Free Vibration Analysis of a Nonlinear Beam Using Homotopy and Modified Lindstedt–Poincare Methods

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ABSTRACT

In this paper, homotopy perturbation and modified Lindstedt-Poincare methods are employed for nonlinear free vibrational analysis of simply supported and double-clamped beams subjected to axial loads. Mid-plane stretching effect has also been accounted in the model. Galerkin's decomposition technique is implemented to convert the dimensionless equation of the motion to nonlinear ordinary differential equation. Homotopy and modified Lindstedt-Poincare (HPM) are applied to find analytic expressions for nonlinear natural frequencies of the beams. Effects of design parameters such as axial load and slenderness ratio are investigated. The analytic expressions are valid for a wide range of vibration amplitudes. Comparing the semi-analytic solutions with numerical results, presented in the literature, indicates good agreement. The results signify the fact that HPM is a powerful tool for analyzing dynamic and vibrational behavior of structures analytically.

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1 INTRODUCTION

BEAMS are the most important structure of engineering. They have a wide application in variety of structures from micro/nano dimensions such as micro/nano resonators to macro dimensions such as airplane wings, flexible satellites and long span bridges.

Large amplitude vibration of beams usually leads to material fatigue and structural damage. These effects become more important around the natural frequencies of the structure [1]. In these systems nonlinear effects come into play in large amplitude vibrations. The sources of the nonlinearities may be geometric, inertial or material in nature. The geometric nonlinearity may be caused by nonlinear stretching or large curvatures. Nonlinear inertial effects are caused by the presence of concentrated or distributed masses. Material nonlinearity occurs whenever the stresses are nonlinear functions of strains [2]. Euler-Bernoulli beam theory assumes that plane cross sections, normal to the natural axis before deformation, continue to remain plane and continue to remain normal to the neutral axis and do not undergo any strain in their planes [3]. In fact it assumes that warping, transverse shear deformation effects and transverse normal strains are considered to be negligible and can be neglected [4]. Pillai and Rao [5] examined the problem of large amplitude free vibrations of simply supported uniform beams and found the frequency response of the system by several methods, the elliptic function method, the harmonic balance method and the method which one assumes simple harmonic oscillations. Pirbodaghi et al [1] used HAM to investigate

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nonlinear vibrational behavior of Euler Bernoulli beams subjected to axial loads and provided analytical expressions for geometrically nonlinear vibration of beams.

Foda [7] used the method of multiple scales to analyze the nonlinear vibrations of a beam with pinned ends considering the effect of shear deformation and rotary inertia. Ramezani et al [6] used the same method for the same problem with doubly clamped boundary conditions. They concluded that when the theory of beams is used for the study of micro/nano electromechanical structures, shear deformation and rotary inertia effects should be considered for an accurate dynamic analysis. In general it is extremely difficult to find an exact solution for the nonlinear vibration of beams. Consequently one can use approximate analytical approach or numerical techniques for this purpose. Besides all advantages of numerical methods, due to convenience for parametric studies and accounting for the physics of the problem, an analytical solution appears more appealing than the numerical one. Also analytical solutions give a reference frame for verification and validation of the numerical approaches [1]. Although it is difficult to have an analytical approach for nonlinear vibrational analysis of beams, there are some analytic approaches for this problem such as perturbation techniques [7]. In general, the analytical methods have their own limitations. For example, perturbation methods, the most extensively used analytical techniques, are generally restricted to the case of weak non-linearity and are carried out with respect to a small parameter in the equation. Most of non-linear problems, especially those having strong non-linearity, have no small parameters at all [1]. Based on the homotopy method in topology, Liao has proposed homotopy analysis method (HAM) to present analytic solutions for strongly nonlinear problems [8]. Another useful method for strongly nonlinear problems is homotopy perturbation method (HPM), which has been proposed by He [9]. Although it was shown by several researchers that the homotopy perturbation method is a special case of the homotopy analysis method [10-12], it has been utilized in the present study due to its easier formulation.

The current paper makes use of the HPM to analyze nonlinear free vibration analysis of clamped-clamped and simply supported beams. He [13] presented a new perturbation technique which does not depend upon the assumption of small parameters. He illustrated the well-known duffing equation as an example and found that even using a first order approximation, the maximal relative error of the period is less than 7% even the parameter e approaches infinity. He [14] proposed this new perturbation method which does not require a small parameter in an equation. His new method takes full advantages of the traditional perturbation methods and homotopy techniques. Blendez et al [15] solved the nonlinear differential equations which govern the nonlinear oscillation of a simple pendulum and showed that even only one iteration leads to the relative error of less than 2% for the approximated period even when for amplitudes as high as 130°. Blendez et al [16] find improved approximate solutions to conservative truly nonlinear oscillators using He's homotopy perturbation method. They found that for the second order approximation the relative error in the analytical approximate frequency is approximately 0.03% for any parameter values involved.

As it is seen in the literature of the HPM, this method overcomes the limitations of classical perturbation methods and at the same time provides an accurate prediction of the behavior of the nonlinear systems. So here, this method has been used in conjunction with the modified Lindstedt-Poincare method to solve the problem of nonlinear free vibration of micro beams considering the midplane stretching.

2 PROBLEM FORMULATION

The nonlinear partial differential equation of the beam, when the effects of mid-plane stretching are not negligible, is given by:

$$EI \frac{\partial^4 \hat{w}}{\partial \hat{x}^4} + m \frac{\partial^2 \hat{w}}{\partial \hat{t}^2} - N_0 \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{EA}{2L} \int_0^L \left(\frac{\partial \hat{w}}{\partial \hat{x}} \right)^2 dx \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} = 0 \tag{1}$$

In this equation E is the Young's modulus of elasticity of the beam material, I is the second moment of area of the cross section with respect to the bending axis, $\hat{w}$ is the beam deflection, m is the longitudinal density, $\hat{t}$ is the time, A is the cross sectional area of the beam, N_0 is the pretension of the beam and L is the length of the beam. Where assuming that the beam is vibrating with one of its natural frequencies and introducing the nondimensionalized variables t , x , w and T which are defined in Eqs. (2) to (4), the Eq. (1) is nondimensionalized as Eq. (6).

$$t = \frac{\hat{t}}{T} \tag{2}$$

$$x = \frac{\hat{x}}{L} \tag{3}$$

$$w = \frac{\hat{w}}{L} \tag{4}$$

$$T = \frac{1}{\beta^2} \sqrt{\frac{mL^4}{EI}} \tag{5}$$

$$\frac{\partial^4 w}{\partial x^4} + \beta^4 \frac{\partial^2 w}{\partial t^2} - \frac{N_0}{EA} \left(\frac{L}{r}\right)^2 \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx = 0 \tag{6}$$

where

$$r^2 = \frac{I}{A} \tag{7}$$

Now the solution of the Eq. (6) can be assumed as

$$w(x,t) = \phi(x) q(t) \tag{8}$$

Note that for simply supported boundary condition's, when the

$$\beta_n = n\pi \tag{9}$$

$\phi(x)$ is the first linear undamped vibrational mode of the beam. For simply support boundary condition's $\phi(x)$ can be stated as Eq. (10)

$$\phi(x) = \sin(n\pi x) \tag{10}$$

And for clamped-clamped beams the first linear undamped vibrational mode of the beam can be stated as equation:

$$Q_n(x) = \cosh\beta_n x - \cos\beta_n x - \frac{\cosh\beta_n - \cos\beta_n}{\sinh\beta_n - \sin\beta_n}(\sinh\beta_n x - \sin\beta_n x) \tag{11}$$

$$\phi_n(x) = \frac{Q_n(x)}{Q_n(0.5)} \tag{12}$$

For $n = 1$, $\beta_1 = 4.730040745$

According to the Galerkin procedure, by substituting Eq. (8) into Eq. (6) and integrating the residual by weight $\phi(x)$ over the problem domain one may arrive to the following nonlinear ODE:

$$\ddot{q} + \omega_0^2 q + \alpha q^3 = 0 \tag{13}$$

$$q(0) = \frac{W_{max}}{L}, \quad \dot{q}(0) = 0 \tag{14}$$

where

$$\omega_0^2 = \frac{1}{\beta^4 F_1}\left(F_3 + \frac{N_0}{EA}\left(\frac{L}{r}\right)^2 F_2\right) \tag{15}$$

$$\alpha = \frac{1}{2\beta^4 F_1}\left(\frac{L}{r}\right)^2 F_2^2 \tag{16}$$

In Eqs. (15) and (16), F_i 's, $1 \leq i \leq 3$ are defined as follows

$$F_1 = \int_0^1 \phi^2 \mathrm{d}x, \ \ F_2 = \int_0^1 \phi'^2 \mathrm{d}x, \ \ F_3 = \int_0^1 \phi''^2 \mathrm{d}x \tag{17}$$

From Eq. (13), ω_0 is the linear frequency of the beam. Now the method of homotopy perturbation is applied to solve the Eq. (13). The homotopy form is constructed as follows

$$(1-P)\left[\ddot{q} + \omega_0^2 q\right] + P\left[\ddot{q} + \omega_0^2 q + \alpha q^3\right] = 0 \tag{18}$$

Using the modified Lindstedt- Poincare method, q and ω_0^2 are perturbed using perturbation parameter P

$$q = q_0 + P q_1 \tag{19a}$$
$$\omega_0^2 = \omega^2 + P \omega_{11} \tag{19b}$$

Substituting Eqs. (19) into Eq. (18) and setting the coefficient of each power of P equal to zero leads to the following sets of equations

$$\ddot{q}_0 + \omega^2 q_0 = 0, \ q_0(0) = \frac{W_{\max}}{L}, \ \dot{q}_0(0) = 0 \tag{20}$$

$$\ddot{q}_1 + \omega^2 q_1 + \omega_{11} q_0 + \alpha q_0^3 = 0, \ q_1(0) =, \ \dot{q}_1(0) = 0 \tag{21}$$

Solving Eq. (20) yields Eq. (22) for q_0

$$q_0 = \frac{W_{\max}}{L}\cos \omega t \tag{22}$$

Substituting q_0 from Eq. (22) to Eq. (21) one can conclude Eq. (23)

$$\ddot{q}_1 + \omega^2 q_1 + \omega_{11}\frac{W_{\max}}{L}\cos \omega t + \alpha\left(\frac{W_{\max}}{L}\cos \omega t\right)^3 = 0, \ q_1(0) =, \ \dot{q}_1(0) = 0 \tag{23}$$

Eliminating secular terms in Eq. (23) yields to Eqs. (24)

$$\omega_{11}\frac{W_{\max}}{L} + \frac{3}{4}\alpha\left(\frac{W_{\max}}{L}\right)^3 = 0 \Rightarrow \omega_{11} = -\frac{3}{4}\alpha\left(\frac{W_{\max}}{L}\right)^2 \tag{24}$$

Letting $P = 1$ in Eq. (19b)

$$\omega_0^2 = \omega^2 + \omega_{11} \tag{25}$$

Eqs. (25) are then solved for finding the natural frequencies ω

$$\omega = \sqrt{\omega_0^2 + \frac{3}{4}\alpha\left(\frac{W_{max}}{L}\right)^2} \tag{26}$$

Solving Eq. (21) yields

$$q_1(t) = \frac{1}{32\omega^2}\alpha\left(\frac{W_{max}}{L}\right)^3\left[\cos 3\omega t - \cos \omega t\right] \tag{27}$$

And the first-order approximation of the $q(t)$ becomes as:

$$q = q_0(t) + q_1(t) = \frac{W_{max}}{L}\cos \omega t + \frac{1}{32\omega^2}\alpha\left(\frac{W_{max}}{L}\right)^3\left[\cos 3\omega t - \cos \omega t\right] \tag{28}$$

when $\frac{3}{4}\frac{\alpha}{\omega_0^2}\left(\frac{W_{max}}{L}\right)^2 << 1$, the frequency of the beam can be expressed as follows:

$$\omega_s \simeq \omega_0\left(1 + \frac{3}{8}\frac{\alpha}{\omega_0^2}\left(\frac{W_{max}}{L}\right)^2\right) = \omega_0 + \frac{3}{8}\frac{\alpha}{\omega_0}\left(\frac{W_{max}}{L}\right)^2 \tag{29}$$

which is in agreement with results of Ramezani et al. [6]. It should be noted that their results are valid only for small $\frac{3}{4}\frac{\alpha}{\omega_0^2}\left(\frac{W_{max}}{L}\right)^2$.

3   RESULTS AND DISCUSSION

In order to demonstrate the accuracy and effectiveness of the procedure explained in the previous section, a numerical example is implemented for the case of simply supported and clamped-clamped beams. Figs. 1 and 2 show a comparison between results of HPM, and numerical results for a clamped-clamped and simply supported beam respectively. The results completely agree with each other.
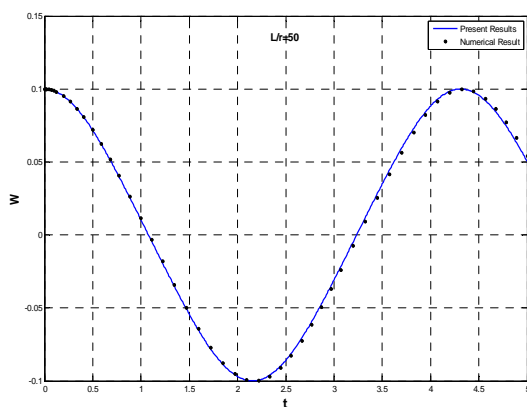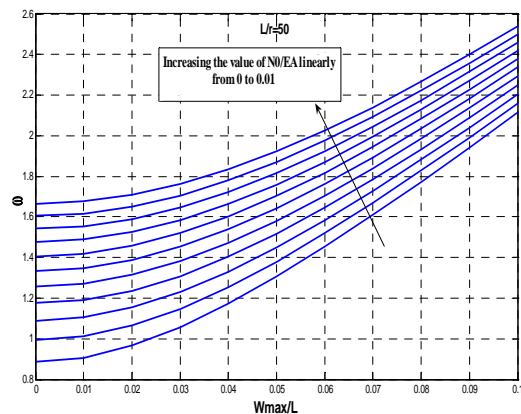

Fig. 1
Deflection variation of double-clamped beam versus *t* for $L/r = 50$ and $N_0/EA = 0$.
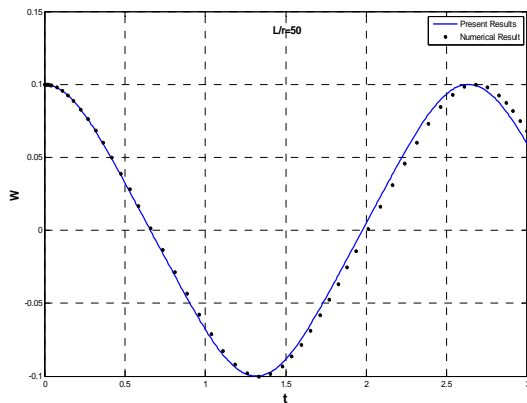
Fig. 2
Deflection variation of simply supported beam versus t for $L/r = 50$ and $N_0/EA = 0$.
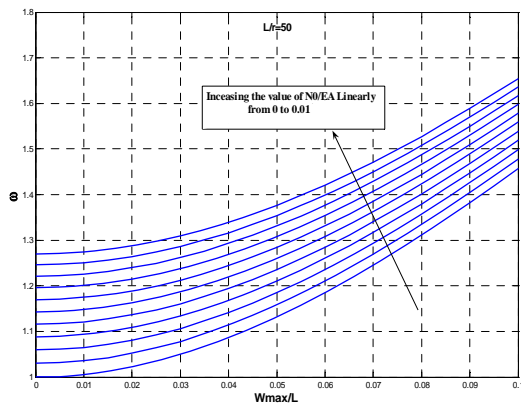
Fig. 3
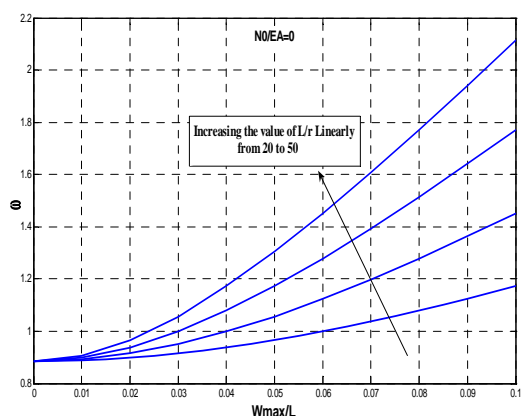Nonlinear natural frequency of double-clamped beam versus W_{max}/L for various N_0/EA.

Fig. 4
Nonlinear natural frequency of simply supported beam versus W_{max}/L for various N_0/EA.

Figs. 3 and 4 illustrate the effects of the parameter $N_0/(EA)$ on the nonlinear frequency versus non-dimensional amplitude for doubly-clamped and simply supported beam. As it can be observed, the increase in non-linear fundamental natural frequency with increasing the displacement is very low at small amplitudes. This leads to the conclusion that at small deflections data from linear and non-linear models agree well with each other. However, as the maximum amplitude increases, the non-linearity effect becomes significant. Applying pre-tensile loads will reduce the nonlinear period or increase the nonlinear frequency of the system.
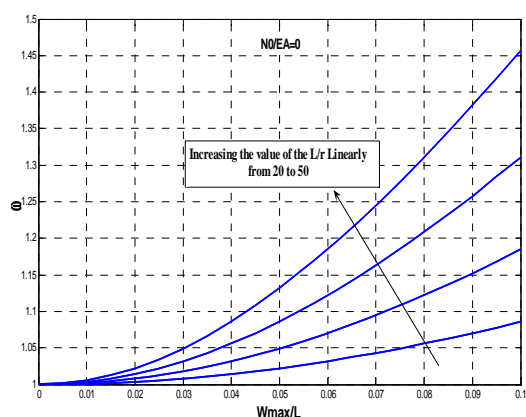
Fig. 5
Nonlinear natural frequency of double-clamped beam versus W_{max}/L for various L/r .

Fig. 6
Nonlinear natural frequency of simply supported beam versus W_{max}/L for various L/r .

Figs. 5 and 6 have been predicted to investigate the effect of the slenderness ratio to the nonlinear frequency of the system. It can be seen that with increasing the slenderness ratio the nonlinear natural frequency would increase regardless of the value of the pre-tensile axial load applied on the beam.

4   CONCLUSION

In this study, the method of homotopy perturbation and modified Lindstedt-Poincare method has been applied in order to find the nonlinear vibrational behavior of beams, considering the effects of midplane stretching. It has been shown that the results of the HPM are significantly more accurate that the previously reported analytical results in the literature. A parametric study has also been applied in order to characterize the behavior of the beam due to changes in applied axial loads and changes in slenderness ratio. It was observed that increasing the applied pre-tensile loads and slenderness ratio would increase the nonlinear natural frequency of the beam.

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