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A Comparative Study of Least-Squares and the Weak-Form Galerkin Finite Element Models for the Nonlinear Analysis of Timoshenko Beams

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ABSTRACT
In this paper, a comparison of weak-form Galerkin and least-squares finite element models of Timoshenko beam theory with the von Kármán strains is presented. Computational characteristics of the two models and the influence of the polynomial orders used on the relative accuracies of the two models are discussed. The degree of approximation functions used varied from linear to the 5th order. In the linear analysis, numerical results of beam bending under different types of boundary conditions are presented along with exact solutions to investigate the degree of shear locking in the newly developed mixed finite element models. In the nonlinear analysis, convergences of nonlinear finite element solutions of newly developed mixed finite element models are presented along with those of existing traditional model to compare the performance.

Keywords: Finite element models; Least-squares model; Weak-form Galerkin model; Geometric nonlinearity; Timoshenko beam theory

1 INTRODUCTION

FOR several decades, finite element models based on the weak-form Galerkin or Ritz formulation [1] have dominated both commercial finite element software and academic research. An advantage of the weak-form Galerkin model as applied to structural mechanics problems is that it is the most natural approach resulting from the application of the principle of virtual displacements. For problems outside of solid and structural mechanics, such principles are not available. It is possible, however, to develop the so-called weak forms or weighted-residual integral statements from the governing differential equations of any physical problem. The weak forms, in most cases, are not equivalent to any principle or minimization of error introduced in the approximation of the variables. Thus, the weak forms of problems for which there is no underlying physical or mathematical principle, are merely a means to computing solutions of the integral statements represented by the weak forms [2]. Consequently, it is possible that such solutions may degenerate for certain combinations of physical characteristics of the problem and finite element approximations used. It is well known that the weak formulations lower the differentiability requirements on the variables of the formulation. The traditional displacement based (i.e., principle of virtual displacements) finite element models can provide high level of accuracy only for the generalized displacements, increasing errors in the post-computed variables involving the derivatives of the generalized displacements. To remedy this, the so-called *mixed formulations* [3] are adopted. The mixed models include secondary variables as independent degrees of freedom and they yield increased accuracy of the secondary variables, sometimes at the expense of the primary variables and increased computational effort. Thus weak-form, mixed, Galerkin finite element models can provide higher level of accuracy for secondary variables included in the model. But to establish

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mixed Galerkin finite element models, each of the governing equation should be multiplied by proper weight functions, and then should be integrated over the computational domain. The Lagrange multiplier method can be used to find proper weight functions, which is still burdensome.

In recent times, the least-squares method is considered as a good alternative method because of its simplicity in applications to a variety of problems. It can also overcome many drawbacks of the weak-form Galerkin formulations. A brief idea of the least-squares method is to minimize sums of squares of the residuals over the finite element space. In the least-squares method, procedures of finding proper weight function for each residual is unnecessary. Because variations of residuals can be used as weight function from the weighted integral viewpoint. Despite of the computational advantage and simplicity that the least squares method can provide, applying this method to the analysis of practical structural problems with mixed formulation is not easy, especially in the case of the structural problems which include geometric nonlinearity. For example, in structural mechanics problems, the displacements are numerically very small compared to forces or stresses. Thus, squares of the residuals of equations involving force like variables are very large compared to those containing displacements in the least-squares functional. To handle these magnitude differences, variables can be properly nondimensionalized or residuals themselves should be multiplied by proper scale factors. The latter approach was adopted in the present paper.

Both the least-squares and weak-form Galerkin methods may have their own merits and demerits. The purpose of this paper is to provide some insights and proper strategies on the development of nonlinear finite element models in the structural problems using the two different formulations and to compare the computational aspects of various finite element models of Timoshenko beam theory with the von Kármán nonlinearity.

2 GOVERNING EQUATIONS

To develop new mixed finite models and verify their performance, the Timoshenko beam theory (TBT) with the *von Kármán nonlinearity* was considered for. First, the governing equations are reviewed briefly (see Reddy [1-4] for details). In the TBT, the displacement field is adopted as to include the transverse shear deformation (strain) in the simplest way, as shown in Fig. 1. The components u_1, u_2 and u_3 of the displacement field are

$$u_1 = u_0(x) + z\phi_x(x), \qquad u_2 = 0, \qquad u_3 = w_0(x) \tag{1}$$

where the u_0 is axial displacement of centroid line of cross section along the x-axis, the w_0 is vertical displacement of the center line along the z-axis, and z is a vertical distance between the centroid line and arbitrary point of the beam.

The von Kármán nonlinear strains of the TBT are

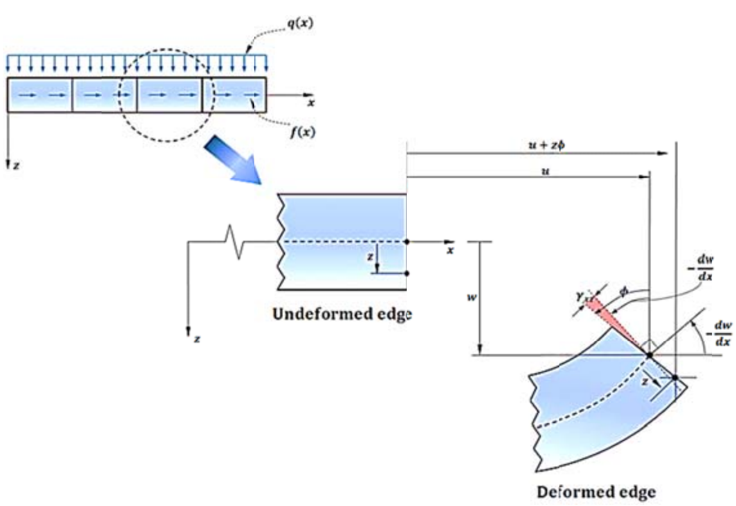


Fig. 1
Description of undeformed and deformed edges of the TBT beam, source from.

$$\varepsilon_{xz} = 0.5\ (w_0' + \phi_x), \qquad \varepsilon_{xx} = u_0' + 0.5\ (w_0')^2 + z\phi_x' \tag{2}$$

The constitutive relations are

$$\sigma_{xx} = E\varepsilon_{xx}, \qquad \sigma_{xz} = 2G\varepsilon_{xz} \tag{3}$$

where E is Young's modulus and G is shear modulus. The stress resultants are

$$\begin{aligned} N_{xx} &= \int_A \sigma_{xx} \mathrm{d}y\ \mathrm{d}z = EA\ [u_0' + 0.5\ (w_0')^2] \\ Q_x &= \int_A \sigma_{xz}\ \mathrm{d}y\ \mathrm{d}z = K_s GA\ (\phi_x + w_0') \\ M_{xx} &= \int_A z\sigma_{xx} \mathrm{d}y\ \mathrm{d}z = EI\phi_x' \end{aligned} \tag{4}$$

where the K_s (5/6) is shear correction factor, the N_{xx} is axial force, the V_x is the transverse shear force, and M_{xx} is bending moment at any cross section of the beam. The equilibrium equations of the TBT are

$$N_{xx}' + f(x) = 0, \qquad (Q_x + w_0' N_{xx})' + q(x) = 0, \qquad M_{xx}' - Q_x = 0 \tag{5}$$

where $f(x)$ and $q(x)$ are distributed axial force and transverse loads, respectively. The resultant forces and bending moment can be included in the finite element models to develop mixed finite element models, which will be discussed in the next section.

3 FINITE ELEMENT MODELS

Traditionally, the three equilibrium equations in (5) are written in terms of the generalized displacements (u_0, w_0, ϕ_x) to develop the displacement finite element model of the TBT. Here, we discuss weighted-residual finite element models of these equations. Since we approximate the variables of the formulation, the differential equations in (4) and (5) are not satisfied exactly everywhere in the domain. Thus, there is error introduced into the differential equations, called residuals. All approximation methods try to minimize the residuals in some suitable sense. For example, consider the three equations in Eq. (5), expressed in terms of the displacements. The residuals in the three equations are

$$\begin{aligned} R_1 &= -\{EA\ [u_0' + 0.5\ (w_0')^2]\} - f(x) \neq 0 \\ R_2 &= -\{K_s GA\ (\phi_x + w_0') + w_0' EA\ [u_0' + 0.5\ (w_0')^2]\}' - q(x) \neq 0 \\ R_3 &= -EI\phi_x'' + K_s GA\ (\phi_x + w_0') \neq 0 \end{aligned} \tag{6}$$

The resulting weighted-residual or weak-form model will contain only the displacement variables. On the other hand, one may also consider Eqs. (4) and (5) to be six independent equations and treat three displacements (u_0, w_0, ϕ_x) and there stress resultants (N_{xx}, V_x, M_{xx}) as independent variables. The resulting finite element model is termed a *mixed model* of the TBT because it contains displacements as well as stress resultants:

$$\begin{aligned} R_1 &= N_{xx}' + f(x) \neq 0, \\ R_2 &= (Q_x + w_0' N_{xx})' + q(x) \neq 0, \\ R_3 &= M_{xx}' - Q_x \neq 0, \\ R_4 &= (K_s GA)^{-1} Q_x - (\phi_x + w_0') \neq 0, \\ R_5 &= (EA)^{-1} N_{xx} - (u_0' + 0.5 w_0' w_0') \neq 0, \\ R_6 &= (EI)^{-1} M_{xx} - \phi_x' \neq 0. \end{aligned} \tag{7}$$

An advantage of mixed formulations is that they yield increased accuracy of the resultants and they can be computed at the nodes as opposed to the Gauss points in displacement finite element models. As a special case of the weighted -residual method, one may construct least-squares finite element models. In Eq. (7), one can find that the differentiability requirements on u_0, w_0 and ϕ_x are lowered by inclusion of the stress resultants N_{xx}, V_x and M_{xx}.

3.1 Traditional displacement based Galerkin weak -form model

In this section, a traditional displacement based Galerkin model is reviewed to provide better comparison of the characteristics and behavior of the newly developed mixed models herein. The traditional displacement based model includes only u_0, w_0 and ϕ_x as nodal variables, yields the smallest stiffness matrix. The weak formulation of the traditional displacement model is based on the statements [1, 3]

$$
\begin{aligned}
&\int_0^{L_e} [EA\boldsymbol{\delta} u_0'[u_0' + 0.5(w_0')^2] - f(x)\boldsymbol{\delta} u_o]\,\mathrm{d}x - \left[\boldsymbol{\delta} u_o Q_1\right]\Big|_0^{L_e} \\
&\int_0^{L_e} \{K_s GA\boldsymbol{\delta} w_0'(\phi_x + w_0') + w_0' EA\boldsymbol{\delta} w_0'[u_0' + 0.5(w_0')^2] - q(x)\boldsymbol{\delta} w_o\}\,\mathrm{d}x - \left[\boldsymbol{\delta} w_o Q_2\right]\Big|_0^{L_e} \\
&\int_0^{L_e} [EI\boldsymbol{\delta}\phi_x'\phi_x' + K_s GA\boldsymbol{\delta}\phi_x(\phi_x + w_0')]\,\mathrm{d}x - \left[\boldsymbol{\delta}\phi_x Q_3\right]\Big|_0^{L_e}
\end{aligned}
\tag{8}
$$

where $Q_1=N_{xx}$, $Q_2 = Q_x + w_0' N_{xx} = V_x$, and $Q_3=M_{xx}$ at the boundaries. In the traditional displacement models, boundary conditions can be imposed exactly only on the displacements and in integral sense on the stress resultants. However, mixed models allow exact imposition of boundary conditions even on the stress resultants that appear in the model. The displacements appearing in Eq. (8) can be approximated by

$$
\begin{aligned}
u_0 &\cong \sum_{j=1}^{l} \psi_j u_j \\
w_0 &\cong \sum_{j=1}^{m} \psi_j w_j \\
\phi_x &\cong \sum_{j=1}^{n} \psi_j \phi_j
\end{aligned}
\tag{9}
$$

and substitution into Eq. (8) yields the following finite element equations:

$$
[K_T(\{\mathbf{U}_T\})]\{\mathbf{U}_T\} = \{\mathbf{F}_T\} \quad \text{or} \quad
\begin{bmatrix} [\mathbf{K}^{11}] & [\mathbf{K}^{12}] & [\mathbf{K}^{13}] \\ [\mathbf{K}^{21}] & [\mathbf{K}^{22}] & [\mathbf{K}^{23}] \\ [\mathbf{K}^{31}] & [\mathbf{K}^{32}] & [\mathbf{K}^{33}] \end{bmatrix}
\begin{Bmatrix} \{\boldsymbol{u}_j\} \\ \{\boldsymbol{w}_j\} \\ \{\boldsymbol{\phi}_j\} \end{Bmatrix}
=
\begin{Bmatrix} \{\mathbf{F}^1\} \\ \{\mathbf{F}^2\} \\ \{\mathbf{F}^3\} \end{Bmatrix}
\tag{10}
$$

where

$$
\begin{aligned}
&K_{ij}^{11} = \int_0^L (EA\psi_i'\psi_j')\,\mathrm{d}x, \quad K_{ij}^{12} = \int_0^L (0.5EAw_0'\psi_i'\psi_j')\,\mathrm{d}x \\
&K_{ij}^{23} = \int_0^L (K_s GA\psi_i'\psi_j)\,\mathrm{d}x, \quad K_{ij}^{22} = \int_0^L (K_s GA\psi_i'\psi_j' + 0.5EA(w_0')^2\psi_i'\psi_j')\,\mathrm{d}x \\
&K_{ij}^{21} = \int_0^L (EAw_0'\psi_i'\psi_j')\,\mathrm{d}x, \quad K_{ij}^{33} = \int_0^L (EI\psi_i'\psi_j' + K_s GA\psi_i\psi_j)\,\mathrm{d}x, \quad K_{ij}^{32} = K_{ji}^{23} \\
&f_i^1 = \int_0^L [f(x)\psi_i]\,\mathrm{d}x + [\psi_i Q_1]\Big|_0^{L_e}, \quad f_i^2 = \int_0^L [q(x)\psi_i]\,\mathrm{d}x + [\psi_i Q_2]\Big|_0^{L_e} \ \text{and} \ f_i^3 = [\psi_i Q_2]\Big|_0^{L_e}
\end{aligned}
$$

3.2 Mixed Galerkin model

For a mixed finite element model, the principle of minimum potential energy and the Lagrange multiplier method can be employed simultaneously to minimize the total potential energy and include the stress resultant-displacement relations as constrains through the Lagrange multiplier method. For the problem at hand, the total potential energy functional of the TBT beam can be expressed as

$$L_c = \int_0^{L_e} [0.5(\sigma_{xx}\varepsilon_{xx} + \sigma_{xz}\gamma_{xz}) - fu_0 - qw_0]\,\mathrm{d}x \tag{11}$$

where the L_c is the potential energy functional. Then minimizing L_c subject to the constraints can be expressed as

$$\begin{aligned} L_l(u_0, w_0, \phi_x, N_x, Q_x, M_x) &= L_c + \sum_{k=1}^{3}\left(\int_0^{L_e} \lambda_k G_k \;\mathrm{d}x\right) \\ &= \int_0^{L_e} [0.5(EA)^{-1} N_x^2 + 0.5(EI)^{-1} M_{xx}^2 + 0.5(K_s GA)^{-1} Q_x^2 - fu_0 - qw_0]\,\mathrm{d}x \\ &\quad + \int_0^{L_e} \lambda_1[(EA)^{-1} N_{xx} - (u_0' + 0.5 w_0' w_0')]\,\mathrm{d}x + \int_0^{L_e} \lambda_2\left[(K_s GA)^{-1} Q_x - (\phi_x + w_0')\right]\mathrm{d}x \\ &\quad + \int_0^{L_e} \lambda_3[(EI)^{-1} M_{xx} - \phi_x']\,\mathrm{d}x \end{aligned} \tag{12}$$

where L_l is the mixed functional and λ_i are the Lagrange multipliers. The condition for the minimum of L_l is δL_l=0, where the $\boldsymbol{\delta}$ is variational operator. Then we can obtain the following relations, which can be used to develop the mixed Galerkin finite element model. Since L_l contains 6 variables and their first derivatives only, coefficients of variations of the variables can be computed from the Euler-Lagrange equations [2] as follows:

$$\begin{aligned} &\boldsymbol{\delta} u_o : \frac{\partial L_l}{\partial u_0} - \frac{\partial}{\partial x}\left(\frac{\partial L_l}{\partial u_0'}\right) = \int_0^{L_e} (-f - \lambda_1')\,\mathrm{d}x = 0 \\ &\boldsymbol{\delta} w_o : \frac{\partial L_l}{\partial w_0} - \frac{\partial}{\partial x}\left(\frac{\partial L_l}{\partial w_0'}\right) = \int_0^{L_e} [-q - \lambda_2' - (\lambda_1 w_0')']\,\mathrm{d}x = 0 \\ &\boldsymbol{\delta} \phi_x : \frac{\partial L_l}{\partial \phi_x} - \frac{\partial}{\partial x}\left(\frac{\partial L_l}{\partial \phi_x'}\right) = \int_0^{L_e} (\lambda_2 - \lambda_3')\,\mathrm{d}x = 0 \\ &\boldsymbol{\delta} N_{xx} : \frac{\partial L_l}{\partial N_{xx}} = \int_0^{L_e} [(EA)^{-1} N_{xx} - (EA)^{-1}\lambda_1]\,\mathrm{d}x = 0 \\ &\boldsymbol{\delta} Q_x : \frac{\partial L_l}{\partial Q_x} = \int_0^{L_e} [(K_s GA)^{-1} Q_x - (K_s GA)^{-1}\lambda_2]\,\mathrm{d}x = 0 \\ &\boldsymbol{\delta} M_{xx} : \frac{\partial L_l}{\partial M_{xx}} = \int_0^{L_e} [(EI)^{-1} M_{xx} - (EI)^{-1}\lambda_3]\,\mathrm{d}x = 0 \end{aligned} \tag{13}$$

The Lagrange multipliers can be chosen to be $\lambda_1=N_x$, $\lambda_2=Q_x$ and $\lambda_3=M_x$ satisfying all conditions given in Eq. (7). By using the Lagrange multiplier method, all governing equations of the TBT beam can be fully recovered. Thus, we can develop mixed Galerkin finite element model with properly chosen weight functions for the residuals. We have

$$\begin{aligned} \boldsymbol{\delta} L_l(u_0, w_0, \phi_x, N_x, Q_x, M_x) = \int_0^{L_e} &\{\boldsymbol{\delta} u_o(-f - N_x') + \boldsymbol{\delta} w_o[-q - (Q_x + N_x w_0')'] + \boldsymbol{\delta}\phi_x(Q_x - M_x') \\ &+ \boldsymbol{\delta} N_{xx}\left[(EA)^{-1} N_{xx} - (u_0' + 0.5 w_0' w_0')\right] + \boldsymbol{\delta} Q_x\left[(K_s GA)^{-1} Q_x - (\phi_x + w_0')\right] \\ &+ \boldsymbol{\delta} M_{xx}(EI\phi_x' - M_x)\}\,\mathrm{d}x = 0 \end{aligned} \tag{14}$$

Note that following relation [3] can be used to lower differentiability requirement for the w_0 simplifying force vector terms in the matrix form of finite element equations.

$$V_x = Q_x + w_0' N_{xx}, \;\; Q_x = V_x - w' N_{xx} \tag{15}$$

In the least squares method, which will be discussed in the next section, use of the relation Eq. (15) will allow us to have only first-order differential equation, and to construct simple element-wise force vectors that contain linear terms only. In Eq. (15), V_x is the shear resultant on the undeformed edge, and Q_x is the shear force on the deformed edge. We write

$$\begin{aligned}
&\boldsymbol{\delta} L_l(u_0, w_0, \phi_x, N_x, V_x, M_x) \\
&\quad = \int_0^{L_e} \{\boldsymbol{\delta} u_o [N_{xx}' + f(x)] + \boldsymbol{\delta} w_o [V_x' + q(x)] + \boldsymbol{\delta}\phi_x (M_{xx}' - V_x + w_0' N_{xx}) \\
&\quad + \boldsymbol{\delta} N_{xx} [(EA)^{-1} N_{xx} - (u_0' + 0.5 w_0' w_0')] + (\boldsymbol{\delta} V_x - w_0' \boldsymbol{\delta} N_{xx})[(K_s GA)^{-1}(V_x - w_0' N_{xx}) - (\phi_x + w_0')] \\
&\quad + \boldsymbol{\delta} M_{xx} [(EI)^{-1} M_{xx} - \phi_x']\} \,\mathrm{d}x = 0
\end{aligned} \tag{16}$$

Then the variables and their variations in Eq. (16) can be approximated as linear combinations of nodal values and known interpolation functions, as

$$\begin{aligned}
&u_o \cong \sum_{j=1}^{l} \psi_j \boldsymbol{u}_j, \quad w_0 \cong \sum_{j=1}^{m} \psi_j \boldsymbol{w}_j, \;\; \phi_x \cong \sum_{j=1}^{n} \psi_j \boldsymbol{\phi}_j, \quad N_{xx} \cong \sum_{j=1}^{p} \psi_j \boldsymbol{N}_j, \;\; \cong \sum_{j=1}^{q} \psi_j \boldsymbol{V}_j, \\
&V_x \cong \sum_{i=1}^{q} \psi_i \boldsymbol{V}_i, \;\; M_{xx} \cong \sum_{j=1}^{r} \psi_j \boldsymbol{M}_j
\end{aligned} \tag{17}$$

where $\boldsymbol{u}$, $\boldsymbol{w}$, $\boldsymbol{\phi}$, $\boldsymbol{N}$, $\boldsymbol{Q}$ and $\boldsymbol{M}$ are nodal unknowns of the finite element model, the ψ_i and the ψ_j are i$^{\text{th}}$ and j$^{\text{th}}$ are Lagrange type interpolation functions. By substituting all the approximations given in Eq. (17) into the function δL_l of Eq. (16) and collecting coefficients of the variation of the nodal unknowns (i.e. $\delta\boldsymbol{u}$, $\delta\boldsymbol{w}$, $\delta\boldsymbol{\phi}$, $\delta\boldsymbol{N}$, $\delta\boldsymbol{Q}$ and $\delta\boldsymbol{M}$), the following finite element equations can be obtained:

$$\left[\mathbf{K}_G(\{U_G\})\right]\{\mathbf{U}_G\} = \{\mathbf{F}_G\} \;\text{ or }\; \begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] & [K^{15}] & [K^{16}] \\ [K^{21}] & [K^{22}] & [K^{23}] & [K^{24}] & [K^{25}] & [K^{26}] \\ [K^{31}] & [K^{32}] & [K^{33}] & [K^{34}] & [K^{35}] & [K^{36}] \\ [K^{41}] & [K^{42}] & [K^{43}] & [K^{44}] & [K^{45}] & [K^{46}] \\ [K^{51}] & [K^{52}] & [K^{53}] & [K^{54}] & [K^{55}] & [K^{56}] \\ [K^{61}] & [K^{62}] & [K^{63}] & [K^{64}] & [K^{65}] & [K^{66}] \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{u}_j\} \\ \{\boldsymbol{w}_j\} \\ \{\boldsymbol{\phi}_j\} \\ \{\boldsymbol{N}_j\} \\ \{\boldsymbol{V}_j\} \\ \{\boldsymbol{M}_j\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{F^4\} \\ \{F^5\} \\ \{F^6\} \end{Bmatrix} \tag{18}$$

where $\mathbf{K}_G$ is element stiffness matrix of the mixed Galerkin model, $\mathbf{U}_G$ is the column vector of unknowns, and $\mathbf{F}_G$ is the force vector. Definitions of all nonzero stiffness and force coefficients are given below.

$$\begin{aligned}
&K_{ij}^{14} = \int_0^L (\psi_i \psi_j') \,\mathrm{d}x, \qquad K_{ij}^{25} = \int_0^L (\psi_i \psi_j') \,\mathrm{d}x, \qquad K_{ij}^{36} = \int_0^L (\psi_i \psi_j') \,\mathrm{d}x, \\
&K_{ij}^{35} = \int_0^L (-\psi_i \psi_j) \,\mathrm{d}x, \qquad K_{ij}^{34} = \int_0^L (w_0' \psi_i \psi_j) \,\mathrm{d}x, \qquad K_{ij}^{44} = \int_0^L [(EA)^{-1} \psi_i \psi_j] \,\mathrm{d}x, \\
&K_{ij}^{41} = \int_0^L (-\psi_i \psi_j') \,\mathrm{d}x, \qquad K_{ij}^{42} = \int_0^L (-0.5 w_0' \psi_i \psi_j') \,\mathrm{d}x, \qquad K_{ij}^{43} = K_{ji}^{34}, \\
&K_{ij}^{44} = \int_0^L [(K_s GA)^{-1} w_0'^2 \psi_i \psi_j] \,\mathrm{d}x, \qquad K_{ij}^{45} = \int_0^L [-(K_s GA)^{-1} w_0' \psi_i \psi_j] \,\mathrm{d}x,
\end{aligned}$$

$$K_{ij}^{52}=\int_0^L (w_0'\psi_i\psi_j'-\psi_i\psi_j')\,\mathrm{d}x,\qquad K_{ij}^{53}=\int_0^L(-\psi_i\psi_j)\,\mathrm{d}x,\qquad K_{ij}^{54}=K_{ji}^{45}$$
$$K_{ij}^{55}=\int_0^L[(K_sGA)^{-1}\psi_i\psi_j]\,\mathrm{d}x,\qquad K_{ij}^{66}=\int_0^L[(EI)^{-1}\psi_i\psi_j]\,\mathrm{d}x,\qquad K_{ij}^{63}=\int_0^L(-\psi_i\psi_j')\,\mathrm{d}x$$
$$f_i^1=\int_0^L[-f(x)\psi_i]\,\mathrm{d}x\qquad\text{and}\qquad f_i^2=\int_0^L[-q(x)\psi_i]\,\mathrm{d}x \tag{19}$$

One advantage of this mixed model is the use of equal interpolation functions for all variables. In the displacement model, which was reviewed in Subsection 3.1, each variable should be approximated with consistent interpolation or reduced integrations must be used to prevent shear and membrane locking. However, increase of the number of unknowns is inevitable in mixed models.

3.3 Mixed least-squares model

In the least-squares method, the sum of the squares of the residuals in the governing equations is minimized. Using relations given in Eq. (15), the residuals in Eq. (7) can be expressed as

$$\begin{aligned}
R_1&=N_{xx}'+f(x)\neq 0\\
R_2&=V_x'+q(x)\neq 0\\
R_3&=M_{xx}'-V_x+w_0'N_{xx}\neq 0\\
R_4&=(K_sGA)^{-1}(V_x-w_0'N_{xx})-(\phi_x+w_0')\neq 0\\
R_5&=(EA)^{-1}N_{xx}-(u_0'+0.5w_0'w_0')\neq 0\\
R_6&=(EI)^{-1}M_{xx}-\phi_x'\neq 0
\end{aligned} \tag{20}$$

Unlike weak-form Galerkin model, the least-squares finite element model can be developed directly from Eq. (20) without consideration of the weight functions because they are naturally defined [5-8]. But before considering the development of the mixed least-squares finite element model, we need to scale the residuals so that all of them have the same magnitudes [5]. For example, the units of R_1 and R_2 are forced per length while unit of the R_3 is force. Residual R_6 has unit of force times length while residuals R_4 and the R_5 have no dimension. When the residuals are of different magnitude, the convergence of each residual to zero is dictated by their relative magnitudes (residuals with large magnitudes converge faster than those with less magnitude). To make all terms in the least-squares functional to be dimensionally the same, the following weights are chosen [9]:

$$\varpi_1=L_0\,/\,EA_0,\qquad \varpi_2=L_0^3\,/\,EI_0,\qquad \varpi_3=L_0^2\,/\,EI_0,\qquad \varpi_4=\varpi_5=1,\qquad \varpi_6=L_0 \tag{21}$$

where L_0, A_0 and I_0 are characteristic length, characteristic cross-sectional area, and characteristic second moment of inertia about the y-axis, respectively. In the present study, L_0, A_0 and I_0 are chosen to be 50.0 in, 1.0 in^2 and 1/12 in^4, respectively. Then, the least-squares functional becomes

$$I(u_o,w_o,\phi_x,N_x,V_x,M_x)=\int_0^{L_e}(\varpi_1R_1^2+\varpi_2R_2^2+\varpi_3R_3^2+\varpi_4R_4^2+\varpi_5R_5^2+\varpi_6R_6^2)\,\mathrm{d}x \tag{22}$$

To obtain symmetric coefficient matrix of in the nonlinear mixed least-squares finite element model, linearization of nonlinear terms in I is important. The linearization can be explained by separating the residuals including nonlinear terms (i.e. R_3, R_4 and R_5) and linear terms (i.e. R_1, R_2 and R_6). Then, the least-squares functional I can be rewritten as

$$I(u_o,w_o,\phi_x,N_x,V_x,M_x)=\frac{1}{2}\int_{x_1}^{x_2}[\boldsymbol{\mathcal{L}}+\boldsymbol{\mathcal{N}}]\,\mathrm{d}x \tag{23}$$

where

$$\mathcal{L} = (\varpi_1 R_1^2 + \varpi_2 R_2^2 + \varpi_6 R_6^2), \qquad \mathcal{N} = (\varpi_3 R_3^2 + \varpi_4 R_4^2 + \varpi_5 R_5^2)$$

We can minimize the above least-squares equation, which includes nonlinear terms, by taking first variation of it and setting it to zero:

$$\begin{aligned}
0 = \delta I(u_o, w_o, \phi_x, N_x, V_x, M_x) &= \int_{x_1}^{x_2} (\delta\mathcal{L} + \delta N)\,\mathrm{d}x \\
&= \int_{x_1}^{x_2} (\delta\mathcal{L} + \varpi_3^2 \delta R_3 R_3 + \varpi_4^2 \delta R_4 R_4 + \varpi_5^2 \delta R_5 R_5)\,\mathrm{d}x \\
&= \int_{x_1}^{x_2} \{\delta\mathcal{L} + \varpi_3^2 (\delta M_{xx}' - \delta V_x + \delta w_0' N_{xx} + w_0' \delta N_{xx})(M_{xx}' - V_x + w_0' N_{xx})[(EA)^{-1} N_{xx} - \\
&\quad (u_0' + 0.5 w_0' w_0')] + \varpi_4^2 [(EA)^{-1} \delta N_{xx} - (\delta u_0' + w_0' \delta w_0')][(K_s GA)^{-1}(V_x - w_0' N_{xx}) - (\phi_x + w_0')] \\
&\quad + \varpi_5^2 [(K_s GA)^{-1}(\delta V_x - \delta w_0' N_{xx} - w_0' \delta N_{xx}) - (\delta\phi_x + \delta w_0')]\,\mathrm{d}x\}
\end{aligned} \tag{24}$$

But if we linearize I (i.e., treat w_0' as known from the previous iteration) before taking its variation, then we have

$$\begin{aligned}
\bar{I}(u_o, w_o, \phi_x, N_x, V_x, M_x) &= \int_{x_1}^{x_2} (\delta\mathcal{L} + \delta\overline{N})\,\mathrm{d}x \\
&= \int_{x_1}^{x_2} \{\delta\mathcal{L} + \varpi_3^2 (\delta M_{xx}' - \delta V_x + \overline{w_0'} \delta N_{xx})(M_{xx}' - V_x + \overline{w_0'} N_{xx}) \\
&\quad + \varpi_4^2 [(EA)^{-1} \delta N_{xx} - (\delta u_0' + 0.5 \overline{w_0'} \delta w_0')][(EA)^{-1} N_{xx} - (u_0' + 0.5 \overline{w_0'} w_0')] \\
&\quad + \varpi_5^2 [(K_s GA)^{-1}(\delta V_x - \overline{w_0'} \delta N_{xx}) - (\delta\phi_x + \delta w_0')][(K_s GA)^{-1}(V_x - w_0' N_{xx}) - (\phi_x + w_0')]\}\,\mathrm{d}x
\end{aligned} \tag{25}$$

where quantities with bars indicate that they are linearized values. Then $\delta\bar{I} = 0$ is not the same condition as $\delta I = 0$ unless

$$\begin{aligned}
&\varpi_3^2 \delta w_0' N_{xx} \left(M_{xx}' - V_x + w_0' N_{xx}\right) + 0.5 \varpi_4^2 w_0' \delta w_0' [(EA)^{-1} N_{xx} - (u_0' + 0.5 w_0' w_0')] \\
&\quad - \varpi_5^2 \left(K_s GA\right)^{-1} \delta w_o' N_{xx} [(K_s GA)^{-1}(V_x - w_0' N_{xx}) - (\phi_x + w_0')] = 0
\end{aligned} \tag{26}$$

Since the sufficient condition for the minimum of I is $\delta^2 I > 0$,

$$\begin{aligned}
\delta^2 I = \int_{x_1}^{x_2} \{&\delta\mathcal{L} + \varpi_3^2 (\delta M_{xx}' - \delta V_x + \delta w_0' N_{xx} + w_0' \delta N_{xx})(\delta M_{xx}' - \delta V_x + w_0' \delta N_{xx}) \\
&+ \varpi_4^2 [(EA)^{-1} \delta N_{xx} - (\delta u_0' + w_0' \delta w_0')][(EA)^{-1} \delta N_{xx} - (\delta u_0' + 0.5 w_0' \delta w_0')] \\
&+ \varpi_5^2 [(K_s GA)^{-1}(\delta V_x - \delta w_0' N_{xx} - w_0' \delta N_{xx}) - (\delta\phi_x + \delta w_0')][(K_s GA)^{-1}(\delta V_x - w_0' \delta N_{xx}) \\
&- (\delta\phi_x + \delta w_0')]\}\,\mathrm{d}x > 0
\end{aligned} \tag{27}$$

Thus, in order for $\delta I^2 > 0,$ we must have

$$\begin{aligned}
&\varpi_3^2 \delta w_0' N_{xx} (M_{xx}' - V_x + w_0' N_{xx}) + 0.5 \varpi_4^2 w_0' \delta w_0' [(EA)^{-1} N_{xx} - (u_0' + 0.5 w_0' w_0')] \\
&\quad - \varpi_5^2 \left(K_s GA\right)^{-1} \delta w_0' N_{xx} [(K_s GA)^{-1}(V_x - w_0' N_{xx}) - (\phi_x + w_0')] \geq 0
\end{aligned} \tag{28}$$

or Eq. (26) should hold. The linearization of the least-squares function before taking its first variation results in $\delta\bar{I} = 0$ and $\delta^2\bar{I} > 0,$ which does not alter the minimizing conditions of $\delta I = 0$ and $\delta^2 I > 0$ as given in Eq. (26). Now we can develop symmetric bilinear form of the mixed finite element model from the first variation of the linearized $\boldsymbol{I}$

$$
\begin{aligned}
0=\int_0^{L_e} & \{\varpi_1^2\delta N'_{xx}\left[N'_{xx}+f(x)\right]+\varpi_2^2\delta V'_x\left[V'_x+q(x)\right]+\varpi_3^2(\delta M'_{xx}-\delta V_x+w'_0\delta N_{xx})(M'_{xx}-V_x+w'_0N_{xx}) \\
&+\varpi_4^2[(EA)^{-1}\delta N_{xx}-(\delta u'_0+0.5w'_0\delta w'_0)][(EA)^{-1}N_{xx}-(u'_0+0.5w'_0w'_0)] \\
&+\varpi_5^2[(K_sGA)^{-1}(\delta V_x-w'_0\delta N_{xx})-(\delta\phi_x+\delta w'_0)][(K_sGA)^{-1}(V_x-w'_0N_{xx})-(\phi_x+w'_0)] \\
&+\varpi_6^2[(EI)^{-1}\delta M_{xx}-\delta\phi'_x][(EI)^{-1}M_{xx}-\phi'_x]\}\,\mathrm{d}x.
\end{aligned}
\tag{29}
$$

Note that all bars are omitted for consistency with the Galerkin mixed model. Then the variables involved in Eq. (29) can be approximated and replaced by using Eq. (17) which are the same functions used in the Galerkin model. By using matrix notation, above equations can be rewritten simply as

$$
[\mathbf{K}_L(\{\mathbf{U}_L\})]\{\mathbf{U}_L\}=\{\mathbf{F}_L\}\quad\text{or}\quad
\begin{bmatrix}
[K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] & [K^{15}] & [K^{16}] \\
[K^{21}] & [K^{22}] & [K^{23}] & [K^{24}] & [K^{25}] & [K^{26}] \\
[K^{31}] & [K^{32}] & [K^{33}] & [K^{34}] & [K^{35}] & [K^{36}] \\
[K^{41}] & [K^{42}] & [K^{43}] & [K^{44}] & [K^{45}] & [K^{46}] \\
[K^{51}] & [K^{52}] & [K^{53}] & [K^{54}] & [K^{55}] & [K^{56}] \\
[K^{61}] & [K^{62}] & [K^{63}] & [K^{64}] & [K^{65}] & [K^{66}]
\end{bmatrix}
\begin{Bmatrix}
\{\boldsymbol{u}_j\} \\ \{\boldsymbol{w}_j\} \\ \{\boldsymbol{\phi}_j\} \\ \{\boldsymbol{N}_j\} \\ \{\boldsymbol{V}_j\} \\ \{\boldsymbol{M}_j\}
\end{Bmatrix}
=
\begin{Bmatrix}
\{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{F^4\} \\ \{F^5\} \\ \{F^6\}
\end{Bmatrix}
\tag{30}
$$

where $\mathbf{K}_L$ is element stiffness matrix of the mixed least-squares model, $\mathbf{U}_L$ is the vector of unknowns, and $\mathbf{F}_L$ is the force vector. The nonzero coefficients of Eq. (30) are defined as

$$
\begin{aligned}
&K_{ij}^{11}=\int_0^L\varpi_4^2(\psi'_i\psi'_j)\,\mathrm{d}x,\ K_{ij}^{12}=\int_0^L 0.5\varpi_4^2w'_0(\psi'_i\psi'_j)\,\mathrm{d}x,\ K_{ij}^{14}=\int_0^L-\varpi_4^2(EA)^{-1}(\psi'_i\psi_j)\,\mathrm{d}x, \\
&K_{ij}^{21}=K_{ji}^{12},\ K_{ij}^{22}=\int_0^L[0.25\varpi_4^2(w'_0)^2(\psi'_i\psi'_j)+\varpi_5^2(\psi'_i\psi'_j)]\,\mathrm{d}x,\ K_{ij}^{23}=\int_0^L\varpi_5^2(\psi'_i\psi_j)\,\mathrm{d}x, \\
&K_{ij}^{24}=\int_0^L[-0.5\varpi_4^2(EA)^{-1}w'_0(\psi'_i\psi_j)+\varpi_5^2(K_sGA)^{-1}w'_0(\psi'_i\psi_j)]\,\mathrm{d}x,\ K_{ij}^{25}=\int_0^L-\varpi_5^2(K_sGA)^{-1}(\psi'_i\psi_j)\,\mathrm{d}x, \\
&K_{ij}^{32}=K_{ji}^{23},\ K_{ij}^{33}=\int_0^L[\varpi_5^2(\psi_i\psi_j)+\varpi_6^2(\psi'_i\psi'_j)]\,\mathrm{d}x,\ K_{ij}^{34}=\int_0^L\varpi_5^2(K_sGA)^{-1}w'_0(\psi_i\psi_j)\,\mathrm{d}x, \\
&K_{ij}^{35}=\int_0^L-\varpi_5^2(K_sGA)^{-1}(\psi_i\psi_j)\mathrm{d}x,\ K_{ij}^{36}=\int_0^L-\varpi_6^2(EI)^{-1}(\psi'_i\psi_j)\,\mathrm{d}x, \\
&K_{ij}^{41}=K_{ji}^{14},\ K_{ij}^{42}=\int_0^L[-0.5\varpi_4^2w'_0(\psi_i\psi'_j)+\varpi_5^2(K_sGA)^{-1}w'_0(\psi_i\psi'_j)]\,\mathrm{d}x,\quad K_{ij}^{43}=K_{ji}^{34}, \\
&K_{ij}^{44}=\int_0^L[\varpi_1^2(\psi'_i\psi'_j)+\varpi_3^2(w'_0)^2(\psi_i\psi_j)+\varpi_4^2(EA)^{-2}(\psi_i\psi_j)+\varpi_5^2(K_sGA)^{-2}(w'_0)^2(\psi_i\psi_j)]\,\mathrm{d}x, \\
&K_{ij}^{45}=\int_0^L[-\varpi_3^2w'_0(\psi_i\psi_j)-\varpi_5^2(K_sGA)^{-2}w'_0(\psi_i\psi_j)]\,\mathrm{d}x,\ K_{ij}^{46}=\int_0^L\varpi_3^2w'_0(\psi_i\psi'_j)\,\mathrm{d}x, \\
&K_{ij}^{55}=\int_0^L[\varpi_3^2(\psi_i\psi_j)+\varpi_2^2(\psi'_i\psi'_j)+\varpi_5^2(K_sGA)^{-2}(\psi_i\psi_j)]\,\mathrm{d}x,\ K_{ij}^{56}=\int_0^L-\varpi_3^2(\psi_i\psi'_j)\,\mathrm{d}x, \\
&K_{ij}^{52}=K_{ji}^{25},\ K_{ij}^{53}=K_{ji}^{35},\quad K_{ij}^{54}=K_{ji}^{45}, \\
&K_{ij}^{63}=K_{ji}^{36},\ K_{ij}^{64}=K_{ji}^{46},\ K_{ij}^{65}=K_{ji}^{56},\ K_{ij}^{66}=\int_0^L[\varpi_3^2(\psi'_i\psi'_j)+\varpi_6^2(EI)^{-2}(\psi_i\psi_j)]\,\mathrm{d}x, \\
&f_i^4=\int_0^L[-\varpi_1^2f(x)\psi'_i]\,\mathrm{d}x\ \text{ and }\ f_i^5=\int_0^L[-\varpi_2^2q(x)\psi'_i]\,\mathrm{d}x
\end{aligned}
\tag{31}
$$

Comparing Eqs. (30) and (31) with Eqs. (18) and (19), it can be seen that the $\mathbf{K}_L$ is symmetric while the $\mathbf{K}_G$ is not. In Eqs. (30) and (31), force vector contains only linear terms, by replacing Q_x with $V_x+w'_0N_{xx}$.

4 NUMERICAL RESULTS
4.1 Description of problem

A steel beam with the geometry shown in Fig. 2 was chosen for the study. The properties given in Eq. (32) were used, with uniform applied distributed load $q(x)$, which was set to vary from 1.0 to 10.0 lb/in with ten steps to archive proper nonlinear convergence.

$$E = 30\times 10^6 \text{ psi},\ \ \nu = 0.25,\ \ K_s = 5/6,\ \ f(x) = 0,\ \ q(x) = 1.0 \sim 10.0 \text{ lb/in} \tag{32}$$

Three different types of boundary conditions shown in Fig. 3, i.e. H-H (hinged-hinged), P-P (Pined-Pined) and C-C (clamped-clamped), were considered to evaluate the performance of the elements. All boundary conditions allow modeling half of the beam.

The C-C and P-P boundary conditions can be used to test the nonlinear behavior of newly developed beam finite elements, and the H-H boundary condition can be used to determine membrane locking is experienced [10]. Since the beam has no horizontal support under the H-H boundary condition, $u_0' + 0.5(w_0')^2 = 0$ should be satisfied to have N_{xx}=0. In the traditional displacement model, this condition is barely satisfied because of the inconsistency in polynomial orders between the u_0' and the $(w_0')^2$ terms. Also, the inconsistency in polynomial orders between the ϕ_x and the w_0' in $K_sGA(\phi_x + w_0')$ may cause shear locking [3, 10] which results in inaccurate linear solution. In displacement based models, performance of beam element is mostly depends on the relations of the displacements

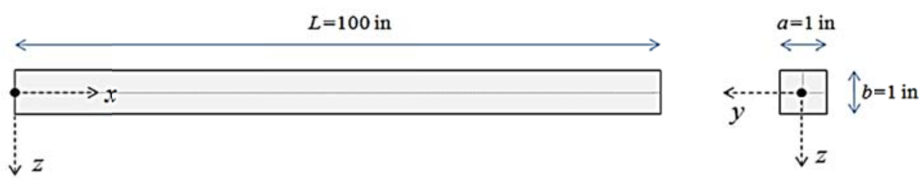


Fig.2
Geometry of beam and chosen coordinate system for the numerical analysis [source from 3].

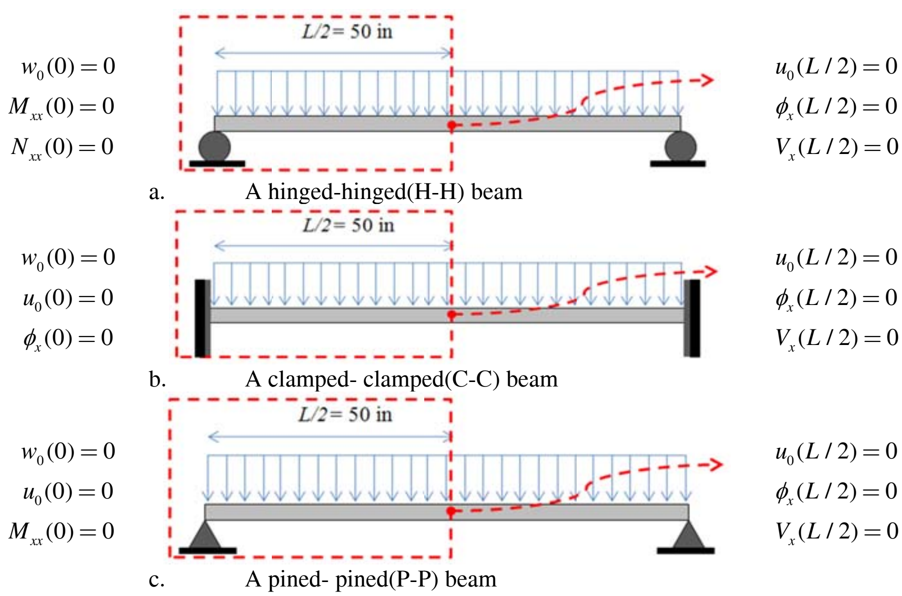


Fig. 3
Description of symmetry boundary conditions.

and their derivatives, thus they experience sever locking compared with mixed models. Some techniques like reduced integration and use of consistency approximations of the displacement field may be employed, but it is not easy (but possible) to implement them into computers. On the other hand, the locking can be weakened or even eliminated by the inclusion of resultants which can be done by mixed formulations, because certain conditions which cause locking are not solely depend on the relations of displacement but also on the resultant in mixed models. The numerical results that show the effects of mixed formulation are presented in the next sections.

4.2 Linear analysis

In this section linear solutions of the newly developed mixed models are compared with known exact solutions and solutions of traditional displacement based model. By taking all nonlinear terms to be zero, we can eliminate geometric nonlinearity in the beam element to study linear behavior of beam element. Linear study is important because finding nonlinear solutions can be started from linear solutions in most of structural problems. Linear study of the TBT beam bending includes interesting phenomena like shear locking. For shear locking, accuracy level of the solutions can be possibly calculated by using known exact solutions. As mentioned in previous section, displacement based models experience severe shear locking even with the use of higher interpolations. But present models developed with mixed formulations showed less locking, especially for the mixed Galerkin model, if higher order of interpolation functions were used. The solutions of the newly developed models are presented in the Table. 1, to investigate shear locking. With the results presented in the Table 1, we can find that accuracy level of the solution does depend on methods and formulations adopted. The mixed Galerkin model showed the best accuracy in finite element solutions obtained under both C-C and P-P boundary conditions, while traditional displacement models showed the most inaccurate results. With the results presented in the Table 1, errors which mean degree of shear locking are compared in the Fig. 3.

The degree of shear locking is measured using the definition

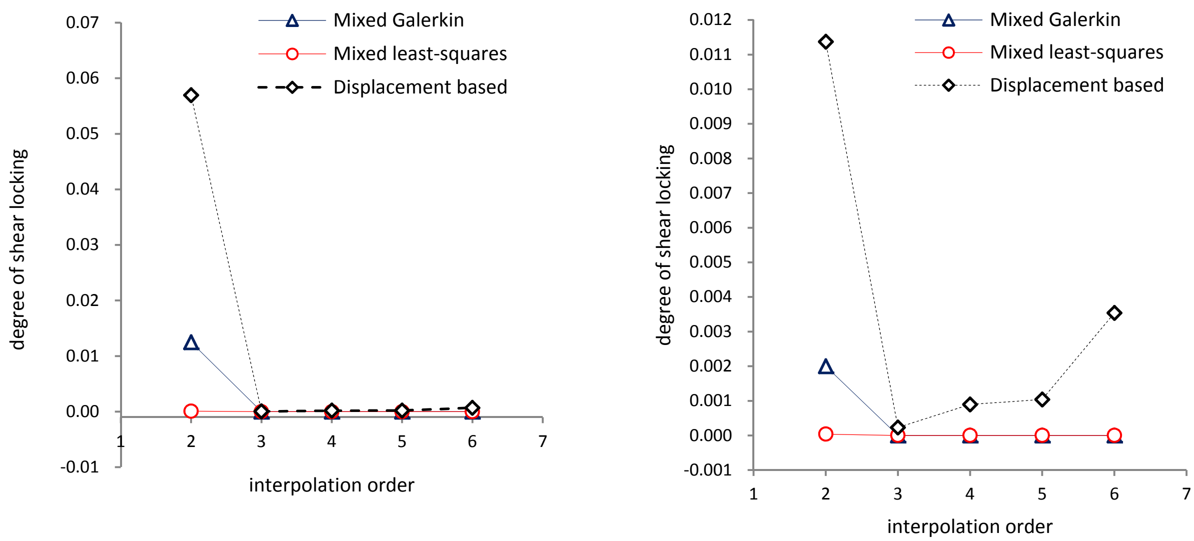
$$\text{Degree of shear locking} = \frac{|w_{exact} - w_{sl}|}{w_{exact}} \tag{33}$$

Table.1
Linear results of beam obtained with 4-element mesh under C-C boundary condition, $q(x)$ =1.0 lb/in, with full integration

Interpolation order	Center deflection of the beam (w_0), in					
	C-C boundary condition(exact: .104291667)			P-P boundary condition(exact: .520958333)		
	Mixed Galerkin	Mixed Least-squares	Traditional displacement	Mixed Galerkin	Mixed Least-squares	Traditional displacement
1st	0.112972222	0.096004271	0.001964678	0.526745370	0.497652359	0.009691326
2nd	0.105593750	0.104283529	0.098352041	0.522000000	0.520938133	0.515034294
3rd	0.104291667	0.104291651	0.104296375	0.520958335	0.520958540	0.521078238
4th	0.104291667	0.104291612	0.104273233	0.520958333	0.520958178	0.520491516
5th	0.104291667	0.104291670	0.104270593	0.520958335	0.520959431	0.520417581
6th	0.104291668	0.104291730	0.104363783	0.520958337	0.520958056	0.522800451

Table 2
Linear results of beam obtained with 1-element mesh under C-C boundary condition, $q(x)$ =1.0 lb/in, with full integration

Interpolation order	Center deflection of the beam (w_0), in					
	C-C boundary condition(exact: .104291667)			P-P boundary condition(exact: .520958333)		
	Mixed Galerkin	Mixed Least-squares	Traditional displacement	Mixed Galerkin	Mixed Least-squares	Traditional Displacement
1st	0.139013889	0.072895696	0.000498209	0.544106482	0.433308428	0.002364252
2nd	0.109500000	0.104161909	0.078860218	0.525125000	0.520637115	0.495521997
3rd	0.104291667	0.104291651	0.104291468	0.520958334	0.520958051	0.520951347
4th	0.104291667	0.104291668	0.104290570	0.520958333	0.520958341	0.520926382
5th	0.104291665	0.104291663	0.104283217	0.520958326	0.520958009	0.520744177
6th	0.104291671	0.104291639	0.104299478	0.520958357	0.520957624	0.521167093



a. C-C boundary condition b. H-H boundary condition

Fig. 4
Decay of error verses the interpolation order with 4-element uniform mesh, full integration.

Table 3
Converged center deflection w_0 (iterations taken) of beam obtained with 4-element mesh, forth order interpolation, under C-C boundary condition

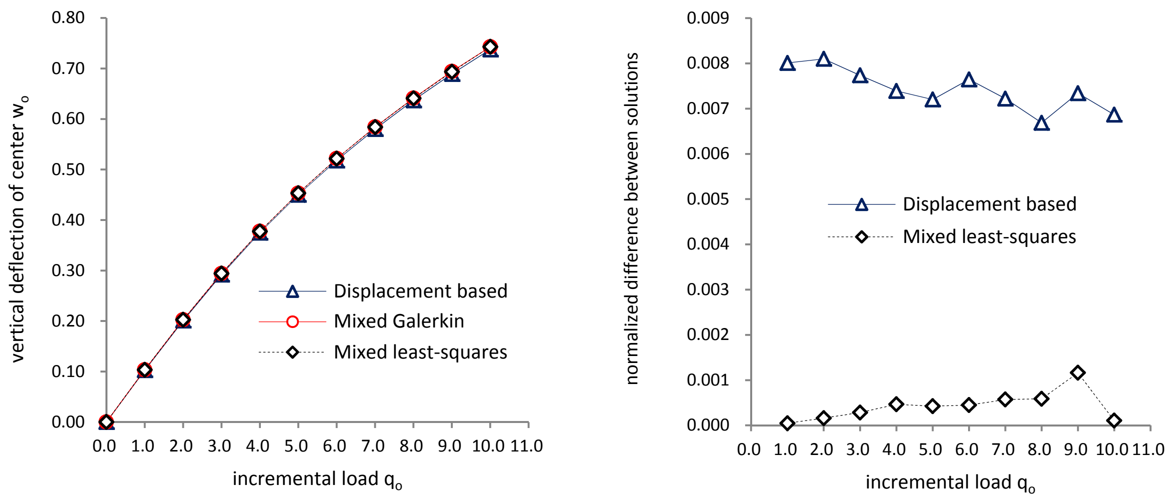
Load	Mixed Galerkin	Mixed Least-squares	Traditional Displacement (full integration)	Traditional Displacement (reduced integration)
1.0	0.1035(3)	0.1035(3)	0.1026(2)	0.1034(3)
2.0	0.2025(4)	0.2025(4)	0.2008(4)	0.2023(4)
3.0	0.2942(5)	0.2941(5)	0.2919(4)	0.2939(4)
4.0	0.3777(5)	0.3775(6)	0.3749(5)	0.3774(5)
5.0	0.4532(6)	0.4530(6)	0.4499(6)	0.4530(6)
6.0	0.5217(7)	0.5215(7)	0.5177(6)	0.5216(6)
7.0	0.5842(7)	0.5838(7)	0.5799(7)	0.5841(7)
8.0	0.6411(8)	0.6407(8)	0.6368(8)	0.6414(8)
9.0	0.6941(9)	0.6933(8)	0.6890(8)	0.6943(9)
10.0	0.7427(10)	0.7426(9)	0.7376(10)	0.7433(10)

where w_{sl} is center vertical deflection of beam which is obtained by linear finite element analysis with full integration, and w_{exact} is mathematically exact solution. Since w_{sl} is obtained with full integration, it will contain certain degree of shear locking caused by inconsistent approximation of the variables. Thus, degree of shear locking can be determined from Eq. (33).

4.3 Nonlinear analysis

Under C-C boundary condition, each model showed similar convergence, as shown in the Fig. 4a, while the converged solutions of the current mixed Galerkin and the least-squares models are more accurate than those of the displacement model (see Fig. 4b). The converged solutions of the displacement model close to the solutions predicted by the current mixed models when proper reduced integration techniques or consistent approximations of variables are used.

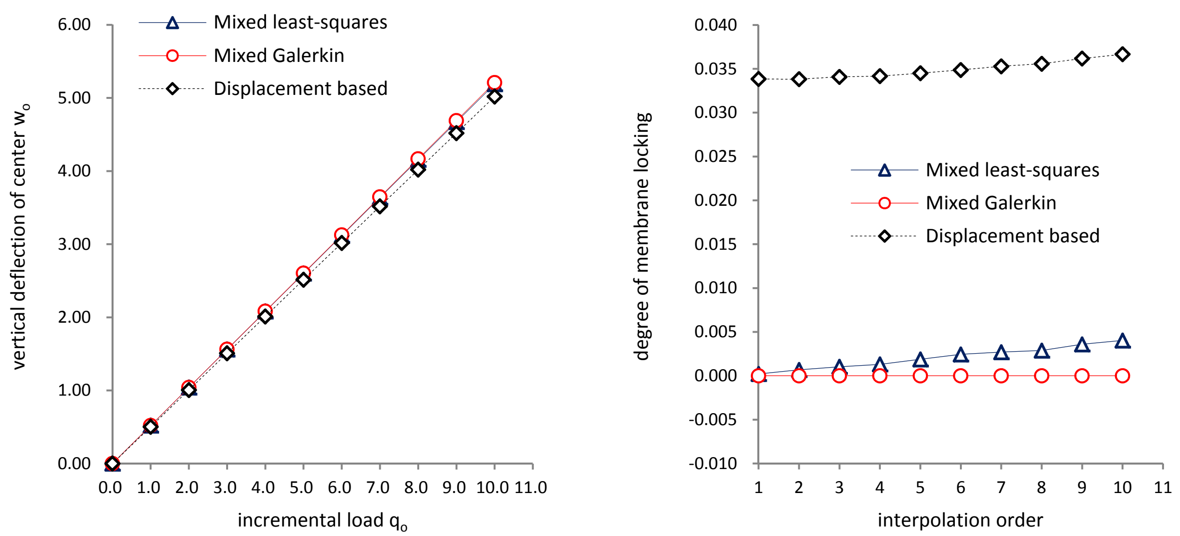
Converged solutions are presented in Table 3. The direct iterative method was used to get the solutions. Two mixed models showed good result with full integrations, while the traditional displacement model showed some degree of locking. Normally, locking of element is sever in finite element solution of lower interpolations, while it is likely to disappear in higher interpolations. Two mixed models showed closest converged solutions, while the displacement based model did not as presented in the Fig. 5b. As mentioned before, membrane locking [3] is caused by the use of inconsistent order of interpolation of the terms like $u_0' + 0.5(w_0')^2 = (EA)^{-1} N_{xx}$. In particular, when the beam undergoes no extensional deformation, it should produce $u_0' + 0.5(w_0')^2 = (EA)^{-1} N_{xx} \cong 0$.



(a) Load verses center defevtion under C-C boundary condition

(b) Nomalixed difference of converged solutions respect to that of the mixed Galerkin model

Fig. 5
Nonlinear behavior of beams under C-C boundary condition, with 4 cubic elements.



(a) Load verses center defevtion

(b) Esitimated degree of menbrane locking

Fig. 6
Nonlinear behavior of beams under H-H boundary condition, with 4 cubic elements.

This is satisfied only when the axial displacement and transverse displacements are interpolated such a way that the membrane strain has the possibility of becoming zero. Thus, the beam element may behave in a manner physically unrealistic. Traditionally, this phenomenon is overcome using reduced integration of the nonlinear stiffness coefficients. In the case of the mixed model presented herein, membrane locking completely disappears. As in Eq. (33), the degree of membrane locking can be measured as

$$\text{Degree of membrane locking} = \frac{\left| w_{exact} - w_{ml} \right|}{w_{exact}} \tag{34}$$

where w_{ml} is converged center vertical deflection of the beam obtained with full integration, and w_{exact} is mathematically exact solution of the same. Judging from the nonlinear results obtained, both models showed good nonlinear convergence while the mixed Galerkin model showed least degree of membrane locking.

5 CONCLUSIONS

Developing procedures of nonlinear beam bending finite element models were presented with two different methods. All element wise coefficient matrices and force vectors are also presented. Mixed formulation provided superior accuracy in linear solutions and better performance in nonlinear analysis with the use of same order interpolations and full integrations. Two types of locking phenomena were discussed and current mixed models showed less locking compared with traditional displacement based model.

ACKNOWLEDGEMENTS

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