Journal of Solid Mechanics Vol. 11, No. 1 (2019) pp. 14-25
DOI: 10.22034/JSM.2019.664212

Size-Dependent Green's Function for Bending of Circular Micro Plates Under Eccentric Load

M. Shahrokhi, E. Jomehzadeh, M. Rezaeizadeh *

Faculty of Mechanical and Material Engineering, Graduate University of Advanced Technology, Kerman, Iran

Received 2 September 2018; accepted 19 November 2018

ABSTRACT

In this paper, a Green's function is developed for bending analysis of micro plates under an asymmetric load. In order to consider the length scale effect, the modified couple stress theory is used. This theory can accurately predict the behavior of micro structures. A thin micro plate is considered and therefore the classical plate theory is utilized. The size dependent governing equilibrium equation of a circular micro plate under an eccentric load is obtained by using the minimum total potential energy principle. This equation is a partial differential equation and it is hard to solve it for an arbitrary loading. A transformation of the coordinate system is introduced to obtain the asymmetric exact solution for deflection of circular micro-plates. By using the obtained size dependent Green's function, the bending behavior of micro plates under arbitrary loads can be easily defined. The results are presented for different asymmetric loads. Also, it is concluded that the length scale has a significant effect on bending of micro plates.

Keywords : Green's function; Micro plate; Length scale effect.

1 INTRODUCTION

THE micro and nano structures have a wide applications in recent years. The micro and nano-structures have a wide use in recent years. The thin films in these scales can be used in electrical, lenses of glasses and corrosion resistance parts. They can be used in different shapes with different loading and the obtaining their deflection is important in order to an optimum design [1]. In the modern industries, micro structures have played an essential role technology. The behavior of material in the micro-scale depends on the length parameter [2]. Since in the micro/nano scale size the classical theory of plate cannot predict the behavior of the plate, some nonclassical continuum mechanics theories like the first strain gradient theory, the second strain gradient theory and modified couple stress theory can be used to consider the effect of size. The second strain gradient theory is developed by Mindlin [3]. This theory states that the strain energy density for material is considered by dependency of both the first and second derivatives of the strain tensor, along with the strain tensor itself. This theory has ten constants more than the classical theory in the strain tensor. By eliminating the second derivatives of the strain tensor, the first strain gradient theory will be achieved with five more constants than the classical theory. Based on this fact that in micro

*Corresponding author. Tel.: +98 34 33776611; Fax: +98 34 33776617.
E-mail address: m.rezaeizadeh@kgut.ac.ir (M. Rezaeizadeh).

scale the material properties depend to the size, the modified couple stress theory introduced a parameter called length scale parameter. This parameter can be used in relations. In fact in the modified couple stress theory, both strain and gradient of rotation are considered. Yang et al. [2] modified the couple stress theory by dictating a new equilibrium equation. Jomehzadeh et al. [4] studied a new model for vibration analysis of rectangular and circular micro plates using a modified couple stress theory. Reddy and Berry [5] derived the governing equations of motion for a circular micro plate under axisymmetric force using the modified couple stress theory. Kumar et al. [6] studied two-dimensional axisymmetric behavior of a circular plate with a heat source by using modified couple stress theory. Gholami et al. [7] investigated the size dependent nonlinear post-buckling of circular cylindrical micro and nano-scale shells. Ansari et al. [8] studied about the effect of Surface stress and surface inertia on mechanical behavior of nano-plates with a high surface to volume ratio based on Gurtin–Murdoch theory and Hamilton principle. By using the couple stress theory, Gholami et al. [9] developed a nonclassical first-order shear deformation shell model for análisis of the axial buckling and dynamic stability of micro shells made of functionally graded materials. Ansari et al. [10] investigated the size-depended free vibration of magneto-electro-thermo-elastic (METE) rectangular nano plates based on. The Mindlin plate theory, von Kármán hypothesis and the nonlocal theory. Wang et al. [11] studied the deflection of circular microplates under uniform load based on nonclassical Kirchhoff plate by using the couple stress theory. Stolken and Evans [12] found a versatile micro bend test method to determine the length scale parameter. In this method, after bending of a thin annealed foil around a small diameter cylindrical mandril, the radius of curvature will measure. This method is fully characterized so it is independent from the elastic modulus of the foil material. By having different radius of mandrills and different thickness of foil, the length scale parameter can be found. Circular plates under various loading are widely used in many applications. Saidi et al. [13] presented a closed-form solution for static analysis of FG circular plate under asymmetric loading according to the Kirchhoff plate theory. Liang et al. [14] studied the bending, buckling and vibration of size-dependent functionally graded annular micro-plates. Zhang et al. [15] developed an efficient size-dependent plate model based on the strain gradient elasticity theory and a refined shear deformation theory. Ansari et al. [16] developed Mindlin micro-plate model based on the modified strain gradient elasticity theory to predict axisymmetric bending, buckling, and free vibration characteristics of circular/annular micro-plates made of functionally graded materials. Park and Gao [17] developed a model for bending of a Bernoulli-Euler beam by using a modified couple stress theory. The dynamic behaviors of simply supported Bernoulli-Euler beams were investigated by Kong et al. [18] on the basis of the modified couple stress theory. Bending of a functionally graded micro-scale Timoshenko beam was studied by Simsek et al. [19]. They analytically solved governing equations of a simply supported micro-beam subjected to a point and uniformly distributed loads. Nateghi et al. [20] studied the buckling analysis of functionally graded micro-beams by considering classical, first and third order shear deformation beam theories. Roque et al. [21] employed modified couple stress theory to study the bending of simply supported laminated composite micro-beams subjected to transverse loads by using meshless numerical method. Ke et al. [22] investigated nonlinear free vibration of micro-beams made of functionally graded materials using von Karman geometric nonlinearity. Ansari et al. [23] developed a size-dependent micro model for bending, buckling and free vibration behaviors of micro-plates made of functionally graded materials. Baghani et al. [24] presented an analytical size-dependent solution for large-amplitude vibrations of functionally graded tapered-Nano beams. Karimipour et al. [25] investigated the electromechanical response and instability of the clamped microplate immersed in ionic electrolyte media. The strain gradient elasticity is employed to model the size-dependent structural behavior of the clamped microplate. Karimipour et al. [26] used modified strain gradient theory to investigate the size-dependent nonlinear pull-in instability and conduct a stress analysis of thin microplates.

In this paper, a size-dependent closed form solution is obtained for a circular micro-plate under eccentric force according to the couple stress theory. A new coordinate system is utilized to reformulate the equilibrium equations of micro-plate. The Green's function for deflection of circular micro-plate under eccentric load is obtained based on this reformulation. The novelty of this paper is to find the deflection of a micro-plate under asymmetric load by using a new coordinate system also the deflection of the micro-plate under any arbitrary load can be easily obtained as well.

2 DISPLACEMENT FIELD

Let us consider a circular micro-plate with radius R and thin thickness h. The displacement components according to the classical theory of plates are given by [11]:

$$U_r(r,\theta,z) = u_r(r,\theta) - z\,\frac{\partial w}{\partial r} \tag{1a}$$

$$U_\theta(r,\theta,z) = u_\theta(r,\theta) - \frac{z}{r}\frac{\partial w}{\partial \theta} \tag{1b}$$

$$U_z(r,\theta,z) = w(r,\theta) \tag{1c}$$

where u_r, u_θ and w are displacement components of middle surface of the micro-plate in the cylindrical coordinate system. Based on this field, the strain components can be written as [11]:

$$\varepsilon_r = \frac{\partial u_r}{\partial r} - z\,\frac{\partial^2 w}{\partial r^2} \tag{2a}$$

$$\varepsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} - z\,(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}) \tag{2b}$$

$$2\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} - z\,(\frac{2}{r}\frac{\partial w}{\partial \theta} + \frac{2}{r^2}\frac{\partial^2 w}{\partial r \partial \theta}) \tag{2c}$$

Strain components are zero.

3 EQUATIONS OF MODIFIED COUPLE STRESS THEORY

The modified couple stress theory is a popular theory for considering the length scale effect in micro-structures. This theory introduces just one parameter, so its equations just have a little change in compare with the classic theory and it makes the comparison easier. By considering the couple stress theory just the stiffness of micro-plates will change. Also, by length scale parameter, the behavior of structure in micro-scales can be predicted [11]. The experimental works on Cu wires show that thin (15 μm) wires required substantially higher torsions than thicker wires to cause equivalent rotations. Also for measurement of the crack growth in metal-oxide interfaces it is needed to have and consider the length scale [12].

In the classical theory both stress and strain energy depend on the strain tensor, but in the modified couple stress theory besides the strain tensor, gradient of the rotation vector has effect on the strain energy [4]. In compare with the higher order stress theory, the effects of the dilatation gradient and the deviatoric stretch gradient are assumed to be zero [1].

According to the modified couple stress theory, the strain energy of the micro-plate is written as [4]:

$$U_{SE} = \int (\sigma\varepsilon + m\,\chi) dV \tag{3}$$

where σ is the Cauchy stress tensor, ε is the Green strain tensor, m is the deviatoric part of the symmetric couple stress tensor and χ is the symmetric curvature tensor as [4]:

$$m = 2\mu l^2 \chi \tag{4a}$$

$$\chi = \frac{1}{2}[\nabla\theta + \nabla\theta^T] \tag{4b}$$

where μ is the Lame`s coefficient and θ is the rotation vector that can be written in term of displacement vector (U) as:

$$\theta = \frac{1}{2}\nabla \times U \tag{5}$$

The parameter l in Eq. (4a) is the length scale parameter and can capture the size dependency of the system. By replacing Eq. (5) into Eq. (4b) and using Eq. (1), the components of the curvature tensor for a linear case are obtained as follows:

$$\chi_{rr} = -\frac{1}{r^2}\frac{\partial w}{\partial \theta} + \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} \tag{6a}$$

$$\chi_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial^2 w}{\partial r^2}\right) \tag{6b}$$

$$\chi_{rz} = \frac{1}{2}\left(-\frac{1}{2r^2}u_\theta + \frac{1}{2r}\frac{\partial u_\theta}{\partial r} + \frac{1}{2r^2}\frac{\partial u_r}{\partial \theta} + \frac{1}{2}\frac{\partial^2 u_\theta}{\partial r^2} - \frac{1}{2r}\frac{\partial^2 u_r}{\partial r \partial \theta}\right) \tag{6c}$$

$$\chi_{\theta\theta} = \frac{1}{2}\left(\frac{1}{r^2}\frac{\partial w}{\partial \theta} - \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta}\right) \tag{6d}$$

$$\chi_{\theta z} = \frac{1}{2}\left(\frac{1}{2r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{1}{2r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{1}{2r}\frac{\partial^2 u_\theta}{\partial r \partial \theta}\right) \tag{6e}$$

$$\chi_z = 0 \tag{6f}$$

In order to derive the equilibrium equations, the principle of minimum total potential energy is used as:

$$\delta U_{SE} - \delta W = 0 \tag{7}$$

In which, W is the potential energy as:

$$W = \int p w \, d\Omega \tag{8}$$

where p is the external pressure in transverse direction. By introducing the resulting parameters as [13]:

$$(N_r, N_\theta, N_{r\theta}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_r, \sigma_\theta, \tau_{r\theta}) dz \tag{9a}$$

$$(M_r, M_\theta, M_{r\theta}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_r, \sigma_\theta, \tau_{r\theta}) z dz \tag{9b}$$

$$P = \int_{-\frac{h}{2}}^{\frac{h}{2}} m dz \tag{9c}$$

The equilibrium equations of a micro-plate can be obtained as follows:

$$\delta u_r : \frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_{r\theta}}{\partial \theta} + \frac{N_r - N_\theta}{r} + \frac{1}{2r^2}\frac{\partial P_{rz}}{\partial \theta} + \frac{1}{2r^2}\frac{\partial^2 P_{\theta z}}{\partial \theta^2} + \frac{1}{2r}\frac{\partial^2 P_{rz}}{\partial r \partial z} = 0 \tag{10a}$$

$$\delta u_\theta : \frac{2N_{r\theta}}{r} + \frac{\partial N_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + \frac{P_{rz}}{2r^2} - \frac{1}{2r}\frac{\partial P_{rz}}{\partial r} + \frac{1}{2r^2}\frac{\partial P_{\theta z}}{\partial \theta} - \frac{1}{2}\frac{\partial^2 P_{rz}}{\partial r^2} - \frac{1}{2r}\frac{\partial^2 P_{\theta z}}{\partial r \partial \theta} = 0 \tag{10b}$$

$$\delta w : \frac{2\partial M_r}{r\partial r} + \frac{\partial^2 M_r}{\partial r^2} + \frac{2}{r}\frac{\partial^2 M_{r\theta}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial M_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial M_{r\theta}}{\partial \theta} + \frac{1}{r^2}\frac{\partial M_\theta}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial P_\theta}{\partial \theta} + \frac{3}{r}\frac{\partial P_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial^2 P_{r\theta}}{\partial r^2} + \frac{1}{r}\frac{\partial^2 P_\theta}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial P_r}{\partial \theta} - \frac{1}{r}\frac{\partial^2 P_r}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial^2 P_{r\theta}}{\partial \theta^2} = F \tag{10c}$$

The corresponding boundary conditions can also be obtained as:

$$\delta u_r = 0 \quad or \quad N_r e_r + (N_{r\theta} + \frac{P_{rz}}{2r} + \frac{\partial P_{rz}}{2\partial r} + \frac{1}{2r}\frac{\partial P_{\theta z}}{\partial \theta})e_\theta = 0 \tag{11a}$$

$$\frac{\partial \delta u_r}{\partial \theta} = 0 \quad or \quad P_{rz}e_r + P_{\theta z}e_\theta = 0 \tag{11b}$$

$$\delta u_\theta = 0 \quad or \quad (N_\theta + \frac{P_{\theta z}}{2r})e_\theta + (N_{r\theta} - \frac{\partial P_{rz}}{2\partial r} - \frac{1}{2r}\frac{\partial P_{\theta z}}{\partial \theta})e_r = 0 \tag{11c}$$

$$\frac{\partial \delta u_\theta}{\partial r} = 0 \quad or \quad rP_{rz}e_r + \frac{P_{\theta z}}{r}e_\theta = 0 \tag{11d}$$

$$\delta w = 0 \quad or \quad (\frac{M_r}{r} + \frac{\partial M_r}{\partial r} + \frac{2}{r}\frac{\partial M_{r\theta}}{\partial \theta} - \frac{M_\theta}{r} + \frac{2P_{r\theta}}{r} + \frac{\partial P_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial P_\theta}{\partial \theta})e_r + (\frac{2}{r}M_{r\theta} + \frac{1}{r}\frac{\partial M_\theta}{\partial \theta} - \frac{P_r}{r} - \frac{\partial P_r}{\partial r} - \frac{P_\theta}{r} - \frac{1}{r}\frac{\partial P_{r\theta}}{\partial \theta})e_\theta = 0 \tag{11e}$$

$$\frac{\partial \delta w}{\partial r} = 0 \quad or \quad (M_r + P_{r\theta})e_r + (2M_{r\theta} + P_\theta)e_\theta = 0 \tag{11f}$$

$$\frac{\partial \delta w}{\partial \theta} = 0 \quad or \quad (-2M_{r\theta} + P_r)e_r + (-M_\theta + \frac{P_{r\theta}}{2})e_\theta = 0 \tag{11g}$$

Since the proposed micro-plate is homogeneous, the in-plane and transverse equations are decoupled. By obtaining the stress resultant in terms of displacement components and substituting them into Eq. (10c), the governing bending equilibrium equations of a circular micro-plate can be obtained as:

$$(D + \mu h l^2)(\frac{\partial^4 w}{\partial r^4} + \frac{2}{r^2}\frac{\partial^4 w}{\partial r^2 \partial \theta^2} - \frac{2}{r^3}\frac{\partial^3 w}{\partial r \partial \theta^2} + \frac{2}{r}\frac{\partial^3 w}{\partial \theta^3} - \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{4}{r^4}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r^3}\frac{\partial w}{\partial r} + \frac{1}{r^4}\frac{\partial^4 w}{\partial r^4}) = F \tag{12}$$

The parameter $D + \mu h l^2$ is the equivalent stiffness of a micro-plate by considering the length scale effect. As it can be concluded the modified couple stress theory increases the stiffness of micro-plates in comparison to the classical theory ($\mu = 0$).

4 TRANSFORMATION OF THE COORDINATE SYSTEM

The general solution for micro-structures under an arbitrary concentrated load is one of the fundamental problems in micro-mechanics. This solution can be used as a Green's function for determining deflections of the problems caused by any arbitrary loading.

Since the Green's function of a micro-plate is the solution of a partial differential equation with asymmetric conditions, it is worth to use a different coordinate system in order to find the solution.

Consider a circular micro-plate under the concentrated load F on O_1 as shown in Fig. 1. The points O_1 and O_2 are two fixed points on the Cartesian coordinate system. Let point A^0 be any point on the circumference, and r_1^0 and r_2^0 are distances from the point A^0 to points O_1 and O_2 as shown in Fig.1. It can be shown that if $(OO_1)(OO_2) = R^2$.

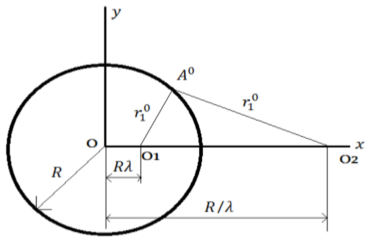

Fig.1
The new coordinate system.

For an arbitrary point on the circumference boundary of the micro-plate, the following relation is satisfied [13]:

$$\frac{r_1^0}{r_2^0} = \lambda \tag{13}$$

where λ is a constant. Locate an arbitrary point by coordinate r_1 and r_2 on the surface of the micro-plate (Fig.2). The coordinates r_1 and r_2 related to polar coordinate by the following expression [13]:

$$r_1^2 = (r\cos\theta - R\lambda)^2 + (r\sin\theta)^2 \tag{14a}$$

$$r_2^2 = (r\cos\theta - R/\lambda)^2 + (r\sin\theta)^2 \tag{14b}$$

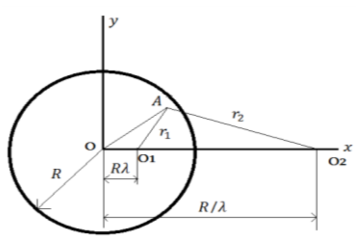

Fig.2
The coordinates of a point on the new coordinate system.

In order to obtain the governing equation of micro-plate in the new coordinate system, the Laplace operation should be obtained in terms of r_1 and r_2. To this end, the derivative with respect to x can be written as:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r_1}\frac{\partial r_1}{\partial x} + \frac{\partial}{\partial r_2}\frac{\partial r_2}{\partial x} = \frac{x-\lambda R}{r_1}\frac{\partial}{\partial r_1} + \frac{x-R/\lambda}{r_1}\frac{\partial}{\partial r_2} \tag{15}$$

Also, the second derivative with respect to x can be obtained in new coordinate system as:

$$\frac{\partial^2}{\partial x^2} = \frac{(x-\lambda R)^2}{r_1^2}\frac{\partial^2}{\partial r_1^2} + \frac{(x-R/\lambda)^2}{r_2^2}\frac{\partial^2}{\partial r_2^2} + [\frac{1}{r_1} - \frac{(x-\lambda R)^2}{r_1^3}]\frac{\partial}{\partial r_1} + [\frac{1}{r_2} - \frac{(x-R/\lambda)^2}{r_2^3}]\frac{\partial}{\partial r_2} \tag{16}$$

Similarly, the second derivative with respect to y can be determined in terms of derivative of r_1 and r_2 as:

$$\frac{\partial^2}{\partial y^2}=\frac{y^2}{r_1^2}\frac{\partial^2}{\partial r_1^2}+\frac{y^2}{r_2^2}\frac{\partial^2}{\partial r_2^2}+[\frac{1}{r_1}-\frac{y^2}{r_1^3}]\frac{\partial}{\partial r_1}+[\frac{1}{r_2}-\frac{y^2}{r_2^3}]\frac{\partial}{\partial r_2} \tag{17}$$

Hence the Laplace operator can be obtained by adding Eqs. (16) and (17) as:

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r_1^2}+\frac{\partial^2}{\partial r_2^2}+\frac{1}{r_1}\frac{\partial}{\partial r_1}+\frac{1}{r_2}\frac{\partial}{\partial r_2} \tag{18}$$

This expression can be rewritten in the following form:

$$\nabla^2=\frac{1}{r_1}\frac{\partial}{\partial r_1}(r_1\frac{\partial}{\partial r_1})+\frac{1}{r_2}\frac{\partial}{\partial r_2}(r_2\frac{\partial}{\partial r_2}) \tag{19}$$

Also biharmonic operator can be obtained by the square of Laplacian equation as [13]:

$$\nabla^4=\frac{1}{r_1}\frac{\partial}{\partial r_1}(r_1\frac{\partial}{\partial r_1}[\frac{1}{r_1}\frac{\partial}{\partial r_1}(r_1\frac{\partial}{\partial r_1})+\frac{1}{r_2}\frac{\partial}{\partial r_2}(r_2\frac{\partial}{\partial r_2})])+\frac{1}{r_2}\frac{\partial}{\partial r_2}(r_2\frac{\partial}{\partial r_2}[\frac{1}{r_1}\frac{\partial}{\partial r_1}(r_1\frac{\partial}{\partial r_1})+\frac{1}{r_2}\frac{\partial}{\partial r_2}(r_2\frac{\partial}{\partial r_2})]) \tag{20}$$

Therefore, both Laplacian and biharmonic operators are expressed in the new coordinate system.

5 THE GREEN'S FUNCTION

The Green's function is the solution of the differential equation to a concentrated load $F=1$. This method is a powerful method to find the responses of plates under asymmetric distributed loads. By using the Green's function, the deflection of micro-plate under arbitrary load can be defined. In order to obtain the Green's function of a circular micro-plate, let us express the governing Eq. (12) in the new coordinate system, i.e.

$$\begin{aligned}(D+\mu hl^2)\nabla^4 w &=(D+\mu hl^2)\{\frac{1}{r_1}\frac{\partial}{\partial r_1}(r_1\frac{\partial}{\partial r_1}[\frac{1}{r_1}\frac{\partial}{\partial r_1}(r_1\frac{\partial w}{\partial r_1})+\frac{1}{r_2}\frac{\partial}{\partial r_2}(r_2\frac{\partial w}{\partial r_2})])+\\ &\frac{1}{r_2}\frac{\partial}{\partial r_2}(r_2\frac{\partial}{\partial r_2}[\frac{1}{r_1}\frac{\partial}{\partial r_1}(r_1\frac{\partial w}{\partial r_1})+\frac{1}{r_2}\frac{\partial}{\partial r_2}(r_2\frac{\partial w}{\partial r_2})])\}=F\end{aligned} \tag{21}$$

By considering the above equation in two parts, the solution can be found by the help of axisymmetric state. A solution of the following equation for a circular micro-plate loaded by a concentrated force F is:

$$w=\frac{F}{8\pi(D+\mu hl^2)}r_1^2\ln r_1+C_1r_1^2\ln r_2+C_2r_1^2+C_3r_2^3 \tag{22}$$

The first term on the right-hand side of the equation represents the deflection surface of the plate symmetrical about point O_1 and having a singularity at that point. The constants of integration C_1, C_2 and C_3 can be determined from the boundary conditions.

Consider the clamped edge for the circular micro-plate. Since on the boundary $r_1=r_1^0$, $r_2=r_2^0$ and also $r_1^0=\lambda r_2^0$ then the boundary condition $w=0$ will be satisfied if we put:

$$C_1=-\frac{F}{8\pi(D+\mu hl^2)},\quad C_2=-\frac{F}{8\pi(D+\mu hl^2)}\ln\lambda-\frac{C_3}{\lambda^2} \tag{23}$$

Therefore, the deflection is simplified as:

$$w = \frac{F}{8\pi(D+\mu hl^2)} r_1^2 \ln\frac{r_1}{\lambda r_2} + C_3(r_2^2 - \frac{r_1^2}{\lambda^2}) \tag{24}$$

The constant C_3 is determined from the condition that on the clamped boundary $\frac{\partial w}{\partial n}=0$. Since on the boundary $w=0$, then to satisfy this boundary condition it is sufficient that $\frac{\partial w}{\partial r_1}=0$ vanish on the boundary for $r_1=\lambda r_2$, therefore:

$$C_3 = -\frac{F}{8\pi(D+\mu hl^2)}\frac{\lambda^2}{2} \tag{25}$$

Finally, the deflection of the micro-plate under eccentric concentrated load is defined as:

$$w = \frac{F}{8\pi(D+\mu hl^2)}[r_1^2 \ln\frac{r_1}{\lambda r_2} + \frac{1}{2}(\lambda^2 r_2^2 - r_1^2)] \tag{26}$$

By the help of Eqs. (14a) and (14b), the deflection can be transformed in the polar coordinate system as:

$$w(r,\theta) = \frac{F}{16\pi(D+\mu hl^2)}[(r^2+2rR\lambda\cos\theta+R^2\lambda^2)\ln(\frac{r^2+2rR\lambda\cos\theta+R^2\lambda^2}{R^2+2rR\lambda\cos\theta+r^2\lambda^2})+(\lambda^2-1)(r^2-R^2)] \tag{27}$$

In fact, Eq. (27) is the size-dependent transverse deflection of micro-plates and this equation is agreed with Ref. [13] with difference in the stiffness. Using this equation, the deflection of micro-plates under arbitrary loads can be defined.

In order to obtain the Green's function and consider the variation of parameters in the circumferential direction, Eq. (27) is rewritten in the following form by substituting $F=1$:

$$G(r,\theta-\theta_0) = \frac{1}{16\pi(D+\mu hl^2)}[(r^2+2rR\lambda\cos(\theta-\theta_0)+R^2\lambda^2)\ln(\frac{r^2+2rR\lambda\cos(\theta-\theta_0)+R^2\lambda^2}{R^2+2rR\lambda\cos(\theta-\theta_0)+r^2\lambda^2})+(\lambda^2-1)(r^2-R^2)] \tag{28}$$

which θ_0 is the position of a concentrated force from axis *x*. The above equation is the Green's function for transverse deflection of circular micro-plates.

By using Eq. (28) as a Green's function, the transverse deflection of a circular micro-plate under arbitrary distributed load $F(r_0,\theta_0)$ can be obtain as [13]:

$$w(r,\theta) = \int_0^R \int_0^\beta F(r_0,\theta_0) G(r,\theta-\theta_0) r_0 d\theta_0 dr_0 \tag{29}$$

Since $r_0 = \lambda R$, Eq. (29) can be rewritten as [13]:

$$w(r,\theta) = \int_0^1 \int_0^\beta F(\lambda,\theta_0) G(r,\theta-\theta_0) R^2 \lambda d\theta_0 d\lambda \tag{30}$$

6 NUMERICAL RESULTS

At the first, the validation of work will be discussed by comparing the result with previous papers. To this end, in Table 1., the maximum deflection of plate, under a line load distributed along a circle is compared with Ref. [13]. It can be seen that the results are in good agreement with related reference.

Table 1
Comparison of the maximum deflection of plate under line load with related reference $E = 420GPa$, $\nu = 0.3$, $h/R = 0.05$ and $F = 5\ KN\ /\ m$.

	$w_{\max}(\times 10^4)$	
λ	Present	Ref. [13]
0.1	0.1227	0.1227
0.2	0.2161	0.2161
0.3	0.2703	0.2703
.04	0.2843	0.2843
0.5	0.2622	0.2622
0.6	0.2123	0.2123
0.7	0.1460	0.1460
0.8	0.0773	0.0773
0.9	0.0225	0.0225

Main result: Assuming a circular micro-plate with the following physical properties:

$$E = 200GPa, \quad \nu = 0.3, \quad \mu = 70GPa, \quad R = 20\mu m, \quad F = 1.25\times 10^{-3} N \tag{31}$$

The center non-dimensional deflection of a micro-plate under an eccentric force on λ =0.5 versus dimensionless length scale is given in Fig.3. It can be seen that the length scale effect causes the decreasing of the deflection. The length scale parameter can be found by some experimental methods for example micro-bend test method can be used for determined this parameter. The result of this test is applicable for wide rang of materials [12]. The length scale parameter is 17.6 μm [1]. The length scale parameter is one of material properties that can shows the difference between classical and couple stress elasticity theory. In comparison with the dimension of circular micro-plate, this parameter is small but by diminish the dimensions of plate to micro size the influences of this parameter might be larger [4].

The center non-dimensional deflection of a micro-plate under an eccentric force on λ =0.5 and three length scales are shown in Fig. 4.

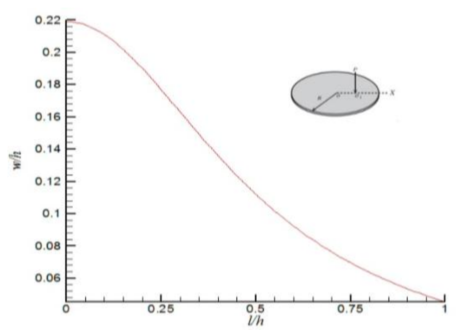

Fig.3
Central non-dimensional deflection of the micro-plate versus the dimensionless length scale.

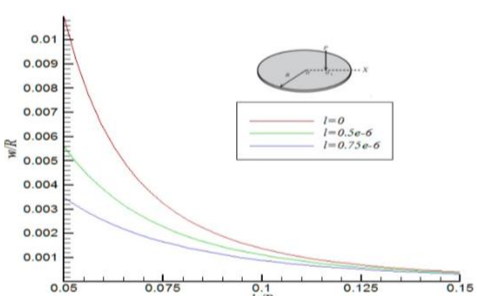

Fig.4
Center deflection of the micro-plate for various length scales.

It can be seen that as the thickness of the micro-plate increases, the size effect decreases, it means that as the thickness of micro-plate become higher, the value of the length scale has lower influence on the deflection. Also, length scale effect causes an additional hardening stiffness. The non-dimensional radial stress versus the radial direction is shown in Fig. 5.

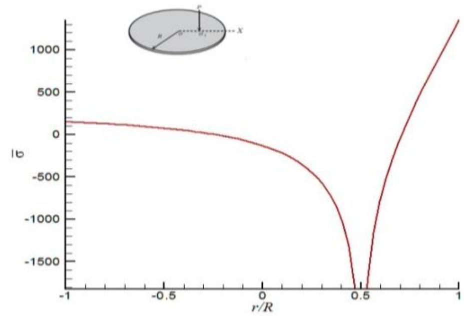

Fig.5
The variation of non-dimensional stress ($\bar{\sigma}=\frac{\sigma\pi r^2}{P}$) in radial direction (l=0, λ=0.5).

It can be seen that the stress under the load point is infinite in this theory. Variation of deflection versus the radial direction for various loading conditions is given in the Figs. 6-8. From these figures, it is obvious that as the parameter l/h increases, the deflection decreases, it is caused by increasing the stiffness of micro-plate $(D+\mu hl^2)$.

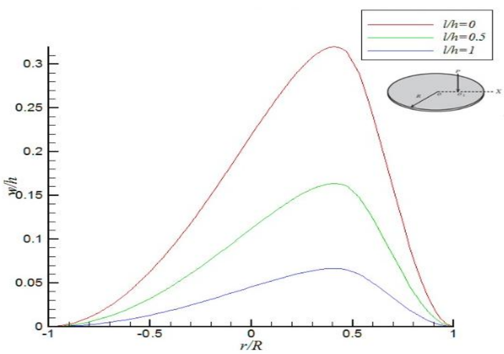

Fig.6
The variation of non-dimensional deflection in radial direction for a point load.

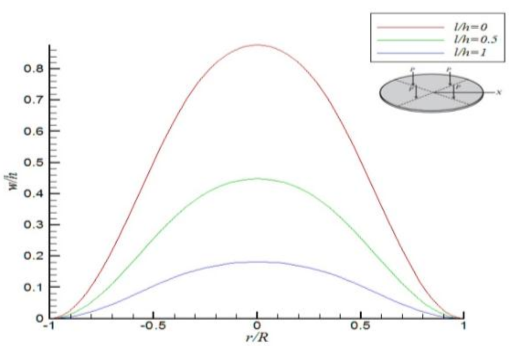

Fig.7
The variation of non-dimensional deflection in radial direction for a quarter loading.

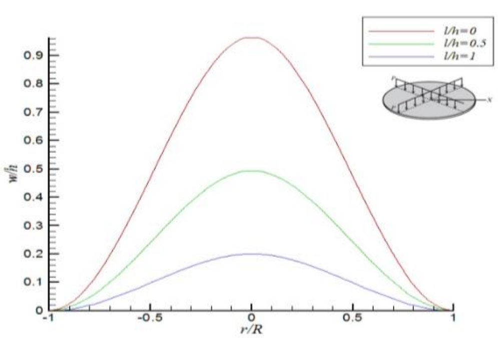

Fig.8
The variation of non- dimensional deflection in radial direction for two lines loading.

Fig. 9 shows the three-dimensional deflection of circular micro-plate with two different loads. It can be seen that the maximum deflection of the micro-plate with a concentrated force is the point of loading whereas it is at the center for four concentrated loads.

Fig.9
The 3D deflection of micro-plates for two kinds of loading.

7 CONCLUSIONS

In this article, the Green's function for bending of circular micro-plate has been developed by using a new coordinate system. Since the experimental researches show that mechanical properties are depended to the size, the modified couple stress theory has been utilized to consider the length scale effect. The deflection of the micro-plate under eccentric force has been defined by solving the equilibrium equation in the new coordinate system. Using this Green' function the transverse deflection of a micro-plates under arbitrary load can be easily obtained. It is concluded that the length scale has a significant effect on bending of micro plate. The result shows that the length scale has a considerable effect. According to the results, existence of the length scale parameter causes increasing in the stiffness and finally decreasing the deflection. Also it is found that when parameter (l) increases the deflection decreases. This trend is in agreement with the results of couple stress in other cases.

REFERENCES

[1] Lam D.C.C., Yang F., Chong A.C.M., Wang J., Tong P., 2003, Experiments and theory in strain gradient elasticity, *Journal of the Mechanics and Physics of Solids* **51**: 1477-1508.

[2] Yang F., Chong A.C.M., Lam D.C.C., Tong P., 2002, Couple stress based strain gradient theory for elasticity, *International Journal of Solids and Structures* **39**: 2731-2743.

[3] Mindlin R.D., 1965, Second gradient of strain and surface tension in linear elasticity, *International Journal of Solids and Structures* **1**: 417-438.

[4] Jomehzadeh E., Noori H.R., Saidi A.R., 2011, The size-dependent vibration analysis of micro-plates based on a modified couple stress theory, *Journal of Physica E* **43**: 877-883.

[5] Reddy J.N., Berry J., 2012, Nonlinear theories of axisymmetric bending of functionally graded circular plates with modified couple stress, *Composite Structures* **94**: 3664-3668.

[6] Kumar R., Devi Sh., Sharma V., 2017, Axisymmetric problem of thick circular plate with heat sources in modified couple stress theory, *Journal of Solid Mechanics* **9**: 157-171.

[7] Gholami R., Darvizeh A., Ansari R., Pourashraf T., 2017, Analytical treatment of the size-dependent nonlinear post-buckling of functionally graded circular cylindrical micro-/ nano-Shells, *Iranian Journal of Science and Technology Transactions of Mechanical Engineering* **42**: 85-97.

[8] Ansari R., Gholami R., 2016 , Surface effect on the large amplitude periodic forced vibration of first-order shear deformable rectangular nanoplates with various edge supports, *Acta Astronautica* **118**: 72-89.

[9] Gholami R., Ansari R., Darvizeh A., Sahmani S., 2015, Axial buckling and dynamic stability of functionally graded microshells based on the modified couple stress theory, *International Journal of Structural Stability and Dynamics* **15**(04): 1450070.

[10] Ansari R., Gholami R., 2016, Nonlocal free vibration in the pre- and post- buckled states of magneto-electro-thermo elastic rectangular nanoplates with various edge conditions, *Smart Materials and Structures* **25**: 5095033.

[11] Wang Y., Lin W., Zhou C., 2013, Nonlinear bending of size-dependent circular microplates based on the modified couple stress theory, *Archive of Applied Mechanics* **84**: 391-400.

[12] Stolken J.S., Evans A.G., 1998, A microbend test method for measuring the plasticity length scale, *Acta Metallurgica et Materialia* **46**: 5109-5115.

[13] Saidi A. R., Naderi A., Jomehzadeh E., 2009, A closed form solution for bending/stretching analysis of functionally graded circular plates under as symmetric loading using the Green function, *IMECHE Part C Journal of Mechanical Engineering Science* **1**: 1-3.

[14] Liang K., Yang J., Kitipornchai S., Bradford M.A., 2012, Bending, buckling and vibration of size-dependent functionally graded annular micro-plates, *Composite Structures* **94**: 3250-3257.

[15] Zhang B., He Y., Liu D., Shen L., Lei J., 2015, An efficient size-dependent plate theory for bending, buckling and free vibration analyses of functionally graded microplates resting on elastic foundation, *Mathematical Modelling* **39**: 3814-3845.

[16] Ansari R., Hasrati E., Faghih Shojaei M., Gholami R., Mohammadi V., Shahabodini A., 2016, Size-dependent bending, buckling and free vibration analyses of microscale functionally graded Mindlin plates based on the strain gradient elasticity theory, *Latin American Journal of Solids and Structures* **13**(4): 632-664.

[17] Park S.K., Gao X.L., 2006, Bernoulli-Euler beam model based on a modified couple stress theory, *Journal of Micromechanics and Microengineering* **16**: 2355-2359.

[18] Kong S., Zhou S., Nie, Z., Wang K., 2008, The size-dependent natural frequency of Bernoulli-Euler micro-beams, *International Journal of Engineering Science* **46**: 427-437.

[19] Simsek M., Kocat¨urk T., Akbas S.D., 2013, Static bending of a functionally graded microscale Timoshenko beam based on the modified couple stress theory, *Composite Structures* **95**: 740-747.

[20] Nateghi A., Salamat-talab M., Rezapour J., Daneshian B., 2012, Size dependent buckling analysis of functionally graded micro beams based on modified couple stress theory, *Applied Mathematical Modelling* **36**: 4971-4987.

[21] Roque C.M.C., Fidalgo D.S., Ferreira A.J.M., Reddy J.N., 2013, A study of a microstructure-dependent composite laminated Timoshenko beam using a modified couple stress theory and a meshless method, *Composite Structures* **96**: 532-537.

[22] Ke L.L., Wang Y., Yang J., Kitipornchai S., 2012, Nonlinear free vibration of size-dependent functionally graded micro-beams, *International Journal of Engineering Science* **50**: 256-267.

[23] Ansari R., Gholami R., Faghih Shojaei M., Mohammadi V., Sahmani S., 2014, Bending, buckling and free vibration analysis of size-dependent functionally graded circular/annular micro-plates based on the modified strain gradient elasticity theory, *European Journal of Mechanics* **49**: 251-267.

[24] Baghani M., MohammadSalehi M., Dabaghian P.H., 2016, Analytical couple-stress solution for size-dependent large-amplitude vibrations of FG tapered-Nano beams, *Latin American Journal of Solids and Structures* **13**(1): 95-118.

[25] Karimipour I., Tadi Beni Y., Taheri N., 2017, Influence of electrical double-layer dispersion forces and size dependency on pull-in instability of clamped microplate immersed in ionic liquid electrolytes, *Indian Journal of Physics* **91**(10): 1179-1195.

[26] Karimipour I., Tadi Beni Y., Zeighampour H., 2017, Nonlinear size-dependent pull-in instability and stress analysis of thin plate actuator based on enhanced continuum theories including nonlinear effects and surface energy, *Microsystem Technologies* 24: 1811-1839.