A thin-walled FG cylindrical shell, stiffened by closely spaced circular rings attached to the inside of the shell skin and with longitudinal stringers attached to the outside is considered (see Fig. 2). We assumed that the stiffeners and skin are made of functionally graded materials. The constitutive relations of FG stiffened cylindrical shells based on the FSDT are expressed as [5]

$$\begin{aligned}
N_x&=\left(\frac{E_1}{1-\nu^2}+\frac{E_{1s}b_s}{d_s}\right)\varepsilon_x^0+\left(\frac{\nu E_1}{1-\nu^2}\right)\varepsilon_\theta^0+\left(\frac{E_2}{1-\nu^2}+\frac{E_{2s}b_s}{d_s}\right)k_x+\left(\frac{\nu E_2}{1-\nu^2}\right)k_\theta\\
N_\theta&=\left(\frac{\nu E_1}{1-\nu^2}\right)\varepsilon_x^0+\left(\frac{E_1}{1-\nu^2}+\frac{E_{1r}b_r}{d_r}\right)\varepsilon_\theta^0+\left(\frac{\nu E_2}{1-\nu^2}\right)k_x+\left(\frac{E_2}{1-\nu^2}+\frac{E_{2r}b_r}{d_r}\right)k_\theta\\
N_{x\theta}&=\left(\frac{E_1}{2+2\nu}\right)\gamma_{x\theta}^0+\left(\frac{E_2}{2+2\nu}\right)k_{x\theta}\\
M_x&=\left(\frac{E_2}{1-\nu^2}+\frac{E_{2s}b_s}{d_s}\right)\varepsilon_x^0+\left(\frac{\nu E_2}{1-\nu^2}\right)\varepsilon_\theta^0+\left(\frac{E_3}{1-\nu^2}+\frac{E_{3s}b_s}{d_s}\right)k_x+\left(\frac{\nu E_3}{1-\nu^2}\right)k_\theta\\
M_\theta&=\left(\frac{\nu E_2}{1-\nu^2}\right)\varepsilon_x^0+\left(\frac{E_2}{1-\nu^2}+\frac{E_{2r}b_r}{d_r}\right)\varepsilon_\theta^0+\left(\frac{\nu E_3}{1-\nu^2}\right)k_x+\left(\frac{E_3}{1-\nu^2}+\frac{E_{3r}b_r}{d_r}\right)k_\theta\\
M_{x\theta}&=\left(\frac{E_2}{2+2\nu}\right)\gamma_{x\theta}^0+\left(\frac{E_3}{1+\nu}+\frac{1}{2}\left(\frac{G_sJ_s}{d_s}+\frac{G_rJ_r}{d_r}\right)\right)k_{x\theta}\\
Q_x&=\left(\frac{E_1}{2+2\nu}\right)\gamma_{xz}\\
Q_\theta&=\left(\frac{E_1}{2(1+\nu)}\right)\gamma_{\theta z}
\end{aligned} \tag{3}$$

where

$$\varepsilon_x^0 = u_{0,x} + \frac{1}{2}w_{0,x}^2, \qquad \varepsilon_\theta^0 = \frac{v_{0,\theta} + w_0}{a} + \frac{w_{0,\theta}^2}{2a^2}$$

$$\gamma_{x\theta}^0 = \frac{u_{0,\theta}}{a} + v_{0,x} + \frac{w_{0,x}w_{0,\theta}}{a}, \quad k_x = u_{1,x}, \quad k_\theta = \frac{v_{1,\theta}}{a} \tag{4a}$$

$$k_{x\theta} = \frac{u_{1,\theta}}{a} + v_{1,x}, \quad \gamma_{xz} = u_1 + w_{0,x}, \quad \gamma_{\theta z} = v_1 + \frac{w_{0,\theta}}{a}$$

$$E_1 = E_m h + \frac{E_{cm} h}{k+1}, \qquad E_2 = \frac{kE_{cm}h^2}{2(k+1)(k+2)}$$

$$E_3 = \frac{E_m h^3}{12} + E_{cm}h^3\left(\frac{1}{4(k+1)} - \frac{1}{k+2} + \frac{1}{k+3}\right) \tag{4b}$$

$$E_{1s} = \int_{-\frac{h_s}{2}}^{\frac{h_s}{2}} E(z)\ \mathrm{dz}, \qquad E_{1r} = \int_{-\frac{h_r}{2}}^{\frac{h_r}{2}} E(z)\ \mathrm{dz}$$

$$(E_{2s}, E_{3s}) = \int_{e_s - \frac{h_s}{2}}^{e_s + \frac{h_s}{2}} E(z)\ (z, z^2)\ \mathrm{dz}$$

$$(E_{2r}, E_{3r}) = \int_{e_r - \frac{h_r}{2}}^{e_r + \frac{h_r}{2}} E(z)\ (z, z^2)\ \mathrm{dz} \tag{4c}$$

where subscripts s and r refer to the stringers and rings, respectively. The thickness and width of stringers are respectively denoted by h_s and b_s and for rings are h_r and b_r. Also, d_s and d_r are the distances between two stringers and rings, respectively. The eccentricities e_s and e_r represent the distance from the shell middle surface to the centroid of the stiffener cross section. In Eqs. (3), the stress resultants N_i and M_i are expressed as

$$(N_i, M_i) = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_i (1, z)\,\mathrm{dz} \qquad\qquad i = x, \theta, x\theta$$

$$Q_i = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{iz}\ \mathrm{dz} \qquad\qquad i = x, \theta \tag{5}$$

Using the minimum potential energy criterion [5], the equilibrium equations of FG stiffened cylindrical shells are given by

$$N_{x,x} + \frac{1}{a}N_{x\theta,\theta} = 0$$

$$N_{x\theta,x} + \frac{1}{a}N_{\theta,\theta} = 0$$

$$M_{x,x} + \frac{1}{a}M_{x\theta,\theta} - Q_x = 0 \tag{6}$$

$$M_{x\theta,x} + \frac{1}{a}M_{\theta,\theta} - Q_\theta = 0$$

$$Q_{x,x} + \frac{1}{a}Q_{\theta,\theta} + N_x w_{,xx} + \frac{1}{a^2}N_\theta w_{,\theta\theta} + \frac{2}{a}N_{x\theta}w_{,x\theta} - \frac{1}{a}N_\theta = -P$$

The stability equations of FG cylindrical shell may be derived by the variational approach. If V is the total potential energy of the shell, the first variation δV is associated with the state of equilibrium. The stability of the original configuration of the shell in the neighborhood of the equilibrium state can be determined by the sign of second variation $\delta^2 V$. However, the condition of $\delta^2 V$ =0 is used to derive the stability equations of many practical problems on the buckling of shells [5]. Thus, the stability equations are represented by the Euler equations for the integrand in the second variation expression as

$$
\begin{aligned}
&N_{x1,x} + \frac{1}{a} N_{x\theta 1,\theta} = 0 \\
&N_{x\theta 1,x} + \frac{1}{a} N_{\theta 1,\theta} = 0 \\
&M_{x1,x} + \frac{1}{a} M_{x\theta 1,\theta} - Q_{x1} = 0 \\
&M_{x\theta 1,x} + \frac{1}{a} M_{\theta 1,\theta} - Q_{\theta 1} = 0 \\
&Q_{x1,x} + \frac{1}{a} Q_{\theta 1,\theta} + N_{x0} w^1_{,xx} + \frac{1}{a^2} N_{\theta 0} w^1_{,\theta\theta} + \frac{2}{a} N_{x\theta 0} w^1_{,x\theta} - \frac{1}{a} N_{\theta 0} = 0
\end{aligned}
\tag{7}
$$

The terms with the subscript 0 are related to the state of equilibrium and terms with the subscript 1 are those characterizing the state of stability. By substituting Eqs. (3) into (7), the stability equations can be derived in terms of displacement components. To determine the critical buckling loads, the prebuckling mechanical forces should be found from the equilibrium equations and then substituted into the stability equations for the buckling analysis. Under an uniformly distributed axial compressive load P, the cylinder shortens, except at the ends, and increases in diameter. The initial deformation is axisymmetric and the prebuckling mechanical forces are given by [5]

$$
N_{\theta 0} = N_{x\theta 0} = 0, \qquad N_{x0} = -\frac{P}{2\pi a} \tag{8}
$$

Upon substituting the prebuckling forces into the stability equations in terms of displacement components, a set of five differential equations is obtained. To solve this set of equations, the following approximate solutions which satisfy the resulting equations and the simply supported boundary conditions are assumed

$$
\begin{aligned}
&u^1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} \cos \lambda x \sin n\theta \\
&v^1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \sin \lambda x \cos n\theta \\
&w^1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin \lambda x \sin n\theta \\
&u_1^1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_{mn} \cos \lambda x \sin n\theta \\
&v_1^1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Y_{mn} \sin \lambda x \cos n\theta
\end{aligned}
\tag{9}
$$

where $\lambda = m\pi / L$. Substituting relations (9) into the stability equations in terms of the displacement components gives

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{12} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{13} & K_{23} & K_{33}+N_{x0}\lambda^2 & K_{34} & K_{35} \\ K_{14} & K_{24} & K_{34} & K_{44} & K_{45} \\ K_{15} & K_{25} & K_{35} & K_{45} & K_{55} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ w_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} = 0 \tag{12}$$

where

$$\begin{aligned}
&K_{11} = C_{11}\lambda^2 + \frac{C_{33}n^2}{a^2}, \qquad K_{12} = \frac{(C_{12}+C_{33})\lambda n}{a} \\
&K_{13} = -\frac{C_{12}\lambda}{a}, \qquad K_{14} = C_{16}\lambda^2 + \frac{C_{38}n^2}{a^2} \\
&K_{15} = \frac{(C_{26}+C_{38})\lambda n}{a}, \qquad K_{22} = C_{33}\lambda^2 + \frac{C_{22}n^2}{a^2} \\
&K_{23} = -\frac{C_{22}n}{a}, \qquad K_{24} = \frac{(C_{26}+C_{38})\lambda n}{a} \\
&K_{25} = C_{38}\lambda^2 + \frac{C_{27}n^2}{a^2}, \qquad K_{33} = \frac{C_{22}}{a^2} + C_{33}\lambda^2 + \frac{C_{33}n^2}{a^2} \\
&K_{34} = C_{33}\lambda - \frac{C_{26}\lambda}{a}, \qquad K_{35} = -\frac{C_{27}n}{a^2} + \frac{C_{33}n}{a} \\
&K_{44} = C_{46}\lambda^2 + \frac{C_{68}n^2}{a^2} + C_{33}, \qquad K_{45} = \frac{(C_{47}+C_{68})\lambda n}{a} \\
&K_{55} = C_{68}\lambda^2 + \frac{C_{57}n^2}{a^2} + C_{33}
\end{aligned} \tag{13}$$

By setting the determinant of $[K]$ equal to zero to obtain the non-zero solution, the value of P can be found. The critical buckling load can be obtained by minimizing P with respect to m and n, the number of longitudinal and circumferential buckling waves.

3 NUMERICAL RESULTS

For the given values of the inhomogeneity parameter and thickness of shell, the values of *m* and *n* may be chosen by trial to give the smallest value of buckling load *P*. These values can be obtained by a suitable software or optimization program. To investigate the accuracy of the present method, comparison studies are presented. A ceramic-metal FG stiffened cylindrical shell is considered. The FG stiffened shell constituents are zirconia and aluminum. The Young's modulus for zirconia and aluminum are 151 GPa and 70 GPa, respectively. The Poisson's ratio is assumed to be constant and equal to 0.3. As a numerical example, we consider an FG stiffened cylindrical shell with 15 rings and stringers. Let L=387.35×10^{-3} m, a=60.643×10^{-3} m, h_s=0.076×10^{-3} m, b_s=21.155×10^{-3} m, h_r= 0.127 ×10^{-3} m, and b_r=1.27×10^{-3} m. A comparison between the critical buckling loads of unstiffened and stiffened homogeneous shell using the Donnell's shell theory and finite element method [6], and the present work are given in Table 1. In Table 2, the results are presented for unstiffened and stiffened FG cylindrical shells. As can be seen, the agreement between the results is satisfactory.

The analytical predictions are very close to the FEM results with only about 5% difference. When the shell is stiffened, the difference between the analytical and FEM results is only about 3%. It is evident that the buckling loads of shell increase as the shell becomes thicker. It is interesting to note that, when the shell thickness has doubled, the critical buckling loads almost have quadrupled. The critical buckling load is decreased while the inhomogeneity parameter is increased. This decrease is about 29% for $k=0$ and 1. The critical buckling loads for homogeneous and FG stiffened cylindrical shells are generally upper than the corresponding values for the unstiffened cylindrical shells.

Table 1
Comparison of critical buckling loads for simply supported homogeneous cylindrical shell

h (×10^{-3} m)		FSDT	Donnell [6]	FEM [6]
0.305	Unstiffened	0.083	0.083	0.086
	Stiffened	0.086	0.086	0.088
0.381	Unstiffened	0.129	0.129	0.132
	Stiffened	0.133	0.133	0.135
0.457	Unstiffened	0.185	0.186	0.188
	Stiffened	0.190	0.191	0.191
0.533	Unstiffened	0.252	0.253	0.252
	Stiffened	0.258	0.258	0.256
0.610	Unstiffened	0.329	0.331	0.325
	Stiffened	0.335	0.337	0.330
0.686	Unstiffened	0.417	0.419	0.408
	Stiffened	0.423	0.425	0.413
0.762	Unstiffened	0.514	0.517	0.486
	Stiffened	0.521	0.524	0.505

Table 2
Comparison of critical buckling loads for simply supported FG cylindrical shell

h (×10^{-3} m)		FSDT		Donnell [6]		FEM [6]	
		k=0.5	*k*=1	*k*=0.5	*k*=1	*k*=0.5	*k*=1
0.305	Unstiffened	0.066	0.059	0.066	0.059	0.069	0.061
	Stiffened	0.070	0.063	0.067	0.060	0.071	0.063
0.381	Unstiffened	0.103	0.092	0.103	0.092	0.106	0.094
	Stiffened	0.107	0.097	0.104	0.093	0.108	0.096
0.457	Unstiffened	0.148	0.132	0.149	0.132	0.151	0.133
	Stiffened	0.153	0.139	0.150	0.133	0.153	0.136
0.533	Unstiffened	0.202	0.179	0.202	0.180	0.203	0.179
	Stiffened	0.208	0.188	0.204	0.181	0.206	0.182
0.610	Unstiffened	0.263	0.234	0.264	0.235	0.262	0.231
	Stiffened	0.270	0.244	0.266	0.237	0.265	0.233
0.686	Unstiffened	0.333	0.296	0.334	0.298	0.327	0.288
	Stiffened	0.341	0.308	0.336	0.299	0.331	0.291
0.762	Unstiffened	0.411	0.366	0.413	0.368	0.400	0.353
	Stiffened	0.419	0.379	0.415	0.369	0.404	0.356

4 CONCLUSIONS

The present paper addresses to buckling problem of functionally graded stiffened cylindrical shells using the first-order shear deformation theory. The buckling loads are obtained for a various range of the thickness factor and compared with the published results based on classical shell theory and finite element solution. It is found that the difference between the values of buckling loads for classical and first-order shear deformation theories is only obvious for thick cylindrical shells. Both theories provide approximately the same results for thin shells. For FG stiffened shells, the first-order shear deformation theory predicts higher values for buckling loads, while for FG unstiffened shells, the classical shell theory predicts higher values for buckling loads.

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