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Analysis of Nonlinear Vibration of Piezoelectric Nanobeam Embedded in Multiple Layers Elastic Media in a Thermo-Magnetic Environment Using Iteration Perturbation Method

M.G. Sobamowo *

Department of Mechanical Engineering, University of Lagos, Akoka, Lagos, Nigeria
Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria

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ABSTRACT

In this work, analysis of nonlinear vibration of piezoelectric nanobeam in a thermo-magnetic environment embedded in Winkler, Pasternak, quadratic and cubic nonlinear elastic media for simply supported and clamped boundary conditions is presented. With the considerations of Von- Karman geometric nonlinearity effect and with the aids of nonlocal elasticity theory as well as Euler–Bernoulli beam model, the equation of motion for the nanobeam is derived using Hamilton's principle. The nonlinear dynamic model is solved using Galerkin-decomposition coupled with iteration perturbation method. From the parametric studies, it is shown that the frequency of the nanobeam increases at low temperatures but decreases at high temperatures. The nonlocal parameter decreases the frequencies of the piezoelectric nanobeam. An increase in the quadratic nonlinear elastic medium stiffness causes a decrease in the first mode of the nanobeam with clamped-clamped supports and an increase in all modes of the simply supported nanobeam at both low and high temperatures. When the magnetic force, cubic nonlinear elastic medium stiffness, and amplitude increase, there is an increase in all mode frequencies of the nanobeam. An increase in the temperature change at high temperature reduces the frequency ratio but at low or room temperature, an increase in temperature change, increases the frequency ratio of the structure nanotube. The significance of this study is evident in the design and applications of nanobeams in thermal and magnetic environments.

Keywords : Elastic media; Magnetic field; Electric field; Nonlocal elasticity theory; Temperature effect; Iteration perturbation method.

1 INTRODUCTION

THE continuous and increasing wide applications of nanomaterials have been tremendous following the discovery of the novel nanostructure materials by Iijima [1]. Such expanding applications of nanomaterials are evident in the developments of nanoelectronics, nanodevices, nanomechanical systems, nanobiological,

*Corresponding author. Tel.: +23 47034717417.
E-mail address: mikegbeminiyi@gmail.com (M.G. Sobamowo)

nanocomposites due to its excellent properties and high strength to weight ratio. However, large deformations within the elastic limit and high frequency of vibrate at frequency in the order of GHz and THz are experienced in the applications of carbon nanotubes. Consequently, there have been increasing research interests directed towards the investigations into the dynamic behaviours of the novel structures [2-9]. In the pools of the research studies, Sears and Batra [10] studied the buckling behaviour of carbon nanotubes subjected to axial compression. Yoon et al. [11] explored the noncoaxial resonance of an isolated carbon nanotube with multiple walls while Wang and Cai [12] presented an extended study on the same work with the consideration of the effects of initial stress on the nanostructure. Wang et al. [13] analyzed the dynamic behaviour of carbon nanotube with multiple walls using Timoshenko beam model. Zhang et al. [14] examined the impact of compressive axial load on the transverse vibrations of carbon nanotube with double walls. Elishakoff and Pentaras [15] presented the fundamental natural frequencies of carbon nanotube with double walls. Buks and Yurke [16] accessed the nonlinear nanomechanical resonator of mass detection while [17] Postma et al. [18] determined the dynamic range of carbon nanostructure. Fu et al. [18] submitted nonlinear vibration analysis of embedded nanotubes. Vibration of carbon nanotube with electrical actuator was studied by some authors [19-24]. The nonlinear vibrations of the carbon nanotube with double walls was submitted by Hawwa and Al-Qahtani [24]. Hajnayeb and Khadem [25] studied the nonlinear dynamic behaviour and stability of the double-walled nanotube subjected to electrostatic actuation. Xu et al. [26] considers nonlinear intertube van der Waals forces on the dynamic response of carbon nanotube with double walls. With the aids of nonlocal Timoshenko beam model. Lei et al. [27] explored surface effects on the frequency of vibration of carbon nanotube with double walls. Ghorbanpour et al. [28] used shell model to analyze nonlinear nonlocal vibration of fluid-conveying embedded carbon nanotubes with double walls. The analyses of the carbon nanotubes were extended to multi-walled carbon nanotubes (MWCNTs) [29-35]. Sobamowo [36, 37 and 38], Sobamowo et al. [39] as well as Arefi and Nahvi [40] studied nonlinear vibration in nano-structures with slightly and initial curvature while Cigeroglu and Samandari [41] analyzed the dynamic behaviour of curved nanobeams. Studies on vibrations of nanotubes as presented in literatures using experimental measurements, density functional theory, molecular dynamics simulations, and classical continuum theories and non-classical continuum theories such as nonlocal stress theory, modified couple stress theory, gradient strain theory, and surface elasticity theory. There are some difficulties in the experiment investigations at the nanoscale level. Therefore, majority of the past works are based on theoretical investigations using classical continuum models (which do not consider the small-scale effects). However, due to their scale-free models as they cannot incorporate the small-scale effects in their formulations, the classical continuum theories are inadequate for the accurate predictions of the dynamic behaviours of the nanotubes. Such inadequacy in the classical continuum models is corrected in the works of Eringen [42-44] and that of Erigen and Edelen [45], where the author developed nonlocal continuum mechanics based on nonlocal elasticity theory. Although, some studies in literature have used the nonlocal continuum mechanics to present some theoretical investigations [46-67]. Simsek [68] as well as Murmu and Pradhan [69] adopted nonlocal elasticity theory to study the nonlinear vibration of a carbon nanotube embedded in an elastic medium. In a recent study, Abdullah et al. [70] presented effects of temperature, magnetic field and elastic media on the nonlinear vibration of nanobeams. The authors present very good work and results.

However, the dynamic response of the nanobeam was not explored and the effect of electric field on the vibration characteristics of the nanobeam was not studied. Moreover, to the best of the authors knowledge, a study on the effects of electromechanical and thermomagnetic loadings on the nonlinear vibration of nanobeams embedded in Winkler, Pasternak, quadratic and cubic nonlinear elastic media has not been presented in literature. Therefore, with the aid of iteration perturbation method, the present work focusses on such study. With the considerations of Von Karman geometric nonlinearity effect and with the aids of nonlocal elasticity theory and Euler–Bernoulli beam model, the equation of motion for the nanobeam is derived using Hamilton's principle. Also, the present analysis used four layers (Winkler, Pasternak, and quadratic and cubic nonlinear layers) which generate nonlinearities in the developed dynamic models. Additionally, the impacts of nonlocal parameter, electromechanical parameter, magnetic force, elastic media, temperature and amplitude on the dynamic behaviour of the nanotube are investigated.

2 MODEL DEVELOPMENT FOR THE SINGLE_WALLED NANOTUBE

Consider a nanobeam embedded in linear and nonlinear elastic media as shown in Fig. 1. The nanobeam is subjected to stretching effects and resting on Winkler, Pasternak and nonlinear elastic media in a thermo-magnetic environment as depicted in the figure.

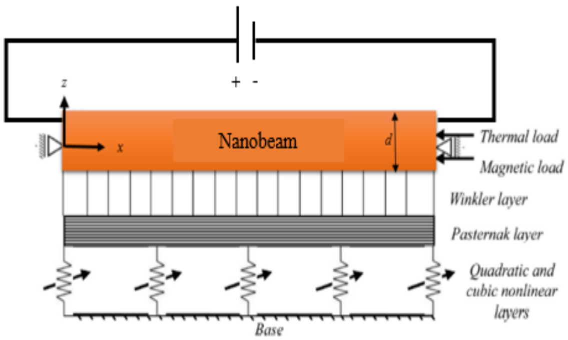


Fig.1
A piezoelectric nanobeam embedded in linear and nonlinear elastic media (Note: only the bottom side of the elastic media is shown).

with the applications of nonlocal theory presented by Erigen [42, 43, 44] and that of Erigen and Edelen [45], the relationship between the nonlocal stress–tenso, Euler-Bernoulli theory and Hamilton's principle the following equations are developed

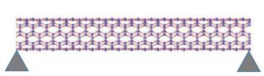
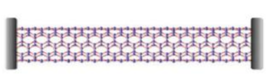
$$\begin{aligned}&EI\left(\frac{\partial^4\bar{w}}{\partial\bar{x}^4}\right)+\rho A_c\frac{\partial^2}{\partial\bar{t}^2}\left[\bar{w}-(e_0a)^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right]+k_w\left[\bar{w}-(e_0a)^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right]-k_p\frac{\partial^2}{\partial\bar{x}^2}\left[\bar{w}-(e_0a)^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right]+k_2\left[\bar{w}^2-(e_0a)^2\frac{\partial^2\left(\bar{w}^2\right)}{\partial\bar{x}^2}\right]\\&+k_3\left[\bar{w}^3-(e_0a)^2\frac{\partial^2\left(\bar{w}^3\right)}{\partial\bar{x}^2}\right]-\eta A_cH_{\bar{x}}^2\frac{\partial^2}{\partial\bar{x}^2}\left[\bar{w}-(e_0a)^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right]+\left(EA_c\frac{\alpha_{\bar{x}}\Delta T}{1-2\nu}\right)\frac{\partial^2}{\partial\bar{x}^2}\left[\bar{w}-(e_0a)^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right]\\&+\zeta E_zA_c\frac{\partial^2}{\partial\bar{x}^2}\left[\bar{w}-(e_0a)^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right]-\left[\left(\frac{EA_c}{2L}\int_0^L\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2d\bar{x}\right)\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2}-(e_0a)^2\frac{\partial^4\bar{w}}{\partial\bar{x}^4}\right)\right]=0\end{aligned} \tag{1}$$

It is assumed that the midpoint of the nanobeam is subjected to the following initial conditions

$$\bar{w}(\bar{x},0)=\bar{w}_o,\quad \frac{\partial^2\bar{w}(\bar{x},0)}{\partial^2\bar{x}}=0 \tag{2}$$

The following boundary conditions for the multi-walled nanotubes are considered in this work:

Table 1
The basic functions corresponding to the above boundary conditions.

Cases	Mode shape, $\phi(x)$	Value of β for the first mode
Simply support 	$sin\left(\frac{\beta x}{L}\right)$	π
Clamped-Clamped support 	$\left(\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right)-\left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right)$	4.730041
	$\frac{1}{2}\left(1-cos\left(\frac{\beta x}{L}\right)\right)$	2π

For simply supported (S-S) nanotube,

$$\bar{w}(0,\bar{t})=0,\quad \frac{\partial^2\bar{w}(0,\bar{t})}{\partial^2\bar{x}}=0,\quad \bar{w}(L,\bar{t})=0,\quad \frac{\partial^2\bar{w}(L,\bar{t})}{\partial^2\bar{x}}=0. \tag{3}$$

For clamped-clamped supported (C-C) nanotube,

$$\bar{w}(0,\bar{t}) = 0, \quad \frac{\partial \bar{w}(0,\bar{t})}{\partial \bar{x}} = 0, \quad w((L,\bar{t}) = 0, \quad \frac{\partial w(L,\bar{t})}{\partial \bar{x}} = 0. \tag{4}$$

Using the following adimensional constants and variables

$$\begin{gathered} x = \frac{\bar{x}}{L};\ w = \frac{\bar{w}}{r};\ t = \sqrt{\frac{EI}{\rho A_c L^4}};\ r = \sqrt{\frac{I}{A_c}};h = \frac{e_0 a}{L};\ \ \alpha_t^d = \frac{N_{thermal} L^2}{EI};\ \ A = \frac{\bar{w}_o}{r},\ \ \xi_e = \frac{\zeta E_z A_c L^2}{EI} \\ K_w = \frac{k_w L^4}{EI};\ K_p = \frac{k_p L 2}{EI};\ \ Ha_m = \frac{\eta A_c H_{\bar{x}}^2 L^2}{EI};\ \ K_2^d = \frac{k_2 r L^4}{EI};\ K_3^d = \frac{k_3 r^2 L^4}{EI}. \end{gathered} \tag{5}$$

The adimensional form of the governing equation of motion for the nanobeam is given as:

$$\begin{gathered} \left[1 + K_p h^2 + Ha_m h^2 - \alpha_t^d h^2 - \xi_e h^2 + \frac{h^2}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^4 w}{\partial x^4} + \left[\alpha_t^d + \xi_e - K_w h^2 - K_p - Ha_m - \frac{1}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2 w}{\partial x^2} \\ + K_w w + \frac{\partial^2 w}{\partial t^2} - h^2 \frac{\partial^4 w}{\partial x^2 \partial t^2} + K_2^d\left[w^2 - h^2 \frac{\partial^2 \left(w^2\right)}{\partial x^2}\right] + K_3^d\left[w^3 - h^2 \frac{\partial^2 \left(w^3\right)}{\partial x^2}\right] = 0 \end{gathered} \tag{6}$$

and the boundary conditions become.

For simply supported (S-S) nanotube,

$$w(0,t) = 0, \quad \frac{\partial^2 w(0,t)}{\partial^2 x} = 0, \quad w(1,t) = 0, \quad \frac{\partial^2 w(1,t)}{\partial^2 x} = 0. \tag{7}$$

For clamped-clamped supported (C-C) nanotube,

$$w(0,t) = 0, \quad \frac{\partial w(0,t)}{\partial x} = 0, \ w(1,t) = 0, \quad \frac{\partial w(1,t)}{\partial x} = 0. \tag{8}$$

3 SOLUTION METHODOLOGY: GALERKIN DECOMPOSITION AND ITERATION PERTURBATION METHOD

The method of solution for the governing equation include Galerkin decomposition and homotopy perturbation methods. As the name implies the Galerkin decomposition method is used to decompose the governing partial differential equation of motion can be separated into spatial and temporal parts. The resulting temporal equations are solved using iteration perturbation method.

The procedures for the analysis of the equations are given in the proceeding sections as follows:

3.1 Galerkin decomposition method

With the application of Galerkin decomposition procedure, the governing partial differential equations of motion can be separated into spatial and temporal parts of the lateral displacement function as:

$$w(x,t) = \phi(x) q(t) \tag{9}$$

Using one-parameter Galerkin decomposition procedure, one arrives at

$$\int_0^1 R(x,t)\phi(x) dx = 0 \tag{10}$$

where $R(x,t)$ is the governing equation of motion for nanobeam i.e.

$$\begin{aligned} R(x,t) = & \left[1+K_p h^2 + Ha_m h^2 - \alpha_t^d h^2 - \xi_e h^2 + \frac{h^2}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^4 w}{\partial x^4} \\ & + \left[\alpha_t^d + \xi_e - K_w h^2 - K_p - Ha_m - \frac{1}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2 w}{\partial x^2} \\ & + K_w w + \frac{\partial^2 w}{\partial t^2} - h^2 \frac{\partial^4 w}{\partial x^2 \partial t^2} + K_2^d\left[w^2 - h^2 \frac{\partial^2 (w^2)}{\partial x^2}\right] + K_3^d\left[w^3 - h^2 \frac{\partial^2 (w^3)}{\partial x^2}\right] = 0 \end{aligned} \tag{11}$$

where $\phi(x)$ is the basis or trial or comparison function or normal function, which must satisfy the kinetic boundary conditions in Eq. (7) and (8), and $q(t)$ is the temporal part (time-dependent function).

Substituting Eqs. (11) into (10), then multiplying both sides of the resulting equation by $\phi(x)$ and integrating it for the domain of (0,1)

$$\int_0^1 \left\{\begin{aligned} & \left[1+K_p h^2 + Ha_m h^2 - \alpha_t^d h^2 - \xi_e h^2 + \frac{h^2}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^4 w}{\partial x^4} \\ & + \left[\alpha_t^d + \xi_e - K_w h^2 - K_p - Ha_m - \frac{1}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2 w}{\partial x^2} \\ & + K_w w + \frac{\partial^2 w}{\partial t^2} - h^2 \frac{\partial^4 w}{\partial x^2 \partial t^2} + K_2^d\left[w^2 - h^2 \frac{\partial^2 (w^2)}{\partial x^2}\right] + K_3^d\left[w^3 - h^2 \frac{\partial^2 (w^3)}{\partial x^2}\right] \end{aligned}\right\} \phi(x) dx = 0 \tag{12}$$

Substitution of Eq. (10) into Eq. (12), gives

$$\int_0^1 \left\{\begin{aligned} & \left[1+K_p h^2 + Ha_m h^2 - \alpha_t^d h^2 - \xi_e h^2 + \frac{h^2}{2}\int_0^1 \left(\frac{\partial(\phi(x)q(t))}{\partial x}\right)^2 dx\right]\frac{\partial^4 (\phi(x)q(t))}{\partial x^4} \\ & + \left[\alpha_t^d + \xi_e - K_w h^2 - K_p - Ha_m - \frac{1}{2}\int_0^1 \left(\frac{\partial(\phi(x)q(t))}{\partial x}\right)^2 dx\right]\frac{\partial^2 (\phi(x)q(t))}{\partial x^2} \\ & + K_w (\phi(x)q(t)) + \frac{\partial^2 (\phi(x)q(t))}{\partial t^2} - h^2 \frac{\partial^4 (\phi(x)q(t))}{\partial x^2 \partial t^2} \\ & + K_2^d\left[(\phi(x)q(t))^2 - h^2 \frac{\partial^2 \left((\phi(x)q(t))^2\right)}{\partial x^2}\right] + K_3^d\left[(\phi(x)q(t))^3 - h^2 \frac{\partial^2 \left((\phi(x)q(t))^3\right)}{\partial x^2}\right] \end{aligned}\right\} \phi(x) dx = 0 \tag{13}$$

$$\int_0^1 \left\{\begin{aligned} & \left[1+K_p h^2 + Ha_m h^2 - \alpha_t^d h^2 - \xi_e h^2 + \frac{h^2 q^2(t)}{2}\int_0^1 \left(\frac{\partial \phi(x)}{\partial x}\right)^2 dx\right] q(t)\frac{\partial^4 \phi(x)}{\partial x^4} \\ & + \left[\alpha_t^d + \xi_e - K_w h^2 - K_p - Ha_m - \frac{q^2(t)}{2}\int_0^1 \left(\frac{\partial \phi(x)}{\partial x}\right)^2 dx\right] q(t)\frac{\partial^2 \phi(x)}{\partial x^2} \\ & + K_w (\phi(x)q(t)) + \phi(x)\frac{\partial^2 q(t)}{\partial t^2} - h^2 \left(\frac{\partial^2}{\partial x^2}\left(\phi(x)\frac{\partial^2 q(t)}{\partial t^2}\right)\right) \\ & + K_2^d\left[(\phi(x)q(t))^2 - h^2 q^2(t)\frac{\partial^2 \left((\phi(x))^2\right)}{\partial x^2}\right] + K_3^d\left[(\phi(x)q(t))^3 - h^2 q^3(t)\frac{\partial^2 \left((\phi(x))^3\right)}{\partial x^2}\right] \end{aligned}\right\} \phi(x) dx = 0 \tag{14}$$

$$\int_0^1 \left\{ \begin{array}{l} \left[1 + K_p h^2 + Ha_m h^2 - \alpha_t^d h^2 - \xi_e h^2 + \dfrac{h^2 q^3(t)}{2} \int_0^1 \left(\dfrac{\partial \phi(x)}{\partial x} \right)^2 dx \right] \dfrac{\partial^4 \phi(x)}{\partial x^4} \\ + \left[\alpha_t^d + \xi_e - K_w h^2 - K_p - Ha_m - \dfrac{q^3(t)}{2} \int_0^1 \left(\dfrac{\partial \phi(x)}{\partial x} \right)^2 dx \right] \dfrac{\partial^2 \phi(x)}{\partial x^2} \\ + K_w \left(\phi(x) q(t) \right) + \phi(x) \dfrac{\partial^2 q(t)}{\partial t^2} - h^2 \left(\dfrac{\partial^2 q(t)}{\partial t^2} \dfrac{\partial^2 \phi(x)}{\partial x^2} \right) \\ + K_2^d \left[\left(\phi(x) q(t) \right)^2 - h^2 q^2(t) \dfrac{\partial^2 \left(\left(\phi(x) \right)^2 \right)}{\partial x^2} \right] + K_3^d \left[\left(\phi(x) q(t) \right)^3 - h^2 q^3(t) \dfrac{\partial^2 \left(\left(\phi(x) \right)^3 \right)}{\partial x^2} \right] \end{array} \right\} \phi(x) dx = 0 \tag{15}$$

$$\int_0^1 \left\{ \begin{array}{l} \left[\phi^2(x) - h^2 \phi(x) \left(\dfrac{\partial^2 \phi(x)}{\partial x^2} \right) \right] \dfrac{\partial^2 q(t)}{\partial t^2} + \left[\begin{array}{l} \left(K_w \phi^2(x) + \left(1 + K_p h^2 + Ha_m h^2 - \alpha_t^d h^2 - \xi_e h^2 \right) \phi(x) \dfrac{\partial^4 \phi(x)}{\partial x^4} \right) \\ + \left(\left(\alpha_t^d + \xi_e - K_w h^2 - K_p - Ha_m \right) \phi(x) \dfrac{\partial^2 \phi(x)}{\partial x^2} \right) \end{array} \right] q(t) \\ + K_2^d \left[\phi^3(x) - h^2 \phi(x) \dfrac{\partial^2 \left(\left(\phi(x) \right)^2 \right)}{\partial x^2} \right] q^2(t) + \left[\begin{array}{l} K_3^d \left[\phi^4(x) - h^2 \phi(x) \dfrac{\partial^2 \left(\left(\phi(x) \right)^3 \right)}{\partial x^2} \right] \\ + \left(\left[\dfrac{h^2}{2} \int_0^1 \left(\dfrac{\partial \phi(x)}{\partial x} \right)^2 dx \right] \phi(x) \dfrac{\partial^4 \phi(x)}{\partial x^4} \right) \\ - \left(\left[\dfrac{1}{2} \int_0^1 \left(\dfrac{\partial \phi(x)}{\partial x} \right)^2 dx \right] \phi(x) \dfrac{\partial^2 \phi(x)}{\partial x^2} \right) \end{array} \right] q^3(t) \end{array} \right\} dx = 0 \tag{16}$$

Therefore, Eq. (16) can be written as:

$$\bar{\gamma}_0 \frac{d^2 q(t)}{dt^2} + \bar{\gamma}_1 q(t) + \bar{\gamma}_2 q^2(t) + \bar{\gamma}_3 q^3(t) = 0 \tag{17}$$

Furthermore, we can express Eq. (13) as:

$$\frac{d^2 q(t)}{dt^2} + \gamma_1 q(t) + \gamma_2 q^2(t) + \gamma_3 q^3(t) = 0 \tag{18}$$

where

$$\gamma_1 = \frac{\bar{\gamma}_1}{\bar{\gamma}_0};\ \gamma_2 = \frac{\bar{\gamma}_2}{\bar{\gamma}_0};\ \gamma_3 = \frac{\bar{\gamma}_3}{\bar{\gamma}_0}; \tag{19}$$

$$\bar{\gamma}_0 = \int_0^1 \left(\phi^2 - h^2 \phi \frac{\partial^2 \phi}{\partial x^2} \right) dx \tag{20}$$

$$\bar{\gamma}_1 = \int_0^1 \left(\begin{array}{l} K_w \phi^2 + \left(1 + K_p h^2 + Ha_m h^2 - \alpha_t^d h^2 - \xi_e h^2 \right) \phi \dfrac{\partial^4 \phi}{\partial x^4} \\ + \left(\alpha_t^d + \xi_e - K_w h^2 - K_p - Ha_m \right) \phi \dfrac{\partial^2 \phi}{\partial x^2} \end{array} \right) dx \tag{21}$$

$$\bar{\gamma}_2 = \int_0^1 K_2^d \left(\phi^3 - h^2 \phi \frac{\partial^2 \left(\phi^2\right)}{\partial x^2} \right) dx \tag{22}$$

$$\bar{\gamma}_3 = \int_0^1 K_3^d \left(\phi^4 - h^2 \phi \frac{\partial^2 \left(\phi^4\right)}{\partial x^2} \right) dx + \frac{h^2}{2} \int_0^1 \left(\frac{\partial \phi}{\partial x} \right)^2 dx \int_0^1 \phi \frac{\partial^2 \phi}{\partial x^2} dx - \frac{1}{2} \int_0^1 \left(\frac{\partial \phi}{\partial x} \right)^2 dx \int_0^1 \phi \frac{\partial^4 \phi}{\partial x^4} dx \tag{23}$$

The initial conditions are given as:

$$q(0) = A, \qquad \frac{dq(0)}{dt} = 0 \tag{24}$$

A is the maximum vibration amplitude of the structure. It can be seen from the above procedures that the apart from the fact that the Galerkin decomposition method decomposes governing equation of motion into spatial and temporal parts, it also helps in converting the space- and time-dependent partial differential equation to a time-dependent ordinary differential equation. The nonlinear ordinary differential equation easily be solved using numerical methods or approximate analytical methods. In this work, iteration perturbation method is adopted due to its simplicity and high level of accuracy.

4 METHOD OF SOLUTION: ITERATION PERTURBATION METHOD

In order to solve the nonlinear model in Eq. (18), iteration perturbation method is adopted in the present study. In order to obtain an iteration perturbation solution for the nonlinear equation, an artificial parameter ε is introduced. Therefore, Eq. (18) becomes:

$$\frac{d^2 q}{dt^2} + \varepsilon \gamma_1 q + \varepsilon \gamma_2 q q_0 + \varepsilon \gamma_3 q q_0^2 = 0 \tag{25}$$

which implies that

$$\frac{d^2 q}{dt^2} + \varepsilon q \left(\gamma_1 + \gamma_2 q_0 + \gamma_3 q_0^2 \right) = 0 \tag{26}$$

where q_0 is the initial approximation solution. With the purpose of finding the periodic solution of Eq. (18), an initial approximation for zero-order deformation is assumed as:

$$q_o(t) = A cos\left(\omega t\right) \tag{27}$$

where ω is an unknown angular frequency which will be determined later.

Substitution of Eq. (27) into Eq. (26), provides

$$\frac{d^2 q}{dt^2} + \varepsilon q \left(\gamma_1 + \gamma_2 A cos\left(\omega t\right) + \gamma_3 A^2 cos^2\left(\omega t\right) \right) = 0 \tag{28}$$

After using trigonometric identities, Eq. (28) can be written as:

$$\frac{d^2 q}{dt^2} + \varepsilon q \gamma_1 + \varepsilon \gamma_2 q A cos\left(\omega t\right) + \frac{\varepsilon \gamma_3 A^2 q}{2} cos\left(2\omega t\right) + \frac{\varepsilon \gamma_3 A^2 q}{2} = 0 \tag{29}$$

Assuming that

$$q = q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + ... \tag{30a}$$

$$\varepsilon\gamma_1 = \omega^2 + \varepsilon c_1 + \varepsilon^2 c_2 + ... \tag{30b}$$

Substitute Eq. (30a) and (30b) into Eq. (29), provides

$$\begin{aligned}&\frac{d^2\left(q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + ...\right)}{dt^2} + \left(\omega^2 + \varepsilon c_1 + \varepsilon^2 c_2 + ...\right)\left(q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + ...\right) + \varepsilon\gamma_2\left(q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + ...\right)A\cos\left(\omega t\right)\\&+\frac{\varepsilon\gamma_3 A^2\left(q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + ...\right)}{2}\cos\left(2\omega t\right) + \frac{\varepsilon\gamma_3 A^2\left(q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + ...\right)}{2} = 0\end{aligned} \tag{31}$$

Also, for the initial conditions in Eq. (30), we have

$$q_0(0) + \varepsilon q_1(0) + \varepsilon^2 q_2(0) + ... = A, \qquad \frac{d\left(q_0(0) + \varepsilon q_1(0) + \varepsilon^2 q_2(0) + ...\right)}{dt} = 0 \tag{32}$$

Arranging Eq. (31) according to the powers of ε, gives
The zeroth-order equation is

$$\varepsilon^0: \quad \frac{d^2 q_0}{dt^2} + \omega^2 q_0 = 0 \tag{33}$$

The initial conditions are

$$q_0(0) = A, \qquad \frac{dq_0(0)}{dt} = 0 \tag{34}$$

The first-order equation is

$$\varepsilon^1: \quad \frac{d^2 q_1}{dt^2} + \omega^2 q_1 + c_1 q_0 + \gamma_2 q_0 A\cos\left(\omega t\right) + \frac{\gamma_3 A^2 q_0}{2}\cos\left(2\omega t\right) + \frac{\gamma_3 A^2 q_0}{2} = 0 \tag{35}$$

The initial conditions are

$$q_1(0) = 0, \qquad \frac{dq_1(0)}{dt} = 0 \tag{36}$$

The second-order equation is

$$\varepsilon^2: \quad \frac{d^2 q_2}{dt^2} + \omega^2 q_2 + c_1 q_1 + c_2 q_0 + \gamma_2 q_1 A\cos\left(\omega t\right) + \frac{\gamma_3 A^2 q_1}{2}\cos\left(2\omega t\right) + \frac{\gamma_3 A^2 q_1}{2} = 0 \tag{37}$$

The initial conditions are

$$q_2(0) = 0, \qquad \frac{dq_2(0)}{dt} = 0 \tag{38}$$

The solution of the zero-order is

$$q_o(t) = A\cos\left(\omega t\right) \tag{39}$$

which still obeys the assumed solution for the initial approximation.

In order to solve the first-order equation in Eq. (35), we substitute Eq. (39) into Eq. (35),

$$\frac{d^2q_1}{dt^2} + \omega^2 q_1 + c_1 A\cos\left(\omega t\right) + \gamma_2 q_0 A^2 \cos^2\left(\omega t\right) + \frac{\gamma_3 A^3}{2}\cos\left(2\omega t\right)\cos\left(\omega t\right) + \frac{\gamma_3 A^3}{2}\cos\left(\omega t\right) = 0 \tag{40}$$

which can be written as:

$$\frac{d^2q_1}{dt^2} + \omega^2 q_1 + c_1 A\cos\left(\omega t\right) + \gamma_2 A^2\left(\frac{1+\cos\left(2\omega t\right)}{2}\right) + \frac{\gamma_3 A^3}{4}\left[\cos\left(3\omega t\right) + \cos\left(\omega t\right)\right] + \frac{\gamma_3 A^3}{2}\cos\left(\omega t\right) = 0 \tag{41}$$

Collection of like terms in the above Eq. (41), provides

$$\frac{d^2q_1}{dt^2} + \omega^2 q_1 = -\left[\left(c_1 A + \frac{3\gamma_3 A^3}{4}\right)\cos\left(\omega t\right) + \frac{\gamma_2 A^2}{2}\cos\left(2\omega t\right) + \frac{\gamma_3 A^3}{4}\cos\left(3\omega t\right) + \frac{\gamma_2 A^2}{2}\right] \tag{42}$$

If the term $\cos\left(\omega t\right)$ exists in the RHS of Eq. (42), the secular term $t\cos\left(\omega t\right)$ will appear in the final solution. Therefore, the coefficient of $\cos\left(\omega t\right)$ should be equal to zero which gives

$$c_1 = -\frac{3\gamma_3 A^2}{4} \tag{43}$$

The zero-order nonlinear natural frequency can be found by substituting Eq. (43) into Eq. (35). We have

$$\varepsilon\gamma_1 \approx \omega_0^2 - \frac{3\varepsilon\gamma_3 A^2}{4} \quad \Rightarrow \omega_0^2 \approx \varepsilon\left(\gamma_1 + \frac{3\gamma_3 A^2}{4}\right) \tag{44}$$

From the definition of iterative perturbation method, for $\varepsilon = 1$

$$\omega_0^2 \approx \left(\gamma_1 + \frac{3\gamma_3 A^2}{4}\right) \quad \Rightarrow \omega_0^2 \approx \sqrt{\left(\gamma_1 + \frac{3\gamma_3 A^2}{4}\right)} \tag{45}$$

Therefore, the zero-order nonlinear natural frequency is given as:

$$\omega_0 \approx \sqrt{\gamma_1 + \frac{3}{4}\gamma_3 A^2} \tag{46}$$

Therefore, the ratio of the zero-order nonlinear natural frequency, ω_o to the linear frequency, ω_b

$$\frac{\omega_0}{\omega_b} \approx \sqrt{1 + \frac{3}{4}\frac{\gamma_3 A^2}{\gamma_1}} \tag{47}$$

where $\omega_b = \sqrt{\gamma_1}$. Also, it can easily be shown that the solution of Eq. (35) is

$$q_1(t) = A\cos(\omega t) - \frac{A^2(3\gamma_3 A - 8\gamma_2)\cos(\omega t)}{24\omega^2} + \frac{A^2\left[18A\gamma_3\cos(\omega t) + 32\gamma_2\cos(2\omega t) + 6A\gamma_3\cos(3\omega t) - 96\gamma_2\right]}{192\omega^2} \quad (48)$$

We can now substitute Eqs. (39) and (48) into Eq. (37), after simplification, we arrived at

$$\frac{d^2q_2}{dt^2} + \omega^2 q_2 = -\frac{1}{384\omega^2}\begin{bmatrix} 3A^3\left(32\gamma_2^2 - 11\gamma_3^2A^2 + 32\gamma_2\gamma_3A^2\right)\cos(\omega t) + 16A^3\gamma_2(4\gamma_2 - 7\gamma_3 A)\cos(2\omega t) \\ +2A^2\left(16\gamma_2^2 - 3\gamma_3^2A^3 + 16\gamma_2\gamma_3A^2\right)\cos(3\omega t) + 22\gamma_2\gamma_3A^4\cos(4\omega t) \\ +3\gamma_3^2A^5\cos(5\omega t) + A^3\gamma_2(58\gamma_3A + 64\gamma_2) \end{bmatrix} \quad (49)$$

No secular term in Eq. (49) as a result of

$$c(c_2;\omega) = 384Ac_2\omega^2 - 160\gamma_2^2A^3 + 3\gamma_3^2A^5 = 0 \quad (50)$$

The solution of Eq. (50) is

$$c_2 = \frac{A^2}{384\omega^2}\left(160\gamma_2^2 - 3\gamma_3^2A^2\right) \quad (51)$$

Substitute Eq. (36) and (50) into Eq. (30b), we have

$$\varepsilon\gamma_1\omega^2 \approx \omega_1^4 - \frac{3\gamma_3A^2}{4}\varepsilon\omega_1^2 + \varepsilon^2\left(\frac{A^2}{384}\left(160\gamma_2^2 - 3\gamma_3^2A^2\right)\right) \quad (52)$$

After re-arrangement, we have

$$\omega_1^4 - \varepsilon\left(\gamma_1 + \frac{3\gamma_3A^2}{4}\right)\omega_1^2 + \varepsilon^2\left(\frac{A^2}{384}\left(160\gamma_2^2 - 3\gamma_3^2A^2\right)\right) \approx 0 \quad (53)$$

The solution of Eq. (53) is

$$\omega_1 = \sqrt{\frac{\varepsilon}{2}\left(\gamma_1 + \frac{3\gamma_3A^2}{4}\right) + \sqrt{\left[\frac{\varepsilon}{2}\left(\gamma_1 + \frac{3\gamma_3A^2}{4}\right)\right]^2 - \varepsilon^2\left(\frac{A^2}{384}\left(160\gamma_2^2 - 3\gamma_3^2A^2\right)\right)}} \quad (54)$$

with $\varepsilon = 1$ and after simplification, we have

$$\omega_1 = \sqrt{\left(\frac{\gamma_1}{2} + \frac{3\gamma_3A^2}{8}\right) + \sqrt{\frac{\gamma_1^2}{4} + \frac{3\gamma_1\gamma_3A^2}{8} + \frac{19\gamma_3^2A^4}{128} - \frac{5\gamma_2^2A^2}{12}}} \quad (55)$$

The ratio of the first-order nonlinear frequency, ω_1 to the linear frequency, ω_b

$$\frac{\omega_1}{\omega_b} = \sqrt{\left(\frac{1}{2} + \frac{3\gamma_3A^2}{8\gamma_1}\right) + \sqrt{\frac{1}{4} + \frac{3\gamma_1\gamma_3A^2}{8\gamma_1^2} + \frac{19\gamma_3^2A^4}{128\gamma_1^2} - \frac{5\gamma_2^2A^2}{12\gamma_1^2}}}$$

From Eqs. (55) and (47), the following facts are established:

$$\lim_{A\to 0}\frac{\omega}{\omega_b} = 1 \quad (56)$$

and

$$\lim_{A \to \infty} \frac{\omega}{\omega_b} = \infty \tag{57}$$

Substitute the equations in Table 1 and Eq. (48) into Eq. (9), one arrives at. For simple supported nanobeam

$$w(x,t) \approx A \begin{bmatrix} cos(\omega t) - \dfrac{A(3\gamma_3 A - 8\gamma_2) cos(\omega t)}{24\omega^2} \\ + \dfrac{A\left[18A\gamma_3 cos(\omega t) + 32\gamma_2 cos(2\omega t) + 6A\gamma_3 cos(3\omega t) - 96\gamma_2\right]}{192\omega^2} \end{bmatrix} sin\left(\frac{\beta x}{L}\right) \tag{58}$$

For clamped-clamped nanobeam

$$w(x,t) \approx A \begin{bmatrix} cos(\omega t) - \dfrac{A(3\gamma_3 A - 8\gamma_2) cos(\omega t)}{24\omega^2} \\ + \dfrac{A\left[18A\gamma_3 cos(\omega t) + 32\gamma_2 cos(2\omega t) + 6A\gamma_3 cos(3\omega t) - 96\gamma_2\right]}{192\omega^2} \end{bmatrix} \left\{ \begin{aligned} &\left(\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right) \\ &-\left(\frac{\sinh\beta + \sin\beta}{\cosh\beta - \cos\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right) \end{aligned} \right\} \tag{59}$$

where

$$\omega = \sqrt{\left(\frac{\gamma_1}{2} + \frac{3\gamma_3 A^2}{8}\right) + \sqrt{\frac{\gamma_1^2}{4} + \frac{3\gamma_1\gamma_3 A^2}{8} + \frac{19\gamma_3^2 A^4}{128} - \frac{5\gamma_2^2 A^2}{12}}}$$

Table 2
Table of Parameters used for the simulations.

S/N	Parameter	Symbol	Value
1	Diameter of armchair single-walled nanotube	d	0.678 *nm*
2	Length of the nanotube	L	6.78 *nm*
3	Aspect ratio	L/h	10, 20, 50
4	Height of the nanotube	h	0, 0.1, 0.3 *nm*
5	Thickness of the nanotube	t	0.066 *nm*
6	Density of the nanotube	ρ	2300 kg/m^3
7	Young Modulus	E	5.5 *TPa*
8	Poisson's ratio	υ	0.19
9	Cross-sectional area	A	0.1406 nm^2
10	Thermal expansion coefficient for room and low Temperatures	α_x	$-1.6\times10^{-6} K^{-1}$
11	Thermal expansion coefficient for high Temperatures	α_x	$1.1\times10^{-6} K^{-1}$
12	Second moment of inertia	I	$8.155\times10^{-3}\ nm^4$
13	Nonlocal parameter	$(e_o a)^2$	0, 1, 2, 3, 4 nm^2
14	Dimensionless Winkler elastic medium stiffness	K_w	0 – 50
15	Dimensionless Pasternak elastic medium stiffness	K_p	0 – 100
16	Dimensionless quadratic nonlinear elastic medium stiffness	K_1	0 – 100
17	Dimensionless cubic nonlinear elastic medium stiffness	$K_{2'}$	0 – 100
18	Magnetic field permeability	η	10^3-$10^8\ N/A^2$
19	Longitudinal magnetic field	H_x	10^4-$10^9\ A/m$
20	Change in temperature	ΔT	0 – 300 K
21	One dimensional piezoelectric constant	ζ	0.95
22	Electric field	E_z	-1.7×10^7-5×10^6

5 RESULTS AND DISCUSSION

With the aid of MATLAB, the developed solutions are simulated, and the results are verified with the results of previous studies as presented in Tables 3 and 4. The comparison of the results as presented in the tables show that the results of the present study agree very well with the results of the previous studies in literatures. Also, this establishes the accuracy of the iteration perturbation method.

It is presented as shown in the Table 5 that the comparison of the results of linear fundamental frequency of the simply-supported nanobeam for various values of aspect ratio and nonlocal parameter and aspect ratio. It is shown in the results that the method is valid for a wide range of vibration amplitudes. Also, the method is relatively simple and cost effective as compared to the other approximate analytical methods.

Table 3
Comparison of results of nonlinear frequency ratio for simply supported when $\beta_w = \beta_p = \beta_1^d = \beta_2^d = Ha_m = \alpha_t = \xi_e = 0$.

		h=0.1			*h*=0.3	
A	Simsek [68]	Abdullah et al. [70]	Present	Simsek [68]	Abdullah et al. [70]	Present
0.5	1.02542	1.0254	1.0254	1.04331	1.0433	1.0433
1.5	1.20975	1.2098	1.2098	1.34037	1.3404	1.3404
3.5	1.51245	1.5125	1.5125	1.79243	1.7924	1.7924

Table 4
Comparison of results of nonlinear frequency ratio for clamped-clamped supported when $\beta_w = \beta_p = \beta_1^d = \beta_2^d = Ha_m = \alpha_t = \xi_e = 0$.

		h=0.1			*h*=0.3	
A	Simsek [68]	Abdullah et al. [70]	Present	Simsek [68]	Abdullah et al. [70]	Present
0.5	1.01974	1.0197	1.0197	1.04331	1.0433	1.0433
1.5	1.16572	1.1672	1.1672	1.34037	1.3404	1.3404
3.5	1.4132	1.4132	1.4132	2.07455	2.0746	2.0746

Table 5
Comparison of results of linear frequency ratio for simply-supported when $\beta_w = \beta_p = \beta_1^d = \beta_2^d = Ha_m = \alpha_t = 0$.

	L/d = 10			L/d = 20			L/d = 50		
$(e_o a)^2\ (nm^2)$	Simsek [68]	Murmu and Pradhan[69]	Present	Simsek [68]	Murmu and Pradhan[69]	Present	Simsek [68]	Murmu and Pradhan[69]	Present
0.0	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696
1.0	9.4158	9.4124	9.4157	9.7500	9.7498	9.7501	9.8501	9.8501	9.8501
2.0	9.0194	9.0133	9.0193	9.6747	9.6343	9.6745	9.8308	9.8308	9.8308
3.0	8.6692	8.6611	8.6690	9.5234	9.5228	9.5234	9.8116	9.8117	9.8116
4.0	8.3569	8.3472	8.3568	9.4158	9.4150	9.4157	9.7926	9.7925	9.7926

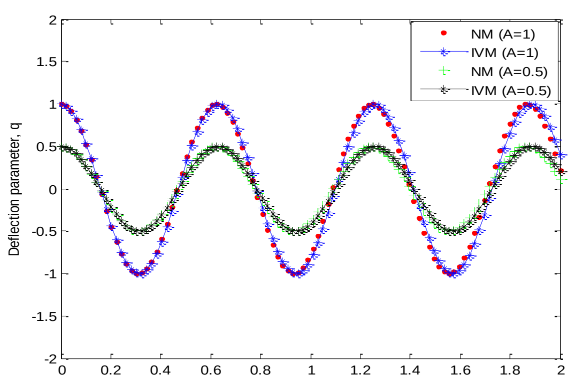


Fig.2
Comparison of results for the normalized deflection parameter *vs* dimensionless time.

The dynamic behaviours of the beam are shown in Fig. 2 Also, the figure shows the comparison of results of numerical method using Fourth-order Runge-Kutta and the results of the present study using homotopy perturbation method. The results show that excellent agreement between the resents of the two methods. Having verified the correctness and the high level of accuracy of the developed approximate analytical solutions, parametric studies are carried out as presented as follows:

5.1 Different buckled and mode shapes of the nanobeam

The different mode shapes (first-five normalized mode shapes) of the beam are shown in Figs. 3 and 4. The figures depict the first-five normalized mode shapes of the beams for the nanotube with simple-simple and clamped-clamped supports. In the figures, it is illustrated that the deflections of the beams along the beams' span at five different buckled and mode shapes.

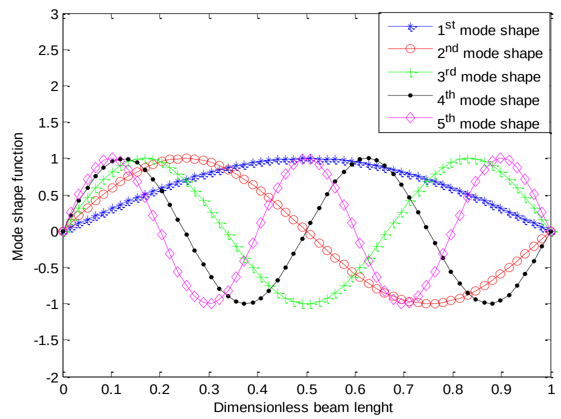


Fig.3
The first five normalized mode shaped of the under simple-simple supports.

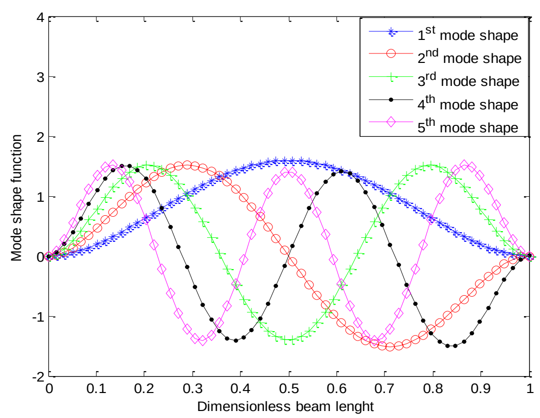


Fig.4
The first five normalized mode shaped of the beams under clamped-clamped supports beams.

5.2 Effects of nonlocal parameter, temperature and aspect ratio on the linear frequency

Figs. 5-9 presents the imparts of nonlocal parameter, change in temperature and aspect ratio on the linear frequencies of the simply and clamped-clamped supported nanotubes. It is shown in the figures that the linear frequencies of the simply and clamped-clamped supported nanotubes decrease at the high temperatures. However, the linear frequencies of the nanotubes under the two types of supports increase at the low temperatures as shown in Figs. 5-8. This is because of the damping effect of temperature which decreases the stiffness of the nanotube at high temperature. Also, it is found that the as the nonlocal parameter increases, the linear frequencies of the nanotubes with simply and clamped-clamped supports decrease at both low and high temperatures.

The effects of aspect ratio (ratio of the length of the beam to its diameter, *L/d*). The figure reveals that the frequency increases as the aspect increases. Also, the figure re-establishes that the linear frequency decreases as the nonlocal parameter increases. However, this impact reduces significantly as the aspect ratio increases

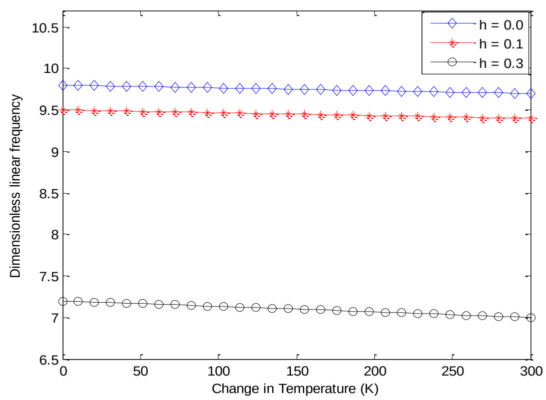


Fig.5
Effect of nonlocal parameter on the fundamental linear frequency of the simply supported nanobeam at high temperature.

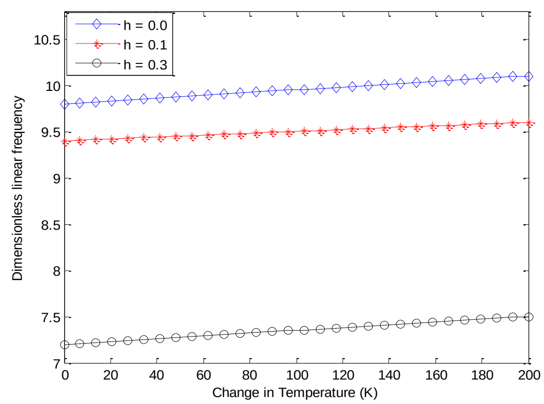


Fig.6
Effect of nonlocal parameter on the fundamental linear frequency of the simply supported nanobeam at low temperature.

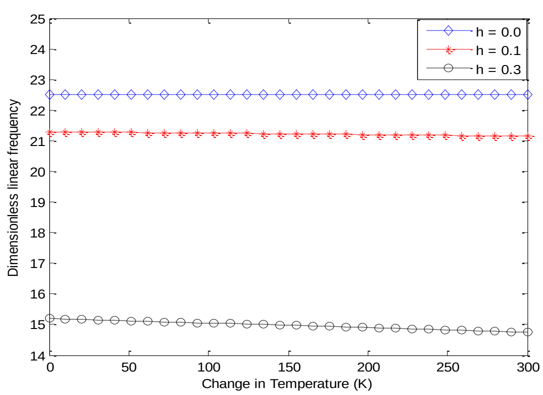


Fig.7
Effect of nonlocal parameter on the fundamental linear frequency of the clamped-clamped supported nanobeam at high temperature.

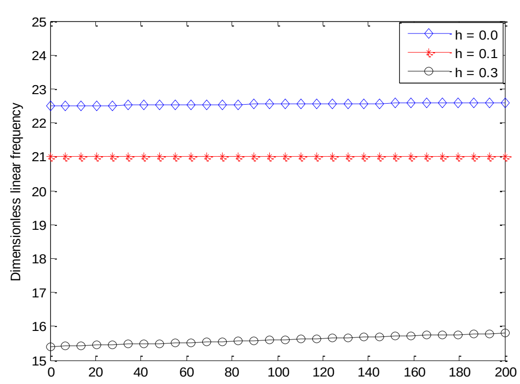


Fig.8
Effect of nonlocal parameter on the fundamental linear frequency of the clamped-clamped supported nanobeam at low temperature.

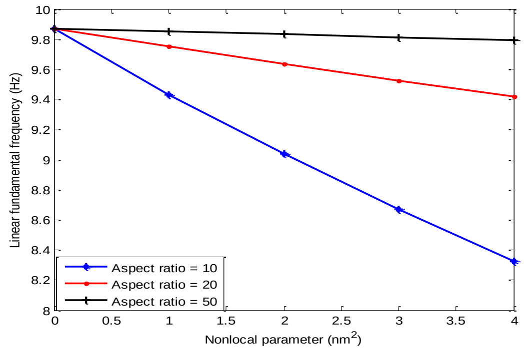


Fig.9
Effect of aspect ratio and nonlocal parameter on the fundamental frequency of the nanobeam.

5.3 Effects of nonlocal parameter, temperature, elastic medium stiffness on the nonlinear frequency

The effects of nonlocal parameter, change in temperature, Winkler, Pasternak, quadratic and cubic stiffnesses on the nonlinear frequencies of the simply and clamped-clamped supported nanotubes are shown in Fig. 10-15. It is depicted in the figures that the nonlinear frequencies of the simply and clamped-clamped supported nanotubes decrease at the high temperatures. However, the nonlinear frequencies of the nanotubes under the two types of supports increase at the low temperatures. This is because of the damping effect on temperature which decreases the stiffness at a high temperature and increases nanobeam stiffness at a low temperature.

It is shown in Figs. 10-13 that the nonlinear frequency increases as the Winkler stiffness (K_w) and Pasternak stiffness (K_p) for both low and high temperatures. This is due to the fact that increase in Pasternak stiffness causes an additional induced stiffness to the elastic medium of the nanotube. It was also shown that the Pasternak stiffness (K_p) has more significant effect on the nonlinear frequencies than the effect of Winkler stiffness (K_w). This is because of the shearing layer of Pasternak medium which bends and moves vertically as compared to the Winkler medium which moves only vertically during the vibration.

The influences of the Hartmann number, quadratic (K_1) and cubic (K_2) nonlinear elastic medium constants on the nonlinear frequencies of the nanobeam for both low and high temperatures in Fig. 14 and 15. The results illustrate that when Hartmann number and cubic (K_2) nonlinear elastic medium constants increase, the nonlinear frequency of the nanobeam increases for both low and high temperatures. However, when the Hartmann number and the quadratic (K_1) nonlinear elastic medium constants increase, the nonlinear frequency of the nanobeam decreases for clamped-clamped beam at both low and high temperatures. Also, the nonlinear frequency of the nanobeam increases as the amplitude of the vibration increases. The nonlinear frequency of the nanobeam increases as the as the magnetic field parameter (Hartmann number) increases because the magnetic field intensity increases the rigidity of the nanobeams. However, it should be stated that it is the quadratic (K_1) nonlinear elastic medium constants that reduces the nonlinear frequency of the nanobeam.

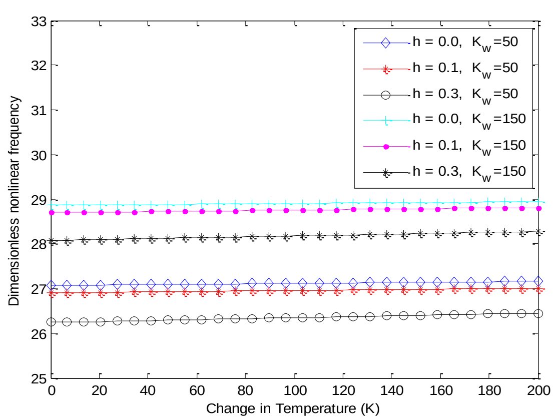


Fig.10
Effect of nonlocal parameter and Winkler elastic medium stiffness on the nonlinear natural frequency of the clamped-clamped supported nanobeam at low temperature.

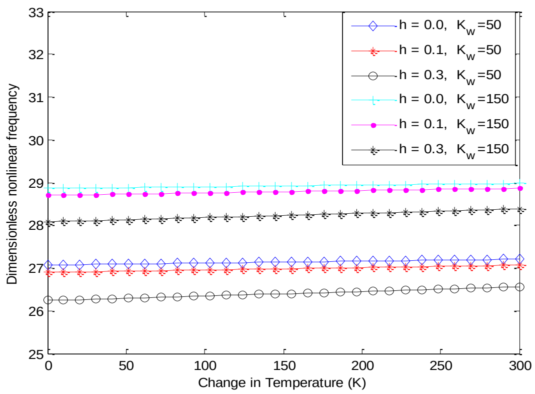


Fig.11
Effect of nonlocal parameter and Winkler elastic medium stiffness on the nonlinear natural frequency of the clamped-clamped supported nanobeam at high temperature.

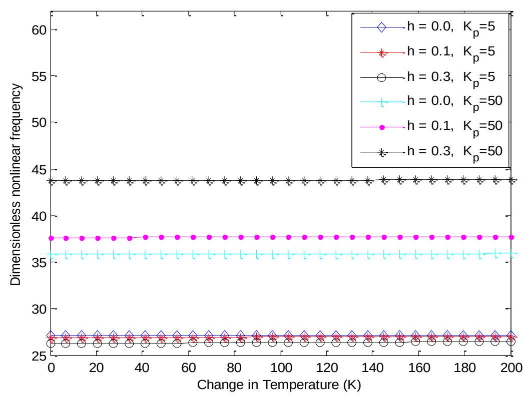


Fig.12
Effect of nonlocal parameter and Pasternak elastic medium stiffness on the nonlinear natural frequency of the clamped-clamped supported nanobeam at low temperature.

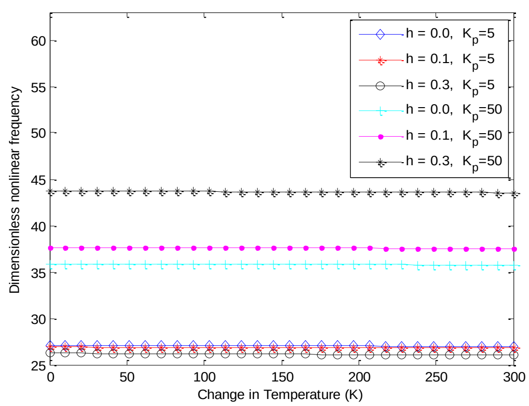


Fig.13
Effect of nonlocal parameter and Pasternak elastic medium stiffness on the nonlinear natural frequency of the clamped-clamped supported nanobeam at high temperature.

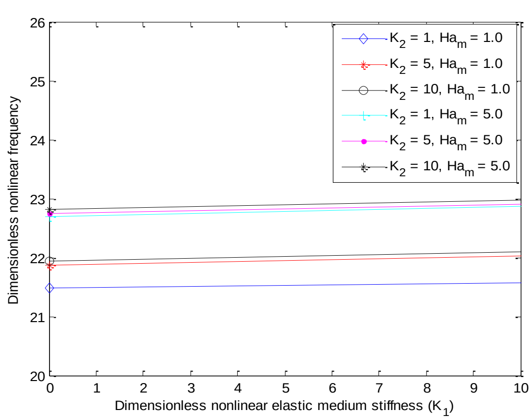


Fig.14
Effect of nonlinear elastic medium stiffness parameters and Hartmann number (magnetic force parameter) on the nonlinear natural frequency of the simply supported nanobeam.

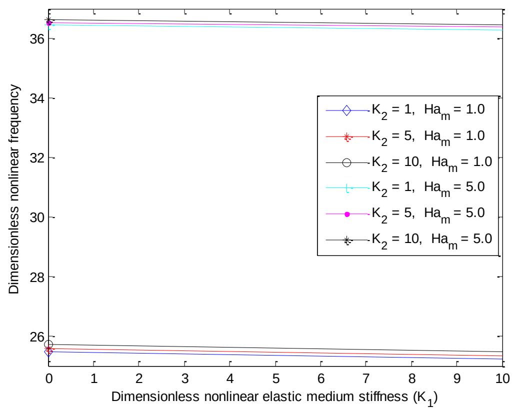


Fig.15
Effect of nonlinear elastic medium stiffness parameters and Hartmann number (magnetic force parameter) on the nonlinear natural frequency of the clamped-clamped supported nanobeam.

5.4 Effects of mode number, temperature, elastic medium stiffness on the nonlinear frequency

While the results in Fig. 2-15 present behaviour of the nanobeam at the first mode of vibration, further investigations as presented in Figs. 16-19 reveal that, at the high values of Winkler stiffness (K_w) and all the values of amplitude as well as Pasternak layer stiffness (K_p), the quantity of increase in the nonlinear frequencies is more significant in simply supported nanobeam than the clamped-clamped nanobeam. This means that simply supported nanobeam is more influenced by the high quantity of the Winkler stiffness than the clamped-clamped. Also, at any value of nonlocal parameter, the change in the first mode is higher than the second mode for the change in temperature. The simply supported nanobeam is more susceptible to the temperature change than the clamped-clamped nanobeam for all the modes. Such behaviour is due to the stiffer nature of the clamped-clamped nanobeam than the simply supported nanobeam.

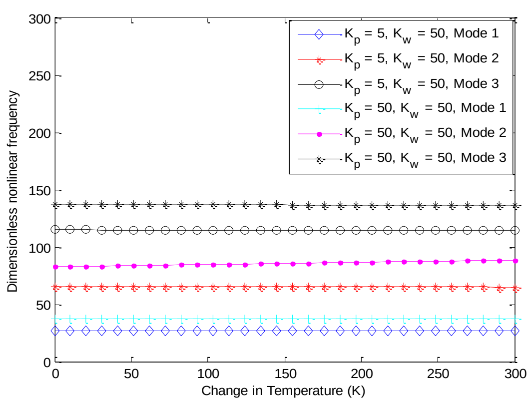


Fig.16
Effect of mode number and elastic medium stiffness on the nonlinear natural frequency of the simply supported nanobeam at high temperature and low Winkler stiffness.

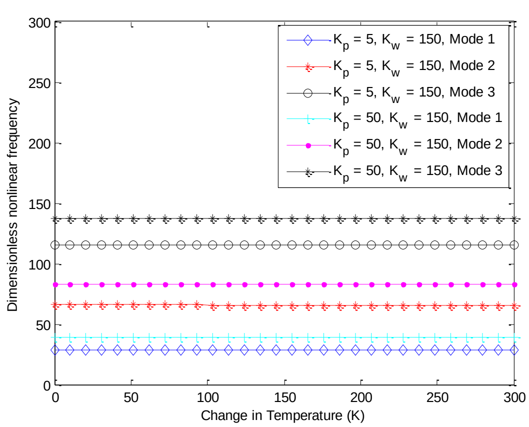


Fig.17
Effect of mode number and elastic medium stiffness on the nonlinear natural frequency of the simply supported nanobeam at high temperature and high Winkler stiffness.

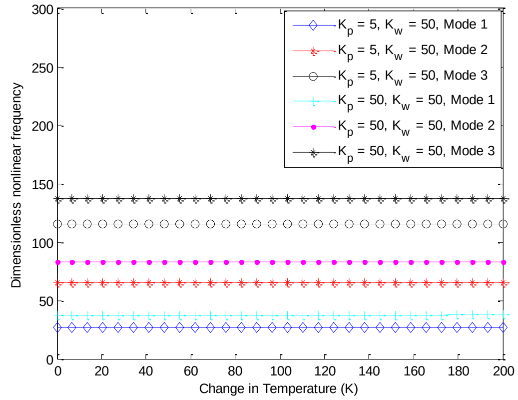


Fig.18
Effect of mode number and elastic medium stiffness on the nonlinear natural frequency of the simply supported nanobeam at low temperature and low Winkler stiffness.

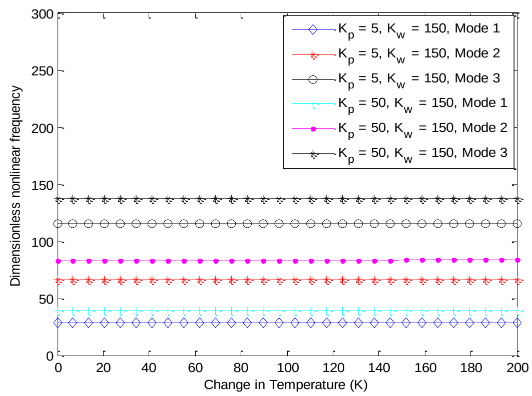


Fig.19
Effect of mode number and elastic medium stiffness on the nonlinear natural frequency of the simply supported nanobeam at low temperature and low Winkler stiffness.

Furthermore, it is observed that when the amplitude increases, the nonlinear frequencies of the all modes increase for the simply supported nanobeam. The nonlinear frequency decreases and increases at high and low temperatures, respectively. The low temperatures increase the third mode frequency for both simply supported and clamped-clamped nanobeams. The first mode nonlinear frequency increases with an increase in the nonlocal parameter. However, at any high values of Pasternak layer stiffness, the second and the third modes nonlinear frequencies decrease with an increase in the nonlocal parameter. It means that the high values of the Pasternak layer stiffness decrease the effect of the nonlocal parameter on the first mode. It could be stated that the first mode is more influenced by low values of the Pasternak layer stiffness while the second and the third modes are significantly influenced by the high values of Pasternak layer stiffness. Furthermore, it was found that the nonlocal parameter increases, the frequencies for all modes decrease. The impact of change in temperature on the nonlinear frequencies rises as the nonlocal parameter rises for all modes for both simply supported and clamped-clamped nanobeam.

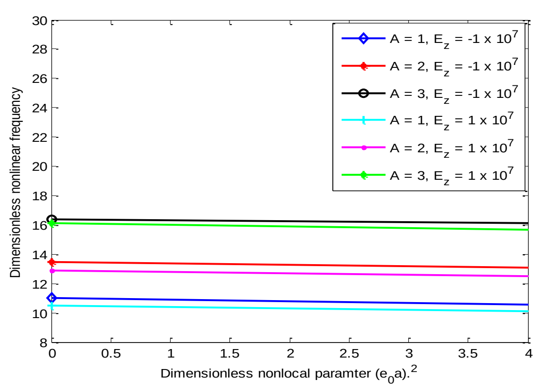


Fig.20
Effect of electric field and vibration amplitude on nonlinear natural frequency of the simply supported nanobeam at low temperature.

The significance of the effect of electric field, E_z on the nonlinear frequency of the nanobeam in Fig. 20. The figure shows that the nonlinear frequency of the beam decreases as the electric field and nonlocal parameter increase. However, the decrease in nonlinear frequency as nonlocal parameter increases is marginal.

5.5 Effects of nonlocal parameter, temperature, elastic medium stiffness on the frequency ratio

The impacts of nonlocal parameter, temperature, elastic medium stiffness on the nonlinear frequency to the linear frequency ratio for both simply and clamped-clamped supported nanobeams are illustrated in Fig. 21-35. In all the results, it is demonstrated that as the dimensionless amplitude increases the frequency ratio increases due to the "hardening spring" behaviour of the nanobeam. Such behaviour in response to the increase in the dimensionless amplitude is caused by the increase in the axial stretching due to the large deflection which leads to a stiffer structure and a larger nonlinear frequency. The results show that the at any given dimensionless amplitude, frequency ratio increases as the values of the dimensionless nonlocal, quadratic and cubic elastic medium stiffness parameters increase as shown in Figs. 21-26. However, at any given dimensionless amplitude, the frequency ratio decreases as the values of the temperature change, magnetic force, one dimensional piezoelectric constant, Winkler and Pasternak layer stiffness parameters increase as shown in Figs. 27-35. It is shown in all the figures that the impact of the dimensionless nonlocal, quadratic, cubic elastic medium stiffness, temperature change, one dimensional piezoelectric constant, magnetic force, Winkler and Pasternak layer stiffness parameters on the nonlinear frequency ratio becomes significant as the dimensionless amplitude increase.

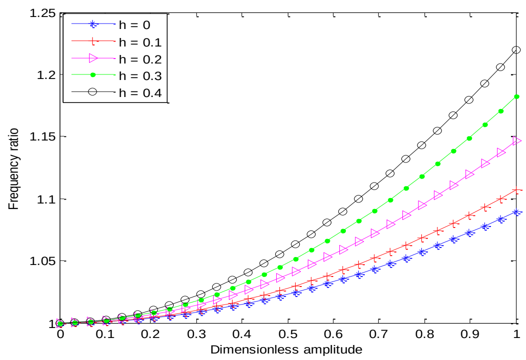


Fig.21
Effects of dimensionless nonlocal parameter on the frequency ratio for simply supported nanobeam.

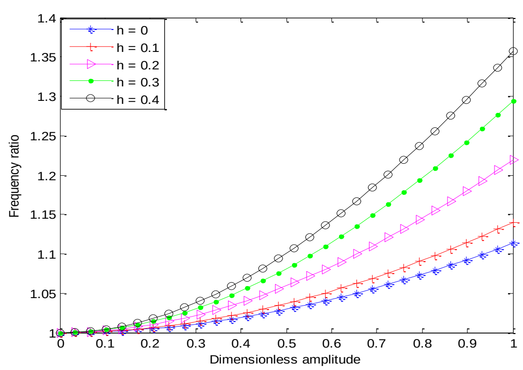


Fig.22
Effects of dimensionless nonlocal parameter on the frequency ratio for clamped-clamped nanobeam.

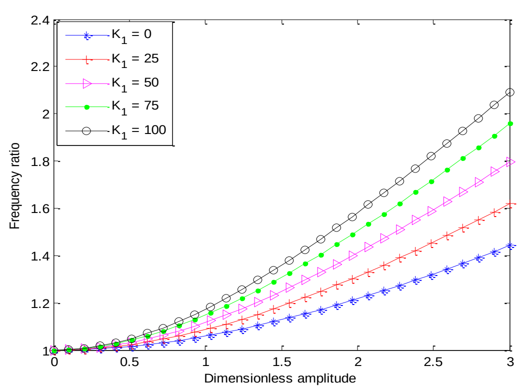


Fig.23
Effects of dimensionless quadratic elastic medium stiffness on the frequency ratio for simply supported nanobeam.

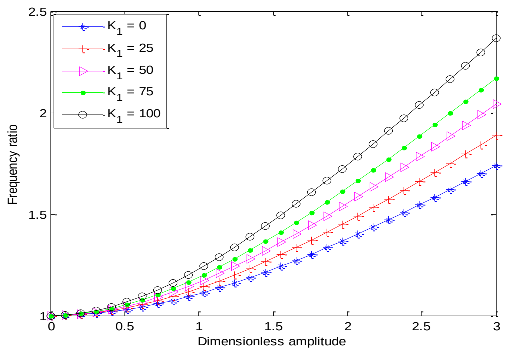


Fig.24
Effects of dimensionless quadratic elastic medium stiffness on the frequency ratio for clamped-clamped supported nanobeam.

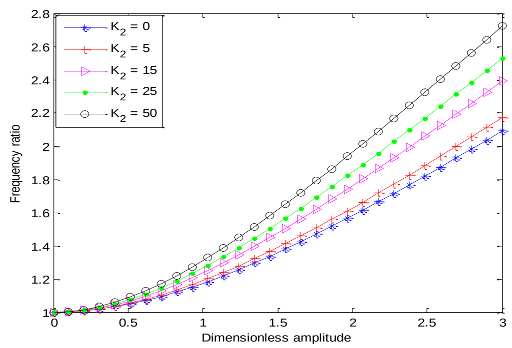


Fig.25
Effects of dimensionless cubic nonlinear elastic medium stiffness on the frequency ratio for simply supported nanobeam.

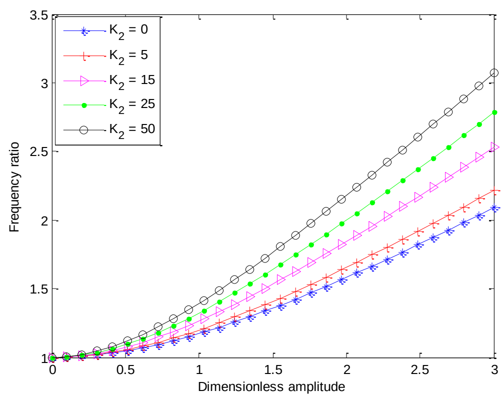


Fig.26
Effects of dimensionless cubic elastic medium stiffness on the frequency ratio for clamped-clamped supported nanobeam.

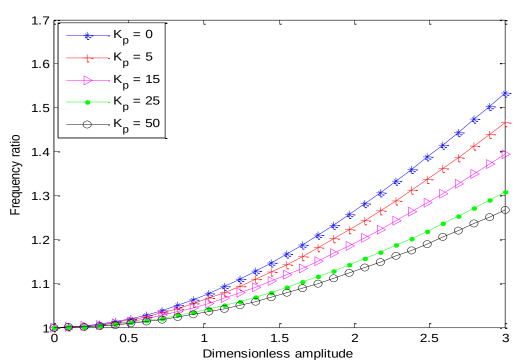


Fig.27
Effects of dimensionless Pasternak elastic medium stiffness on the frequency ratio for simply supported nanobeam

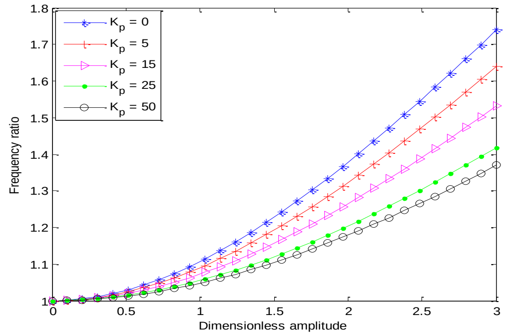


Fig.28
Effects of dimensionless Pasternak elastic medium stiffness on the frequency ratio for clamped-clamped supported nanobeam.

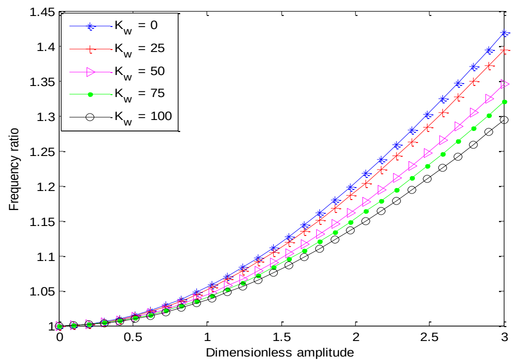


Fig.29
Effects of dimensionless Winkler elastic medium stiffness on the frequency ratio for simply supported nanobeam.

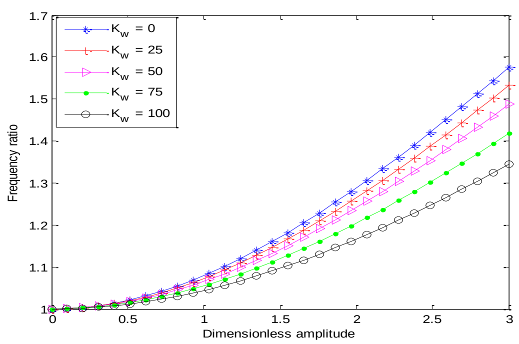


Fig.30
Effects of dimensionless Winkler elastic medium stiffness on the frequency ratio for clamped-clamped supported nanobeam.

It is clearly seen that increase in temperature change at high temperature reduces the frequency ratio as shown in Figs. 31 and 32. Such response is due to the fact that the Young modulus and the flexural rigidity of the nanobeam are functions of temperature. These parameters (Young modulus and the flexural rigidity) increase at high temperature and such causes the nanobeam to become increasingly rigid as the temperature change increases, which consequently decreases the frequency ratio of the vibration of the structure. However, at low or room temperature, increase in temperature change, increases the frequency ratio of the structure nanotube.

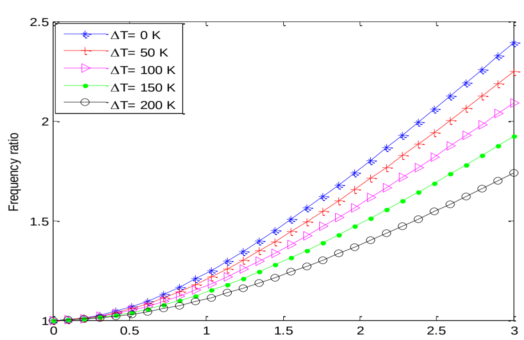


Fig.31
Effects of temperature change on the frequency ratio for simply supported nanobeam.

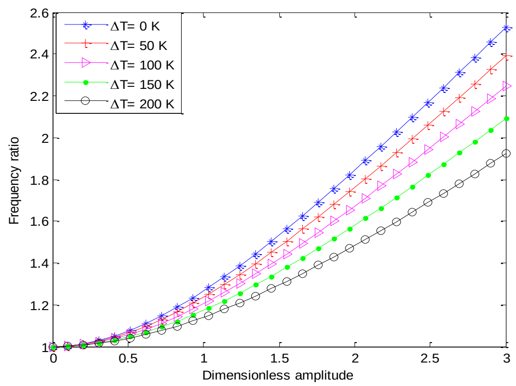


Fig.32
Effects of temperature change on the frequency ratio for clamped-clamped nanobeam.

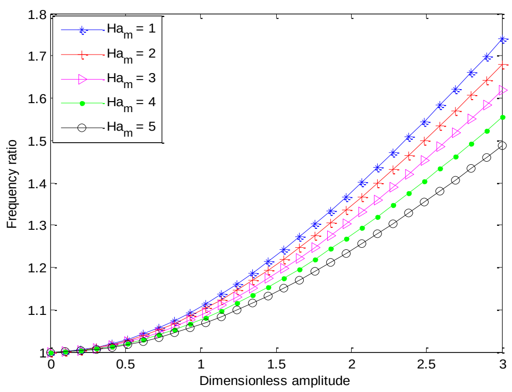


Fig.33
Effects of magnetic force parameter on the frequency ratio for simply supported nanobeam.

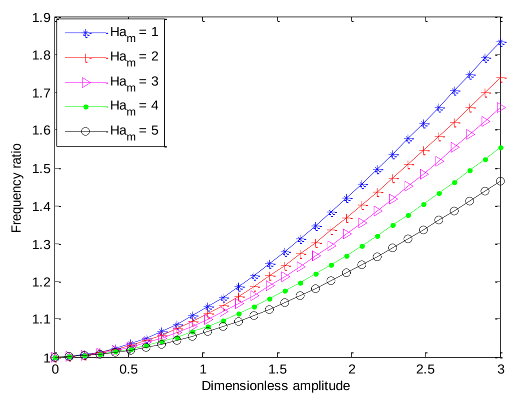


Fig.34
Effects of magnetic force parameter on the frequency ratio for clamped-clamped nanobeam.

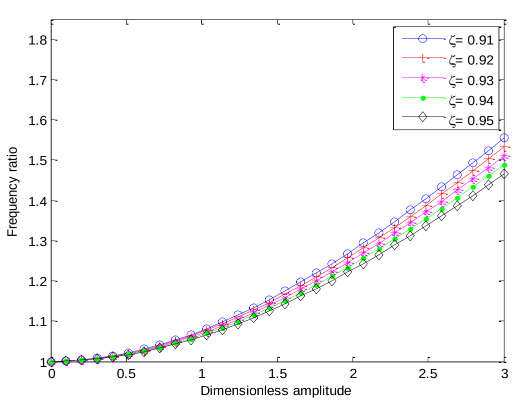


Fig.35
Effects of one dimensional piezoelectric constant on the frequency ratio for clamped-clamped nanobeam.

6 CONCLUSIONS

In this work, the impacts of thermo-magneto-mechanical coupled parameters on the nonlinear vibration of single-walled carbon nanotube embedded in Winkler, Pasternak, and nonlinear elastic media have been analyzed the aids of Galerkin decomposition and iteration perturbation methods. Parametric studies were carried out and the following results were established:

i. The frequency of the nanotube increases at low temperature but decreases at the high temperatures.
ii. The nonlocal parameter and electric field decreases the frequencies of the nanobeam that is simply supported at any amount of the linear and nonlinear elastic medium coefficients.
iii. The effect of the linear Winkler layer stiffness on the nanobeam that is simply supported is more than the nanobeam with clamped-clamped supports.
iv. An increase in the quadratic nonlinear elastic medium stiffness causes a decrease in the first mode of the nanobeam with clamped-clamped supports and an increase in all modes of the simply supported nanobeam at both low and high temperature.
v. When the magnetic force, cubic nonlinear elastic medium stiffness, and amplitude increase, there is an increase in all mode frequency of the nanobeam.
vi. A decrease in Winkler and Pasternak elastic media constants and increase in the nonlinear parameters of elastic medium results in an increase in the frequency ratio.
vii. The frequency ratio increases as the values of the dimensionless nonlocal, quadratic and cubic elastic medium stiffness parameters increase. The dimensionless amplitude increases the frequency ratio increases.
viii. The frequency ratio decreases as the values of the temperature change, magnetic force, one dimensional piezoelectric constant Winkler and Pasternak layer stiffness parameters increase.
ix. An increase in the temperature change at high temperature reduces the frequency ratio but at low or room temperature, increase in temperature change, increases the frequency ratio of the structure nanotube.
x. The impact of the dimensionless nonlocal, quadratic, cubic elastic medium stiffness, temperature change, magnetic force, Winkler and Pasternak layer stiffness parameters on the nonlinear frequency ratio becomes significant as the dimensionless amplitude increase.

Such an extensive analysis as carried in this study work will greatly benefit in the design and applications of nanotube in thermal and magnetic environments.

APPENDIX

Following the nonlocal theory presented by Erigen [42, 43, 44] and that of Erigen and Edelen [45], the relationship between the nonlocal stress–tensor (σ_{ij}) at point x of an isotropic and homogenous nanobeam and the local stress–tensor (t_{ij}) is

$$\left[1-\left(e_0 a\right)^2 \nabla^2\right]\sigma_{ij} = \left[1-\left(\tau l\right)^2 \nabla^2\right]\sigma_{ij} = E\,\varepsilon\left(x\right) = t_{ij} \tag{A.1}$$

Algebraically, Eq. (1) can be written as:

$$\sigma_{\bar{x}\bar{x}} - \left(e_0 a\right)^2 \frac{\partial^2 \sigma_{\bar{x}\bar{x}}}{\partial \bar{x}^2} = E\,\varepsilon_{\bar{x}\bar{x}} = t_{\bar{x}\bar{x}} \tag{A.2}$$

Neglecting the damping of the nanobeam and the damping induced by the surrounding medium. Also, assuming that vibration is independent of time axial forces. Based on Euler-Bernoulli theory, the displacements in the nanobeam are given as:

$$\bar{u}_1 = \bar{u}\left(\bar{x},\bar{t}\right) - \bar{z}\frac{\partial \bar{w}}{\partial \bar{x}}; \qquad \bar{u}_2 = \bar{w}\left(\bar{x},\bar{t}\right); \qquad \bar{u}_3 = 0. \tag{A.3}$$

$\bar{u}_3 = 0,$ since, there is not any motion along the third direction. Also, the strain in the longitudinal direction is given as:

$$\varepsilon_{\overline{xx}} = \frac{\partial \overline{u}_1}{\partial \overline{x}} \tag{A.4}$$

The strain in the longitudinal direction is related to the extension and bending strains as:

$$\varepsilon_{\overline{xx}} = \varepsilon^0_{\overline{xx}} + \overline{z}k\,; \tag{A.5}$$

where extension and bending strains are respectively given as:

$$\varepsilon^0_{\overline{xx}} = \frac{\partial \overline{u}}{\partial \overline{x}}; \quad k = -\frac{\partial^2 \overline{w}}{\partial \overline{x}^2}. \tag{A.6}$$

On substituting Eq. (A.6) into Eq. (A.5), we have

$$\varepsilon_{\overline{xx}} = \frac{\partial \overline{u}}{\partial \overline{x}} - \overline{z}\,\frac{\partial^2 \overline{w}}{\partial \overline{x}^2} \tag{A.7}$$

Considering the Von Karman geometric nonlinearity effect, the extension strain is given as:

$$\varepsilon^0_{\overline{xx}} = \frac{\partial \overline{u}}{\partial \overline{x}} + \frac{1}{2}\left(\frac{\partial \overline{w}}{\partial \overline{x}}\right)^2 \tag{A.8}$$

Substitution of the nonlinear extension strain in Eq. (A.8) and the bending strain in Eq. (A.5), provides geometric nonlinearity in the longitudinal strain as:

$$\varepsilon_{\overline{xx}} = \frac{\partial \overline{u}}{\partial \overline{x}} + \frac{1}{2}\left(\frac{\partial \overline{w}}{\partial \overline{x}}\right)^2 - \overline{z}\,\frac{\partial^2 \overline{w}}{\partial \overline{x}^2} \tag{A.9}$$

Introducing the following stress resultants:

$$N = \int_{A_c} \sigma_{\overline{xx}} dA_c\,; \qquad M = \int_{A_c} \sigma_{\overline{xx}} \overline{z} dA_c\,;\ V = \int_{A_c} \sigma_{\overline{xz}} dA_c\,; \qquad I = \int_{A_c} \overline{z}^{\,2} dA_c\,; \tag{A.10}$$

The required equation of motion for the nanobeam can be derived after taking the variation of the relation

$$\Pi = W + K - U \tag{A.11}$$

where

$$W = \int_0^L \left(f\overline{u} + p\overline{w}\right) dx \tag{A.12}$$

$$K = \int_0^L \left[\frac{m_0}{2}\left(\left(\frac{\partial \overline{u}}{\partial \overline{t}}\right)^2 + \left(\frac{\partial \overline{w}}{\partial \overline{t}}\right)^2\right) + \frac{m_2}{2}\left(\frac{\partial^2 \overline{w}}{\partial \overline{x}\,\partial \overline{t}}\right)^2\right] d\overline{x} \tag{A.13}$$

$$U = \int_V \frac{1}{2}\left(\sigma_{\overline{xx}} \varepsilon_{\overline{xx}}\right) dV = \int_0^L \int_A \frac{E}{2}\left[\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{1}{2}\left(\frac{\partial \overline{w}}{\partial \overline{x}}\right)^2 - \overline{z}\,\frac{\partial^2 \overline{w}}{\partial \overline{x}^2}\right]^2 dA\, d\overline{x} \tag{A.14}$$

Applying Hamilton's principle, the variation of Eq. (A.11) is obtained as:

$$\delta(\Pi)=\delta\int_0^T\left(\bar{W}+\bar{K}-\bar{U}\right)d\bar{t}=0 \tag{A.15}$$

Expansion of the RHS of Eq. (15), gives

$$\delta(\Pi)=\delta\int_0^T\left(\bar{W}\right)d\bar{t}+\delta\int_0^T\left(\bar{K}\right)d\bar{t}-\delta\int_0^T\left(\bar{U}\right)d\bar{t}=0 \tag{A.16}$$

i. Workdone by external forces

For the first term at the RHS of Eq. (A.16), i.e. the variation of the work done by the external forces, substitution of Eq. (A.12) into the first term at the RHS of Eq. (A.16), provides

$$\delta\int_0^T\left(\bar{W}\right)d\bar{t}=\int_0^T\int_0^L\left(\bar{f}\,\delta\bar{u}+\bar{p}\,\delta\bar{w}\right)d\bar{x}d\bar{t} \tag{A.17}$$

ii. Kinetic Energy

Also, for the second term at the RHS of Eq. (16), i.e. the variation of the kinetic energy, substitution of Eq. (13) into the second term at the RHS of Eq. (16), gives

$$\delta\int_0^T\left(\bar{K}\right)dt=\int_0^T\int_0^L\left[m_0\left(\frac{\partial\bar{u}}{\partial\bar{t}}\frac{\partial\delta\bar{u}}{\partial\bar{t}}+\frac{\partial\bar{w}}{\partial\bar{t}}\frac{\partial\delta\bar{w}}{\partial\bar{t}}\right)+m_2\left(\frac{\partial^2\bar{w}}{\partial\bar{x}\,\partial\bar{t}}\right)\left(\frac{\partial^2\delta\bar{w}}{\partial\bar{x}\,\partial\bar{t}}\right)\right]d\bar{x}d\bar{t} \tag{A.18}$$

iii. Strain Energy

Furthermore, for the third term at the RHS of Eq. (16), i.e. the variation of the strain energy, substitution of Eq. (A.14) into the third term at the RHS of Eq. (A.16), results in

$$\delta\int_0^T\left(\bar{U}\right)d\bar{t}=\int_0^T\int_V\delta\left(\frac{1}{2}\left(\sigma_{\bar{x}\bar{x}}\varepsilon_{\bar{x}\bar{x}}\right)\right)dVd\bar{t}=\int_0^T\int_V\delta\left(\frac{1}{2}\left(E\,\varepsilon_{\bar{x}\bar{x}}^2\right)\right)dVd\bar{t}=\int_0^T\int_V\delta\left(\frac{1}{2}\left(\sigma_{\bar{x}\bar{x}}\delta\varepsilon_{\bar{x}\bar{x}}\right)\right)dVd\bar{t} \tag{A.19}$$

which gives

$$\delta\int_0^T\left(\bar{U}\right)d\bar{t}=\int_0^T\int_V\sigma_{\bar{x}\bar{x}}\left(\frac{\partial\delta\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{w}}{\partial\bar{x}}\frac{\partial\delta\bar{w}}{\partial\bar{x}}-z\,\frac{\partial^2\delta\bar{w}}{\partial\bar{x}^2}\right)dVd\bar{t} \tag{A.20}$$

On substituting Eq. (A.10), one can write Eq. (A.20) as:

$$\delta\int_0^T\left(\bar{U}\right)d\bar{t}=\int_0^T\int_0^L\left[\bar{N}\left(\frac{\partial\delta\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{w}}{\partial\bar{x}}\frac{\partial\delta\bar{w}}{\partial\bar{x}}\right)-\bar{M}\,\frac{\partial^2\delta\bar{w}}{\partial\bar{x}^2}\right]d\bar{x}d\bar{t} \tag{A.21}$$

On substituting Eqs. (A.17), (A.18), and (A.21) into Eq. (A.16), we have

$$\begin{aligned}\delta(\Pi)=&\int_0^T\int_0^L\left(\bar{f}\,\delta\bar{u}+\bar{p}\,\delta\bar{w}\right)d\bar{x}d\bar{t}+\int_0^T\int_0^L\left[m_0\left(\frac{\partial\bar{u}}{\partial\bar{t}}\frac{\partial\delta\bar{u}}{\partial\bar{t}}+\frac{\partial\bar{w}}{\partial\bar{t}}\frac{\partial\delta\bar{w}}{\partial\bar{t}}\right)+m_2\left(\frac{\partial^2\bar{w}}{\partial\bar{x}\,\partial\bar{t}}\right)\left(\frac{\partial^2\delta\bar{w}}{\partial\bar{x}\,\partial\bar{t}}\right)\right]d\bar{x}d\bar{t}\\&-\int_0^T\int_0^L\left[\bar{N}\left(\frac{\partial\delta\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{w}}{\partial\bar{x}}\frac{\partial\delta\bar{w}}{\partial\bar{x}}\right)-\bar{M}\,\frac{\partial^2\delta\bar{w}}{\partial\bar{x}^2}\right]d\bar{x}d\bar{t}=0\end{aligned} \tag{A.22}$$

According to Euler–Lagrange, the following equations are obtained

$$-m_0 \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + \frac{\partial N}{\partial \bar{x}} + f\left(\bar{x}, \bar{t}\right) = 0 \tag{A.23}$$

$$-m_0 \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + m_2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}^2} + \frac{\partial^2 M}{\partial \bar{x}^2} + p\left(\bar{x}, \bar{t}\right) + N \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = 0 \tag{A.24}$$

The nonlocal axial force (nonlinear stretching force) and bending moment are given by

$$N - \left(e_0 a\right)^2 \frac{\partial^2 N}{\partial \bar{x}^2} = EA_c \left[\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2 - \frac{\bar{N}}{EA_c}\right] - \zeta E_z A_c \tag{A.25}$$

$$M - \left(e_0 a\right)^2 \frac{\partial^2 M}{\partial \bar{x}^2} = EI\left(-\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right) \tag{A.26}$$

After differentiating Eq. (A.23) wrt x, on arrives at

$$\frac{\partial^2 N}{\partial \bar{x}^2} = m_0 \frac{\partial^3 \bar{u}}{\partial \bar{t}^3} - \frac{\partial f}{\partial \bar{x}} \tag{A.27}$$

Also, from Eq. (24), we have

$$\frac{\partial^2 M}{\partial \bar{x}^2} = -p\left(\bar{x}, \bar{t}\right) - N \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + m_0 \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} - m_2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}^2} \tag{A.28}$$

Substitution of Eqs. (A.27) and (A.28) into Eqs. (A.25) and (A.26), gives the nonlocal axial force and bending moment as:

$$N = EA_c \left[\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2 - \frac{\bar{N}}{EA_c}\right] - \zeta E_z A_c + \left(e_0 a\right)^2 \left(m_0 \frac{\partial^3 \bar{u}}{\partial \bar{x} \partial \bar{t}^2} - \frac{\partial f}{\partial \bar{x}}\right) \tag{A.29}$$

$$M = -EI\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right) + \left(e_0 a\right)^2 \left(-p - N \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + m_0 \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} - m_2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}^2}\right) \tag{A.30}$$

The first derivative of Eq. (A.29) is given as:

$$\frac{\partial N}{\partial \bar{x}} = EA_c \left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{1}{2}\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2 - \frac{\partial}{\partial \bar{x}}\left(\frac{\bar{N}}{EA_c}\right)\right] - \frac{\partial}{\partial \bar{x}}\left(\zeta E_z A_c\right) + \left(e_0 a\right)^2 \left(m_0 \frac{\partial^4 \bar{u}}{\partial \bar{x}^2 \partial \bar{t}^2} - \frac{\partial^2 f}{\partial \bar{x}^2}\right) \tag{A.31}$$

while the second derivative of Eq. (A.30) provides

$$\frac{\partial^2 M}{\partial \bar{x}^2} = -EI\left(\frac{\partial^4 \bar{w}}{\partial \bar{x}^4}\right) + \left(e_0 a\right)^2 \left(-\frac{\partial^2 p}{\partial \bar{x}^2} - N \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + m_0 \frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}^2} - m_2 \frac{\partial^6 \bar{w}}{\partial \bar{x}^4 \partial \bar{t}^2}\right) \tag{A.32}$$

Substituting Eq. (A.31) into Eq. (A.23), one arrives at the horizontal equation of motion as:

$$m_0 \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} - EA_c \left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{1}{2}\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2 - \frac{\partial}{\partial \bar{x}}\left(\frac{\bar{N}}{EA_c}\right)\right] + \frac{\partial}{\partial \bar{x}}\left(\zeta E_z A_c\right) - \left(e_0 a\right)^2 \left(m_0 \frac{\partial^4 \bar{u}}{\partial \bar{x}^2 \partial \bar{t}^2} - \frac{\partial^2 f}{\partial \bar{x}^2}\right) = f\left(\bar{x}, \bar{t}\right) \tag{A.33}$$

which can also be written as:

$$m_0\frac{\partial^2\bar{u}}{\partial\bar{t}^2}-EA_c\left[\frac{\partial^2\bar{u}}{\partial\bar{x}^2}+\frac{1}{2}\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\frac{\partial}{\partial\bar{x}}\left(\frac{\bar{N}}{EA_c}\right)\right]+\frac{\partial}{\partial\bar{x}}\left(\zeta E_z A_c\right)-\left(e_0a\right)^2\left(m_0\frac{\partial^4\bar{u}}{\partial\bar{x}^2\partial\bar{t}^2}-\frac{\partial^2 f}{\partial\bar{x}^2}\right)=f\left(\bar{x},\bar{t}\right) \tag{A.34}$$

where

$$\bar{N}=N_{thermal}-\eta A_c H_{\bar{x}}^2,\quad N_{thermal}=EA_c\frac{\alpha_{\bar{x}}\Delta T}{1-2\nu}\quad\Rightarrow\bar{N}=EA_c\frac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_c H_{\bar{x}}^2 \tag{A.35}$$

Then Eq. (A.29) becomes

$$N=EA_c\left[\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\frac{\left(EA_c\frac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_c H_{\bar{x}}^2\right)}{EA_c}\right]-\zeta E_z A_c+\left(e_0a\right)^2\left(m_0\frac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\frac{\partial f}{\partial\bar{x}}\right) \tag{A.36}$$

Substitution of Eq. (A.35) into Eq. (A.34), gives the horizontal equation of motion of the nanobeam as:

$$m_0\frac{\partial^2\bar{u}}{\partial\bar{t}^2}-EA_c\left[\frac{\partial^2\bar{u}}{\partial\bar{x}^2}+\frac{1}{2}\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\frac{\partial}{\partial\bar{x}}\left(\frac{EA_c\frac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_c H_{\bar{x}}^2}{EA_c}\right)\right]+\frac{\partial}{\partial\bar{x}}\left(\zeta E_z A_c\right)-\left(e_0a\right)^2\left(m_0\frac{\partial^4\bar{u}}{\partial\bar{x}^2\partial\bar{t}^2}-\frac{\partial^2 f}{\partial\bar{x}^2}\right)=f\left(\bar{x},\bar{t}\right) \tag{A.37}$$

Also, substitution of Eqs. (A.32) and (A.36) into Eq. (A.24), provides the vertical equation of motion as:

$$\begin{aligned}&EI\left(\frac{\partial^4\bar{w}}{\partial\bar{x}^4}\right)-\left(e_0a\right)^2\left(-\frac{\partial^2 p}{\partial\bar{x}^2}-E\left(\begin{array}{l}A_c\left[\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\frac{\left(EA_c\frac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_c H_{\bar{x}}^2\right)}{EA_c}\right]\\-\zeta E_z A_c+\left(e_0a\right)^2\left(m_0\frac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\frac{\partial f}{\partial\bar{x}}\right)\end{array}\right)\frac{\partial^4\bar{w}}{\partial\bar{x}^4}+m_0\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}-m_2\frac{\partial^6\bar{w}}{\partial\bar{x}^4\partial\bar{t}^2}\right)\\&-p\left(\bar{x},\bar{t}\right)-E\left(\begin{array}{l}A_c\left[\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\frac{\left(EA_c\frac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_c H_{\bar{x}}^2\right)}{EA_c}\right]\\-\zeta E_z A_c+\left(e_0a\right)^2\left(m_0\frac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\frac{\partial f}{\partial\bar{x}}\right)\end{array}\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+m_0\frac{\partial^2\bar{w}}{\partial\bar{t}^2}-m_2\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}=0\end{aligned} \tag{A.38}$$

which can be written as:

$$\begin{aligned}&EI\left(\frac{\partial^4\bar{w}}{\partial\bar{x}^4}\right)+\left(e_0a\right)^2\left(\frac{\partial^2 p}{\partial\bar{x}^2}+E\left(\begin{array}{l}A_c\left[\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\frac{\left(EA_c\frac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_c H_{\bar{x}}^2\right)}{EA_c}\right]\\-\zeta E_z A_c+\left(e_0a\right)^2\left(m_0\frac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\frac{\partial f}{\partial\bar{x}}\right)\end{array}\right)\frac{\partial^4\bar{w}}{\partial\bar{x}^4}-m_0\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}+m_2\frac{\partial^6\bar{w}}{\partial\bar{x}^4\partial\bar{t}^2}\right)\\&-p\left(\bar{x},\bar{t}\right)-E\left(\begin{array}{l}A_c\left[\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\frac{\left(EA_c\frac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_c H_{\bar{x}}^2\right)}{EA_c}\right]\\-\zeta E_z A_c+\left(e_0a\right)^2\left(m_0\frac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\frac{\partial f}{\partial\bar{x}}\right)\end{array}\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+m_0\frac{\partial^2\bar{w}}{\partial\bar{t}^2}-m_2\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}=0\end{aligned} \tag{A.39}$$

where

$$p=\left(-k_w\bar{w}+k_p\frac{\partial^2\bar{w}}{\partial\bar{x}^2}-k_2\bar{w}^2-k_3\bar{w}^3\right) \tag{A.40}$$

Substitution of Eq. (36) into Eq. (35), gives

$$\begin{aligned}&EI\left(\frac{\partial^4\bar{w}}{\partial\bar{x}^4}\right)+(e_0a)^2\left(\begin{array}{l}\dfrac{\partial^2\left(-k_w\bar{w}+k_p\dfrac{\partial^2\bar{w}}{\partial\bar{x}^2}-k_2\bar{w}^2-k_3\bar{w}^3\right)}{\partial\bar{x}^2}+E\left(\begin{array}{l}A_c\left[\dfrac{\partial\bar{u}}{\partial\bar{x}}+\dfrac{1}{2}\left(\dfrac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\dfrac{\left(EA_c\dfrac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_cH_{\bar{x}}^2\right)}{EA_c}\right]\\-\zeta E_zA_c+(e_0a)^2\left(m_0\dfrac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\dfrac{\partial f}{\partial\bar{x}}\right)\end{array}\right)\dfrac{\partial^4\bar{w}}{\partial\bar{x}^4}\\-m_0\dfrac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}+m_2\dfrac{\partial^6\bar{w}}{\partial\bar{x}^4\partial\bar{t}^2}\end{array}\right)\\&-\left(-k_w\bar{w}+k_p\frac{\partial^2\bar{w}}{\partial\bar{x}^2}-k_2\bar{w}^2-k_3\bar{w}^3\right)-E\left(A_c\left[\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\frac{\left(EA_c\dfrac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_cH_{\bar{x}}^2\right)}{EA_c}\right]-\zeta E_zA_c+(e_0a)^2\left(m_0\frac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\frac{\partial f}{\partial\bar{x}}\right)\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\\&+m_0\frac{\partial^2\bar{w}}{\partial\bar{t}^2}-m_2\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}=0\end{aligned} \tag{A.41}$$

$$\begin{aligned}&EI\left(\frac{\partial^4\bar{w}}{\partial\bar{x}^4}\right)-(e_0a)^2\left(\begin{array}{l}\dfrac{\partial^2\left(k_w\bar{w}-k_p\dfrac{\partial^2\bar{w}}{\partial\bar{x}^2}+k_2\bar{w}^2+k_3\bar{w}^3\right)}{\partial\bar{x}^2}-E\left(\begin{array}{l}A_c\left[\dfrac{\partial\bar{u}}{\partial\bar{x}}+\dfrac{1}{2}\left(\dfrac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\dfrac{\left(EA_c\dfrac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_cH_{\bar{x}}^2\right)}{EA_c}\right]\\-\zeta E_zA_c+(e_0a)^2\left(m_0\dfrac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\dfrac{\partial f}{\partial\bar{x}}\right)\end{array}\right)\dfrac{\partial^4\bar{w}}{\partial\bar{x}^4}\\+m_0\dfrac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}-m_2\dfrac{\partial^6\bar{w}}{\partial\bar{x}^4\partial\bar{t}^2}\end{array}\right)\\&+\left(k_w\bar{w}-k_p\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+k_2\bar{w}^2+k_3\bar{w}^3\right)-E\left(A_c\left[\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\frac{\left(EA_c\dfrac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_cH_{\bar{x}}^2\right)}{EA_c}\right]-\zeta E_zA_c+(e_0a)^2\left(m_0\frac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\frac{\partial f}{\partial\bar{x}}\right)\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\\&+m_0\frac{\partial^2\bar{w}}{\partial\bar{t}^2}-m_2\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}=0\end{aligned} \tag{A.42}$$

Therefore, the vertical equation of motion of the nanobeam is

$$\begin{aligned}&EI\left(\frac{\partial^4\bar{w}}{\partial\bar{x}^4}\right)-(e_0a)^2\left(\begin{array}{l}k_w\dfrac{\partial^2\bar{w}}{\partial\bar{x}^2}-k_p\dfrac{\partial^4\bar{w}}{\partial\bar{x}^4}+k_2\dfrac{\partial^2\left(\bar{w}^2\right)}{\partial\bar{x}^2}+k_3\dfrac{\partial^2\left(\bar{w}^3\right)}{\partial\bar{x}^2}\\-E\left(A_c\left[\dfrac{\partial\bar{u}}{\partial\bar{x}}+\dfrac{1}{2}\left(\dfrac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\dfrac{\left(EA_c\dfrac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_cH_{\bar{x}}^2\right)}{EA_c}\right]-\zeta E_zA_c+(e_0a)^2\left(m_0\dfrac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\dfrac{\partial f}{\partial\bar{x}}\right)\right)\dfrac{\partial^4\bar{w}}{\partial\bar{x}^4}\\+m_0\dfrac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}-m_2\dfrac{\partial^6\bar{w}}{\partial\bar{x}^4\partial\bar{t}^2}\end{array}\right)\\&+\left(k_w\bar{w}-k_p\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+k_2\bar{w}^2+k_3\bar{w}^3\right)-E\left(A_c\left[\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2-\frac{\left(EA_c\dfrac{\alpha_{\bar{x}}\Delta T}{1-2\nu}-\eta A_cH_{\bar{x}}^2\right)}{EA_c}\right]-\zeta E_zA_c+(e_0a)^2\left(m_0\frac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\frac{\partial f}{\partial\bar{x}}\right)\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\\&+m_0\frac{\partial^2\bar{w}}{\partial\bar{t}^2}-m_2\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2}=0\end{aligned} \tag{A.43}$$

Taking $m_0 = \rho A_c$, neglecting the rotary inertial (i.e. $m_2 = 0$) with no axial distributed force (i.e. $f(\bar{x},\bar{t}) = 0$) and zero axial displacements (i.e. $\bar{u} = 0$). After some mathematical processes of Integrating the nonlinear stretching force, N between the limits 0 and L and applying the boundary conditions $\bar{u}(0,t)$ and $\bar{u}(L,t)$ makes the axial normal force in Eq. (A.36) to become

$$N = \left(\frac{EA_c}{2L}\int_0^L \left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2 d\bar{x}\right) - \left(EA_c \frac{\alpha_{\bar{x}}\Delta T}{1-2\nu} - \eta A_c H_{\bar{x}}^2\right) - \zeta E_z A_c \tag{A.44}$$

That is

$$\begin{aligned} &EA_c\left[\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{1}{2}\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2 - \frac{\left(EA_c \frac{\alpha_{\bar{x}}\Delta T}{1-2\nu} - \eta A_c H_{\bar{x}}^2\right)}{EA_c}\right] - \zeta E_z A_c + (e_0 a)^2\left(m_0 \frac{\partial^3 \bar{u}}{\partial \bar{x}\partial \bar{t}^2} - \frac{\partial f}{\partial \bar{x}}\right) \\ &= \left(\frac{EA_c}{2L}\int_0^L \left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2 d\bar{x}\right) - \left(EA_c \frac{\alpha_{\bar{x}}\Delta T}{1-2\nu} - \eta A_c H_{\bar{x}}^2\right) - \zeta E_z A_c \end{aligned} \tag{A.45}$$

Therefore, Eq. (A.37) and (A.43) reduce to

$$\begin{aligned} &EI\left(\frac{\partial^4 \bar{w}}{\partial \bar{x}^4}\right) + \rho A_c \frac{\partial^2}{\partial \bar{t}^2}\left[\bar{w} - (e_0 a)^2 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right] + k_w\left[\bar{w} - (e_0 a)^2 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right] - k_p \frac{\partial^2}{\partial \bar{x}^2}\left[\bar{w} - (e_0 a)^2 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right] + k_2\left[\bar{w}^2 - (e_0 a)^2 \frac{\partial^2 (\bar{w}^2)}{\partial \bar{x}^2}\right] \\ &+ k_3\left[\bar{w}^3 - (e_0 a)^2 \frac{\partial^2 (\bar{w}^3)}{\partial \bar{x}^2}\right] - \eta A_c H_{\bar{x}}^2 \frac{\partial^2}{\partial \bar{x}^2}\left[\bar{w} - (e_0 a)^2 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right] + \left(EA_c \frac{\alpha_{\bar{x}}\Delta T}{1-2\nu}\right)\frac{\partial^2}{\partial \bar{x}^2}\left[\bar{w} - (e_0 a)^2 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right] \\ &+ \zeta E_z A_c \frac{\partial^2}{\partial \bar{x}^2}\left[\bar{w} - (e_0 a)^2 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\right] - \left[\left(\frac{EA_c}{2L}\int_0^L \left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2 d\bar{x}\right)\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - (e_0 a)^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^4}\right)\right] = 0 \end{aligned} \tag{A.46}$$

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