

# An Axisymmetric Contact Problem of a Thermoelastic Layer on a Rigid Circular Base

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## ABSTRACT

We study the thermoelastic deformation of an elastic layer. The upper surface of the medium is subjected to a uniform thermal field along a circular area while the layer is resting on a rigid smooth circular base. The doubly mixed boundary value problem is reduced to a pair of systems of dual integral equations. The both system of the heat conduction and the mechanical problems are calculated by solving a dual integral equation systems which are reduced to an infinite algebraic one using a Gegenbauer's formulas. The stresses and displacements are then obtained as Bessel function series. To get the unknown coefficients, the infinite systems are solved by the truncation method. A closed form solution is given for the displacements, stresses and the stress singularity factors. The effects of the radius of the punch with the rigid base and the layer thickness on the stress field are discussed. A numerical application is also considered with some concluding results.

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**Keywords:** Axisymmetric thermoelastic deformation; Doubly mixed boundary value problem; Hankel integral transforms; Infinite algebraic system; Stress singularity factor.

## 1 INTRODUCTION

THE thermal properties of solid materials allow us to interpret their responses to temperature changes. When a material absorbs thermal energy, its temperature and dimensions increase. If there are temperature gradients, the thermal energy migrates to cooler areas, otherwise the material melts. The contact is a multidisciplinary domain. Indeed, it interacts with mechanics, friction, materials behavior and thermic. It is also a multi-scale problem ranging from microscopic effects to macroscopic phenomena of heat dissipation or structural deformations, etc. The phenomena related to the mechanical contact problem are present in many domestic and industrial applications. The highly complex nature of phenomena related to the purely mechanical or multi-physical interaction between solids still requires special attention in the fields of physics, mathematics and computer science. The mechanical contact is very important for the good resolution of many problems such as shaping (forging, stamping, punching, ...) as well as for the simulation of wear (gears, tire-road, ...) and also for any system comprising several parts in a mechanical or multi-physical context. These problems are important for many industrial sectors, such as production, the

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automobile industry, aeronautics, the nuclear industry and the military. All these areas testify to the growing need for powerful and robust tools to mathematically model and numerically simulate the phenomenon of mechanical contact. Many theoretical and numerical studies have been conducted on the subjects but the experimental works are rare. Therefore the researcher's interest has been devoted to several studies in this field. The papers dealing with heat conduction problems have been considered by many authors. The first works devoted to mixed boundary value problems for infinite slabs were studied by Dhaliwal. The case where a temperature is given on a circular area and a temperature gradient is prescribed on the rest of the surface, where the circular opposite side is considered as a thermal insulator while the other face is insulated, was treated in [1]. In the second work [2], the mixed boundary conditions were prescribed on upper sides of the slab. These problems give rise to a pair of dual integral equations which are reduced to Fredholm integral equations of the second kind. The successive approximation method for large values of the slab thickness was applied for solving the obtained equations. An exact steady state solution for a plate heated by disk source is developed by Mehta and Bose [3] where one surface is considered isothermal and on the opposite side a heat flux is imposed. The solution is given in terms of converging power series very fast. The effect of the plate thickness on the temperature distribution was also studied. Lebedev and Ufliand [4] studied the problem of pressing a stamp of circular cross-section into an elastic layer. They expressed the required displacements and stresses in terms of one auxiliary function, which represents the solution of a Fredholm integral equation with a continuous symmetrical kernel. Zakorko [5] solved the axisymmetric deformation of an elastic layer with a circular line of separation of the boundary conditions on both faces. The corresponding systems of dual integral equations were reduced to a Fredholm integral equation of the second kind. The numerical solution was not given for studied problem. An axisymmetric contact problem for an elastic layer on a rigid foundation with a cylindrical hole has been considered by Dhaliwal and Singh [6]. The problem is reduced to the solution of two simultaneous Fredholm integral equations. A circular load on elastic layer is conducted by Wood [7]. An exact solution was obtained by the Hankel transform method, where the stresses and displacements were given in closed form. Toshiaki *et al.* [8] presented the solution of an elastic layer resting on a rigid base with a cylindrical hole whose radius is different from that of the rigid punch applied on the upper surface of the medium. The problem is reduced to the solution of an infinite system of simultaneous equations by assuming that both the contact stress under the punch and the normal displacement in the region of the hole may be expressed as appropriate Bessel series.

In the work [9], indentation of a penny-shaped crack by a disc-shaped rigid inclusion in an elastic layer has been considered by Sakamoto *et al.* This three part mixed boundary value problem is reduced to a solution of infinite systems of simultaneous equation. An axisymmetric contact problem of an elastic layer subjected to a tensile stress applied to a circular region is studied by Sakamoto and Koboyashi [10]. Their second paper [11] deals with the contact problem of rigid punch on an infinite elastic layer resting on a rigid base with a circular hole. These mixed boundary problems are effectively reduced to an exact solution of infinite systems of simultaneous equation. An analytical solution of an axisymmetric contact problem of an elastic layer on a rigid circular base has been developed by Kebli *et al.* [12]. They determine the solution of the elastic problem by the help Hankel integral transform method using the auxiliary Boussinesq stress functions. The doubly mixed boundary value problem is reduced to a system of dual integral equations. The obtained solution is calculated from the coefficients of the infinite system of simultaneous algebraic equations by means of the Gegenbauer expansion formula of the Bessel function. Dhaliwal [13] studied the steady state thermal stresses in an elastic layer where a heat flux was imposed over a circular area. The problem is reduced to the solution of two simultaneous Fredholm integral equations of the second kind. A similar problem was analyzed by Wadhawan [14]. The expressions for the temperature, displacements and the stresses in the elastic layer were obtained by the application of some differential operators and Mittag-Leffler theorem. The boundary conditions effect for the constriction resistance problem of circular contacts on coated surfaces was studied by Negus *et al.* [15]. Both heat flux and temperature boundary conditions were specified on the contact surface where the layer and the substrate are in perfect contact. Solutions are obtained with the Hankel transform method using a technique of linear superposition for the mixed boundary value problem created by an isothermal contact. Lemczyk and Yovanovich [16] and [17] examined the variation of the thermal constriction resistance with convective boundary conditions. The problem is reduced to a single integro-differential equation, by employing the Hankel integral transform method and an appropriate Abel and Fourier transform relations. The obtained infinite linear set of algebraic equations is also analyzed. A constriction resistance problem was considered by Rao [18] and [19] for a solid covered by a layer with different constant thermal conductivities in the case of an ideal contact. A heat is supplied to the composite medium through a circular spot while the rest of the surface is isolated. The obtained Fredholm integral equation was reduced to a system of algebraic equations by using the method of quadrature and the power series expansion method. The constriction resistance was then displayed for various conductivity ratios. Abed-Halim and Elfalaky [20] treated the thermoelastic problem of an infinite solid

weakened by a penny-shaped crack and subjected to a uniform temperature and a normal stress distribution. A bidimensional analogous problem was studied by means of the Fourier transform method [21].

In the present work an analytical solution of an axisymmetric frictionless contact problem for the case of penetration of rigid punch into an elastic layer on a rigid circular base to has been developed. We determine the solution of the temperature distribution problem and the thermoelastic equilibrium system by the Hankel integral transform method. The doubly mixed boundary value problem is given as two coupled systems of dual integral equations. An analytical procedure of solution is used following the elastostatic analogous treated by Toshiaki [8] and Sakamoto [9]-[10]. The obtained solution is calculated from an infinite system of simultaneous algebraic equations by means of the Gegenbauer expansion formula of the Bessel function. In the numerical application we give some conclusions on the effects of the radius of the punch with the rigid base and the layer thickness on the stresses and thermal field are discussed. Our results are validated on the isothermal case and it shows a good agreement with those obtained by Kebli et al. [12] when the temperature is zero.

## 2 FORMULATION OF THE PROBLEM AND ITS SOLUTION

A cylindrical coordinate system  $(r, 0, z)$  is used in this study. Displacement components along  $r$  and  $z$  are denoted by  $u$  and  $w$ , respectively. The Poisson ratio, the Shear modulus and the coefficient of the linear thermal expansion of the elastic medium are noted by  $\nu$ ,  $G$  and  $\alpha$ , respectively. Components of the stress tensor are expressed by  $\sigma_z$  and  $\tau_{rz}$ . We consider an isotropic elastic layer with thickness  $h$ . A heated rigid, flat-ended circular punch is pressed into the upper boundary  $z=h$  to a depth  $\varepsilon$  by the application of an axial force  $P$  as shown in Fig 1. The magnitude of the penetration  $\varepsilon$  is sufficiently small. The medium is subjected to a uniform thermal field of intensity  $T_0$  imposed along the circular area of radius  $b$  by the punch with a plane base meanwhile the rest of the surface is maintained at a free temperature. The layer is resting on a rigid smooth circular base of radius  $a$ . The doubly mixed boundary value of the elastic layer can be described by the following equations on the rigid base

$$(\sigma_z)_{z=0} = 0, \quad r > a \tag{1}$$

$$(w_z)_{z=0} = 0, \quad 0 \leq r \leq a \tag{2}$$

$$T = 0, \quad r > a \tag{3}$$

$$\frac{\partial T}{\partial z} = 0, \quad r < a \tag{4}$$

on the upper surface

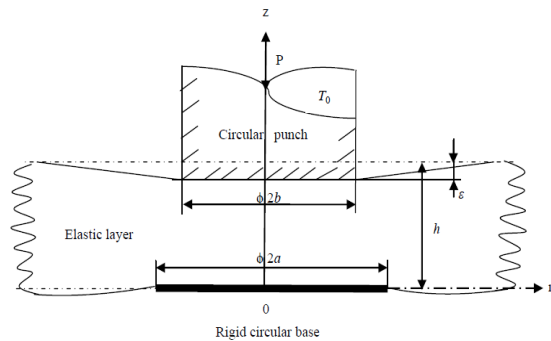
$$(\sigma_z)_{z=h} = 0, \quad r > b \tag{5}$$

$$(w_z)_{z=h} = -\varepsilon, \quad 0 \leq r \leq b \tag{6}$$

$$T(r, h) = \begin{cases} T_0, & r < b \\ 0, & r > b \end{cases} \tag{7}$$

and

$$(\tau_{rz})_{z=0} = (\tau_{rz})_{z=h} = 0, \quad r \geq 0 \tag{8}$$



**Fig.1**  
Geometry of the problem.

The thermoelastic equilibrium equations for an axisymmetric case may be written as [22]

$$(1 + \chi) \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right] + (-1 + \chi) \frac{\partial^2 u}{\partial z^2} + 2 \frac{\partial^2 w}{\partial r \partial z} = 4(1 + \nu) \alpha \frac{\partial T}{\partial r} \quad (9)$$

$$(-1 + \chi) \left[ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right] + (1 + \chi) \frac{\partial^2 w}{\partial z^2} + 2 \left[ \frac{\partial^2 u}{\partial z \partial r} + \frac{\partial u}{r \partial z} \right] = 4(1 + \nu) \alpha \frac{\partial T}{\partial z} \quad (10)$$

where,  $\chi = 3 - 4\nu$

The temperature field  $T$ , in the steady state and in the absence of thermal sources verifies the Laplace equation

$$\Delta T(r, z) = \frac{\partial^2 T(r, z)}{\partial r^2} + \frac{1}{r} \frac{\partial T(r, z)}{\partial r} + \frac{\partial^2 T(r, z)}{\partial z^2} = 0 \quad (11)$$

By the Hooke law, the components of the stress tensor  $\sigma_z$  and  $\tau_{rz}$  associated with the displacement field are given

$$\sigma_{zz} = \frac{2G}{1 - 2\nu} \left[ (1 - \nu) \frac{\partial w}{\partial z} + \nu \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right) - (1 + \nu) \alpha T \right] \quad (12)$$

and

$$\tau_{rz} = G \left[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right] \quad (13)$$

The Hankel integral transform of  $n$  order [23] is defined as:

$$H_n \{ f(\lambda) \} = \int_0^\infty r f(r) J_n(\lambda r) dr \quad (14)$$

where  $J_n$  is the Bessel function of first kind of order  $n$ . The original function can be calculated by the following inverse transforms

$$f(r) = \int_0^\infty \lambda H_n \{ f(\lambda) \} J_n(\lambda r) d\lambda \quad (15)$$

As is standard, when dealing with the type of problem under consideration we use the Hankel transform according to  $r$  of order zero. Then, the transformed equilibrium system and the temperature equation are obtained as:

$$-\lambda^2(1+\chi)U + (-1+\chi)U'' - 2\lambda W' = -4(1+\nu)\alpha\lambda T_H \tag{16}$$

$$2\lambda U' - \lambda^2(-1+\chi)W + (1+\chi)W'' = 4(1+\nu)\alpha\lambda T_H' \tag{17}$$

$$T_H'' - \lambda^2 T_H = 0 \tag{18}$$

2.1 The heat conduction problem

The solution to the Eq. (18) is

$$T_H(\lambda, z) = A(\lambda)e^{-\lambda z} + B(\lambda)e^{\lambda z} \tag{19}$$

where  $A(\lambda)$  and  $B(\lambda)$  are functions of the parameter  $\lambda$  to be determined from the boundary conditions of the temperature field. From the boundary conditions, we get

$$T_H(\lambda, z = h) = T_0 \int_0^\infty r H(b-r) J_0(\lambda r) dr = \frac{bT_0}{\lambda} J_1(\lambda b) \tag{20}$$

$H$  denotes Heaviside unit step function, we find that  $H(x-r) = \begin{cases} 1, & r < x \\ 0, & r > x \end{cases}$

Taking into account the relation (20), we get from (19)

$$B(\lambda) = \left( \frac{bT_0 J_1(\lambda b)}{\lambda} - A(\lambda)e^{-\lambda h} \right) e^{-\lambda h} \tag{21}$$

Substituting  $B$  in the Eq. (19), we find that

$$T_H(\lambda, z) = A(\lambda) \left( e^{-\lambda z} - e^{-\lambda(2h-z)} \right) + bT_0 e^{\lambda(-h+z)} J_1(\lambda b) \tag{22}$$

which gives

$$T(r, z) = \int_0^\infty \lambda \left[ A(\lambda) \left( e^{-\lambda z} - e^{-\lambda(2h-z)} \right) + bT_0 e^{\lambda(-h+z)} J_1(\lambda b) \right] J_0(\lambda r) d\lambda \tag{23}$$

Verifying the doubly mixed boundary value conditions for the temperature field we get the following dual integral equations

$$\int_0^\infty \lambda \phi(\lambda) q(\lambda) J_0(\lambda r) d\lambda = 2bT_0 \int_0^\infty \frac{e^{-\lambda h} J_1(\lambda b)}{1 - e^{-2\lambda h}} J_0(\lambda r) d\lambda, \quad r < a \tag{24}$$

$$\int_0^\infty \phi(\lambda) J_0(\lambda r) d\lambda = 0, \quad r > a \tag{25}$$

where

$$\phi(\lambda) = A(\lambda) \left( 1 - e^{-2\lambda h} \right) + \frac{bT_0 e^{-\lambda h}}{\lambda} J_1(\lambda b) \tag{26}$$

and

$$q(\lambda) = \frac{1 + e^{-2\lambda h}}{1 - e^{-2\lambda h}} \quad (27)$$

A large contribution is made for solving the similar integral equation problems [24]. For the present study we follow the method developed by Sakamoto [10]-[11]. Using the integral formula for the Bessel functions: 6. 522 [25]

$$\int_0^{\infty} \lambda M_n(\lambda) J_0(\lambda r) d\lambda = \begin{cases} \frac{2}{\pi r} \frac{T_{2n+1}(r/a)}{\sqrt{a^2 - r^2}}, & r < a \\ 0, & r > a \end{cases}, n=0, 1, 2, \dots \quad (28)$$

where  $T_{2n+1}$  is the Tchebycheff function of the first kind, and

$$M_n(\lambda a) = J_{n+\frac{1}{2}}(\lambda a/2) J_{-(n+\frac{1}{2})}(\lambda a/2) \quad (29)$$

We seek the solution of the dual integral equations as follows:

$$\phi(\lambda) = \sum_{n=0}^{\infty} \alpha_n M_n(\lambda a) \quad (30)$$

The unknown coefficients  $\alpha_n$  are to be determined. Then the second Eq. (25) is automatically satisfied. Next, we use the following Gegenbauer's formula

$$J_0(\lambda r) = \sum_{m=0}^{\infty} (2 - \delta_{0m}) X_m(\lambda a) \cos m\phi, \quad r = a \sin(\phi/2) \quad (31)$$

where  $\delta_{0m}$  denotes the Kronecker delta

$$\delta_{nm} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \quad (32)$$

$$X_m(\lambda a) = J_m^2\left(\frac{\lambda a}{2}\right) \quad (33)$$

Substituting Eq. (30) into Eq. (24) and using the formula (31), we obtain

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} (2 - \delta_{0m}) \cos m\phi \int_0^{\infty} \lambda M_n(\lambda a) q(\lambda) X_m(\lambda a) d\lambda = \sum_{n=0}^{\infty} (2 - \delta_{0m}) \cos m\phi \int_0^{\infty} \frac{e^{-\lambda h}}{(1 - e^{-2\lambda h})} X_m(\lambda a) J_1(\lambda b) d\lambda \quad (34)$$

where

$$a_n = \frac{\alpha_n}{2bT_0} \quad (35)$$

Matching the coefficients of  $\cos(m\phi)$  on both sides of Eq. (34), we obtain the following infinite system of simultaneous equations for obtaining the unknown coefficients  $a_n$

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} M_n(\lambda a) q(\lambda) Y_m(\lambda a) d\lambda = \left( \int_0^{\infty} \frac{e^{-\lambda h}}{1-e^{-2\lambda h}} X_m(\lambda a) J_1(\lambda b) d\lambda \right) \delta_{0m}, \quad m = 0, 1, 2, \dots \tag{36}$$

where

$$Y_m(\lambda a) = \lambda [X_m(\lambda a) - X_{m+2}(\lambda a)] \quad m = 0, 1, 2, \dots \tag{37}$$

The matrix from of the infinite system of simultaneous equations is

$$\sum_{n=0}^{\infty} a_n A_{mn} = G_{1m} \delta_{0m} \tag{38}$$

where

$$A_{mn} = \int_0^{\infty} M_n(\lambda a) q(\lambda) Y_n(\lambda a) d\lambda \tag{39}$$

$$G_{1m} = \int_0^{\infty} \frac{e^{-\lambda h}}{\lambda(1-e^{-2\lambda h})} Y_m(\lambda a) J_1(\lambda b) d\lambda \tag{40}$$

We write the system (36) in dimensionless form by using

$$t = \lambda b, \quad \delta = \frac{h}{b} \quad \text{and} \quad \eta = \frac{a}{b} \tag{41}$$

Then

$$\sum_{n=0}^{\infty} a'_n A'_{mn} = G'_{1m} \delta_{0m} \tag{42}$$

where

$$a'_n = \frac{\alpha_n}{2T_0} \tag{43}$$

$$A'_{mn} = \int_0^{\infty} q'(t) M_n(\eta t) Y_n(\eta t) dt \tag{44}$$

$$q'(t) = \frac{1+e^{-2\delta t}}{1-e^{-2\delta t}} \tag{45}$$

and

$$G'_{1m} = \int_0^{\infty} \frac{e^{-\delta t}}{t(1-e^{-2\delta t})} Y_m(\eta t) J_1(t) dt \tag{46}$$

For the temperature relation (23) we use the variables  $\zeta$  and  $\rho$  defined by

$$\zeta = \frac{r}{b}, \quad \rho = \frac{z}{b} \quad (47)$$

Then, it can be calculated as follows:

$$\frac{T(\zeta, \rho)}{T_0} = 2 \int_0^{\infty} \left[ \sum_{n=0}^{\infty} a'_n M_n(\eta t) (e^{-\rho t} - e^{t(\rho-2\delta)}) + \frac{J_1(t)}{2t} (e^{t(\rho-\delta)} - e^{-t(\rho+\delta)}) \right] \frac{J_0(\zeta t)}{(1-e^{-2\delta t})} dt \quad (48)$$

We can write the expression of the flux by follows as:

$$\frac{\partial T(\zeta, \rho)}{T_0 \partial \rho} = 2 \int_0^{\infty} -t \left[ \sum_{n=0}^{\infty} a'_n M_n(\eta t) (e^{-\rho t} + e^{t(\rho-2\delta)}) - \frac{J_1(t)}{2t} (e^{t(\rho-\delta)} + e^{-t(\rho+\delta)}) \right] \frac{J_0(\zeta t)}{(1-e^{-2\delta t})} dt \quad (49)$$

## 2.2 Numerical results and discussions

We solve the infinite set of simultaneous Eq.(42) to determine the unknown coefficients  $a'_n$ . For this purpose remarking that for sufficient large value of  $t$ , the infinite integrals in Eq. (42) can be rewritten in the following from

$$A'_{mn} = \int_0^{t_0} q'(t) Y_m(\eta t) M_n(\eta t) dt + A''_{mn} \quad (50)$$

where  $A''_{mn}$  is represented by

$$A''_{mn} = \int_{t_0}^{\infty} q_1'(t) Y_m(\eta t) M_n(\eta t) dt \quad (51)$$

The first term on the right hand side of Eq. (50) is integrated numerically by means of Simpson's rule. Here, we choose 1000 subintervals and  $t_0=1500$  and the second term is integrated by using the approximate form of Bessel functions. Then, the function  $q'(t)$  in the Eq. (51) can be replaced by (-1) and  $Y_m(t)M_n(t)$  can be asymptotically derived as follows:

$$J_\nu(t) \approx \sqrt{\frac{2}{\pi t}} \left[ \cos\left(t - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) - \frac{4\nu^2-1}{8t} \sin\left(t - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O\left(\frac{1}{t^2}\right) \right]_{t \rightarrow \infty} \quad (52)$$

$$J_{\left(n+\frac{1}{2}\right)}\left(\frac{t}{2}\right) J_{-\left(n+\frac{1}{2}\right)}\left(\frac{t}{2}\right) \approx 4\left(\frac{1}{2}+n\right)^2 \frac{\cos(t)}{\pi^2} \quad (53)$$

whereas

$$X_m(\eta t) - X_{m+2}(\eta t) \approx -\frac{16(-1)^m(1+m)}{\pi(\eta t)^2} \cos(\eta t) \quad (54)$$

Next, taking into account of the equivalent  $Y_m(\eta t)M_n(\eta t)$  given by

$$64(-1)^m(m+1)\left(n+\frac{1}{2}\right)^2 \frac{\cos^2(\eta t)}{\pi^2(\eta t)^4} \quad (55)$$



and of the relation obtained by integration par parts

$$\int_{t_0}^{\infty} \frac{\cos^2(\eta t)}{t^2} dt = \frac{\cos^2 \eta t_0}{t_0} + si(2\eta t_0) \tag{56}$$

Then by substituting Eq. (56) into the last integral of Eq. (55), we obtain

$$\int_{t_0}^{\infty} Y_m(\eta t) M_n(\eta t) dt = 64(-1)^m (m+1) \left(n + \frac{1}{2}\right)^2 \left\{ \frac{1 + \cos(2\eta t_0)}{6t_0^3} - \frac{a^2}{3} \left[ \frac{\sin(2\eta t_0)}{2t_0} + \frac{\cos(2\eta t_0)}{t_0} + 2si(2\eta t_0) \right] \right\} \tag{57}$$

where

$$si(x) \text{ is the integral sine function } si(x) = -\int_x^{\infty} \frac{\sin \xi}{\xi} d\xi \tag{58}$$

and

$$ci(x) \text{ is the integral cosine function } ci(x) = -\int_x^{\infty} \frac{\cos \xi}{\xi} d\xi \tag{59}$$

The thermique coefficients  $a'_n$  are shown in the following Tables1-2 of the thickness elastic layer and the radius of the punch with the rigid base

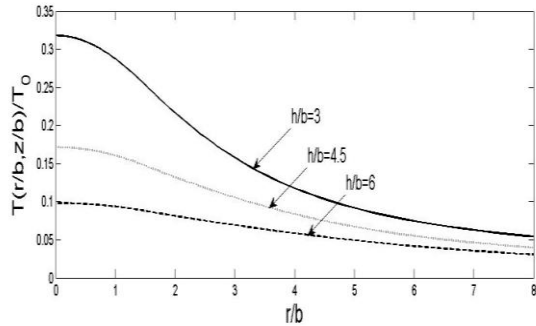
**Table 1**  
Values of the thermique coefficients for  $a/b=0.25$  and various values of  $h/b$ .

$n$	$a'_n$		
	$h/b=0.5$	$h/b=3$	$h/b=5$
0	0.036301204340747	0.002718452344962	0.001017687554760
1	-0.020453148297745	-0.001535629203246	-0.000574883746459
2	-0.004281230437599	-0.000318569931178	-0.000119260135626
3	-0.001528896564065	-0.000114116169693	-0.000042720728610
4	-0.000516153699553	-0.000038523271577	-0.000014421639969
5	-0.000021856254145	-0.000001608519258	-0.000000602160626
6	0.000247783831548	0.000018533417827	0.000006938216259
7	0.000386976541540	0.000028931432125	0.000010830838362
8	0.000421975222645	0.000031544119954	0.000011808928988
9	0.000341498340370	0.000025526870240	0.000009556297172

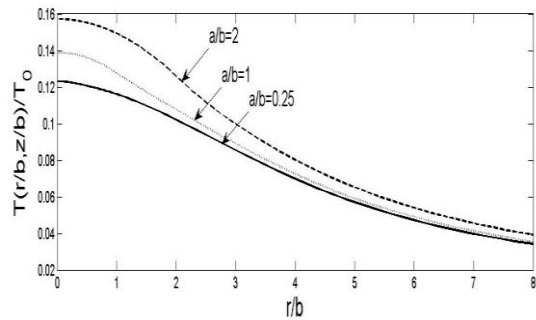
**Table 2**  
Values of the thermique coefficients for  $h/b=5$  and various values of  $a/b$ .

$n$	$a'_n$		
	$a/b=0.25$	$a/b=1$	$a/b=2$
0	0.001017687554760	0.015833767913739	0.056814813499988
1	-0.000574883746459	-0.008742171864567	-0.031278566249732
2	-0.000119260135626	-0.001728828877968	-0.006184507635662
3	-0.000042720728610	-0.000568608140977	-0.002015322813640
4	-0.000014421639969	-0.000157393856213	-0.000545035596517
5	-0.000000602160626	0.000027964875905	0.000113105960128
6	0.000006938216259	0.000117211220151	0.000425753811882
7	0.000010830838362	0.000154516394661	0.000552276169754
8	0.000011808928988	0.000155831010956	0.000551086506536
9	0.000009556297172	0.000121006661581	0.000423931636700

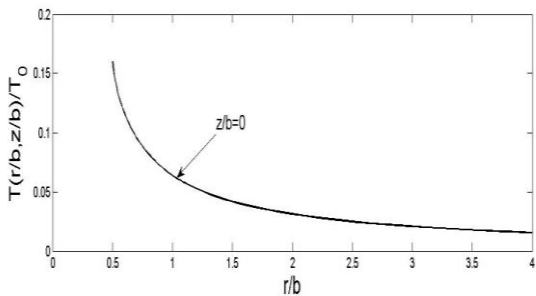
Figs. 2-3 shown the variation of the distribution of the temperature in the plane  $z/b=0.25$  for various values of  $h/b$  and  $a/b$ , respectively. From the graph lines, it is clear that the temperature distribution gets maximum values at the centre of the punch. It increases with decreasing the layer thickness and increasing rigid base radius. In the Fig. 4 the variation of the flux in the plane  $z/b=0$  becomes infinite at the edge of the rigid base. It decreases at different distances from it



**Fig.2**  
The temperature distributions for  $a/b=1.5$  and various values of  $h/b$ .



**Fig.3**  
The temperature distributions for  $h/b=5$  and various values of  $a/b$ .



**Fig.4**  
The flux distributions for  $a/b=0.5$  and  $h/b=1.5$ .

### 2.3 The thermoelastic problem

The solution of the thermoelastic set of Eqs. (16) and (17) can be obtained in the form

$$U = U_h + U_p, \quad W = W_h + W_p$$

where  $U_h, W_h$  are the general solution of the homogeneous of Eqs. (16) and (17) whereas  $U_p, W_p$  are the corresponding particular solutions of the non-homogeneous case. Using the relation (19) we get

$$U_h(\lambda, z) = \left[ C_0(\lambda) + C_1(\lambda) \left( z - \frac{z}{\lambda} \right) \right] e^{-\lambda z} + \left[ C_2(\lambda) + C_3(\lambda) \left( z + \frac{z}{\lambda} \right) \right] e^{\lambda z} \tag{60}$$

$$W_h(\lambda, z) = \left[ (C_0(\lambda) + C_1(\lambda)z) \right] e^{-\lambda z} - \left[ (C_2(\lambda) + C_3(\lambda)z) \right] e^{\lambda z} \tag{61}$$

and

$$U_p(\lambda, z) = 0 \tag{62}$$

$$W_p(\lambda, z) = 2(1+\nu)\frac{\alpha}{\lambda} \left[ B(\lambda) \left( e^{\lambda z} + e^{\lambda(2h-z)} \right) - \frac{T_0 b J_1(\lambda b)}{\lambda} e^{\lambda(h-z)} \right] \tag{63}$$

The final general solution of the thermoelastic equilibrium Eqs. (16) and (17) are

$$U(\lambda, Z) = \left[ C_0(\lambda) + C_1(\lambda) \left( z - \frac{\chi}{\lambda} \right) \right] e^{-\lambda z} - \left[ C_2(\lambda) + C_3(\lambda) \left( z + \frac{\chi}{\lambda} \right) \right] e^{\lambda z} \tag{64}$$

$$W(\lambda, Z) = \left[ C_0(\lambda) + C_1(\lambda)z + \frac{\alpha_0}{\lambda} \left( B(\lambda)e^{\lambda h} - \frac{T_0 b J_1(\lambda b)}{\lambda} \right) e^{\lambda h} \right] e^{-\lambda z} + \left[ C_2(\lambda) + C_3(\lambda)z + \frac{\alpha_0}{\lambda} B(\lambda) \right] e^{\lambda z} \tag{65}$$

where,  $\alpha_0 = 2(1+\nu)\alpha$ . The unknown functions  $C_0(\lambda)$ ,  $C_1(\lambda)$ ,  $C_2(\lambda)$  and  $C_3(\lambda)$  are to be determined from the boundary conditions Eq. (8). We find that

$$C_0(\lambda) = \frac{e^{\lambda h}}{\lambda^2 h} \left\{ 2\lambda C_2(\lambda)(-1+e^{-\lambda h}) - C_3(\lambda) \left[ (2(1-\nu)(\lambda h - 2(1-\nu)) + h\lambda^2) e^{-\lambda h} + 2(2(1-\nu) + \lambda h) \right] + \alpha_0 A(\lambda) \left[ (1+\lambda h)e^{-\lambda h} + (1-\nu)e^{\lambda h} \right] \right\} \tag{66}$$

$$C_1(\lambda) = \frac{e^{\lambda h}}{\lambda h} \left\{ C_2(\lambda) \left[ \lambda(-1+e^{-\lambda h}) - e^{\lambda h}(\lambda(-1+h) - 2(-1+\nu)) \right] - C_3(\lambda) \left[ (2(1+\nu)(-1+e^{-\lambda h}) - h\lambda) \right] + \frac{\alpha_0}{2} A(\lambda) \left[ (2+\nu)e^{\lambda h} - e^{-\lambda h} \right] \right\} \tag{67}$$

Substituting these functions into Eqs. (62) and (63), the solution become

$$U(\lambda, z) = \frac{1}{2\lambda h} \left\{ \left[ 2C_2(\lambda)(-1+2\nu) + 2C_3(\lambda)((h-z)) \right] ((h-z)(2(-1+\nu) - \lambda^2) - 2(-1+\nu)(-1+2\nu)) + \frac{1}{\lambda} \left[ -2\lambda C_2(\lambda) + 2(2(-1+\nu) - \lambda h)C_3(\lambda) + \alpha_0 A(\lambda) \right] (1-2\nu + \lambda z) \right\} \tag{68}$$

and

$$W(\lambda, z) = \frac{1}{2\lambda^2 h} \left\{ \left[ C_2(\lambda)(1-\nu) - 2C_3(\lambda)(\lambda(h-z)(2\nu(1-\nu) + \lambda) - \alpha_0 A(\lambda)(1-\nu-z)) \right] e^{-\lambda z} + 2\lambda h \left[ \lambda C_2(\lambda) + C_3(\lambda)(\chi + \lambda z) + \alpha_0 A(\lambda) \right] e^{\lambda z} + (2(1-\nu) + \lambda z) \left[ 2(-\lambda C_2(\lambda) - C_3(\lambda)(2(1-\nu) - \lambda h)) + \alpha_0 A(\lambda) \right] e^{\lambda(2h-z)} \right\} \tag{69}$$

Next, we apply the Hankel transform of orders zero and one to the relations (12), (13), respectively given by

$$\sigma_H(\lambda, z) = \int_0^\infty r \sigma_z(r, z) J_0(\lambda r) dr \tag{70}$$

$$\tau_H(\lambda, z) = \int_0^\infty r \tau_{rz}(r, z) J_1(\lambda r) dr \tag{71}$$

The transformed normal and shearing stresses were given in terms of  $U$ ,  $W$  and  $T_H$  by the expressions

$$\sigma_H(\lambda, z) = \frac{2G}{1-2\nu} [(1-\nu)W' + \nu\lambda U - (1+\nu)\alpha T_H] \quad (72)$$

$$\tau_H(\lambda, z) = G(U' - \lambda W) \quad (73)$$

Then from the Eqs. (67), (68) and (22), we find that

$$\begin{aligned} \sigma_H(\lambda, z) = & \frac{2G}{1-2\nu} \left\{ C_2(\lambda) \left[ (-1+\nu)(1+\lambda h) \frac{e^{2\lambda h}}{h} - \left( \frac{1}{2} + \lambda(1-\nu)(-h+z) \right) \right] + C_3(\lambda) (2(1-\nu)(1-2\nu) \right. \\ & \left. \left( \frac{1}{\lambda} + (1+\lambda h)e^{2\lambda h} \right) + \lambda h e^{2\lambda h} + 4(1-\nu^2)(-h+z) \right) + \frac{\alpha_0 T_0}{2} \left( \frac{1}{1+e^{-2\lambda h}} \left[ -\sum_{n=0}^{\infty} \alpha_n M_n(\lambda a) + \frac{b}{\lambda} J_1(\lambda b) e^{-\lambda h} \right] \right) \\ & \times \left( -\frac{1-2\nu}{2\lambda} (h+2(1+\nu)-2h(1-\nu)e^{2\lambda h} + (h+2(1+\nu))) - \frac{1}{\lambda} ((1-2\nu)(-1+\lambda(h-z))) e^{-2\lambda h} \right) \\ & + \frac{b}{\lambda} J_1(\lambda) e^{-\lambda h} \left( (-1+2\nu)(1+\nu(-h+\nu)) + \frac{1+\lambda h}{h} + 2\lambda\nu(-h+\nu) \right) + \frac{1-2\nu}{h} (h+z) \left. \right] e^{-\lambda z} \\ & - \left[ \frac{\alpha_0 T_0}{\lambda^2} h b J_1(\lambda) \right] e^{\lambda z} - \left[ C_2(\lambda) \lambda h (-1+h) + C_3(\lambda) h (-1+4\nu(1-\nu)) + \lambda z (1-\nu) \right] e^{\lambda h} \} \end{aligned} \quad (74)$$

and

$$\begin{aligned} \tau_H(\lambda, z) = & -\frac{G}{h} \left\{ \left[ 2(\lambda C_2(\lambda) - 2C_3(\lambda)(1-\nu)) + \alpha_0 A(\lambda) \right] (-h+z) e^{-\lambda z} + 2h \right. \\ & \left. \left[ \lambda C_2(\lambda) + C_3(\lambda) (\lambda z - 2(1-\nu)) - \frac{\alpha_0}{2} A(\lambda) \right] e^{\lambda z} + z \left[ -2\lambda C_2(\lambda) + 2C_3(\lambda) (-\lambda h - 2(1-\nu)) + \alpha_0 A(\lambda) \right] e^{-\lambda(-2h+z)} \right\} \end{aligned} \quad (75)$$

The inverse Hankel integral transforms are

$$\sigma_z(r, z) = \int_0^{\infty} \lambda \sigma_H(\lambda, z) J_0(\lambda r) d\lambda \quad (76)$$

$$w(r, z) = \int_0^{\infty} \lambda W(\lambda, z) J_0(\lambda r) d\lambda \quad (77)$$

The components of normal displacement and stress on two layer boundaries are given by the following system of dual integral equations

$$(\sigma_z)_{z=0} = 0 \Rightarrow \int_0^{\infty} \lambda [C_2(\lambda) g_{11}(\lambda) + C_3(\lambda) g_{12}(\lambda) - g_{13}(\lambda)] J_0(\lambda r) d\lambda = 0, \quad r > a \quad (78)$$

$$(w_z)_{z=0} = 0 \Rightarrow \int_0^{\infty} [C_2(\lambda) g_{21}(\lambda) + C_3(\lambda) g_{22}(\lambda) + g_{23}(\lambda)] J_0(\lambda r) d\lambda = 0, \quad r \leq a \quad (79)$$

$$(\sigma_z)_{z=h} = 0 \Rightarrow \int_0^{\infty} \lambda [C_2(\lambda) g_{41}(\lambda) + C_3(\lambda) g_{42}(\lambda) + g_{43}(\lambda)] J_0(\lambda r) d\lambda = 0, \quad r > b \quad (80)$$

$$\frac{(w_z)_{z=h}}{\varepsilon} = -1 \Rightarrow \int_0^{\infty} [C_2(\lambda) g_{31}(\lambda) + C_3(\lambda) g_{32}(\lambda)] J_0(\lambda r) d\lambda = G_{2m}, \quad 0 \leq r \leq b \quad (81)$$

Remarking from the integral formula (27) that the homogeneous Eqs. (78) and (80) are identically satisfied by setting

$$\begin{cases} C_2(\lambda)g_{11}(\lambda)+C_3(\lambda)g_{12}(\lambda)-g_{13}(\lambda)=\sum_{n=0}^{\infty}\beta_nM_n(\lambda a) \\ C_2(\lambda)g_{41}(\lambda)+C_3(\lambda)g_{42}(\lambda)+g_{43}(\lambda)=\sum_{n=0}^{\infty}\gamma_nM_n(\lambda b) \end{cases} \tag{82}$$

We find that

$$C_2(\lambda)=\Delta^{-1}(\lambda)\left[\sum_{n=0}^{\infty}(-\beta_nM_n(\lambda a)g_{42}(\lambda)+\gamma_nM_n(\lambda b)g_{12}(\lambda))-g_{12}(\lambda)g_{43}(\lambda)-g_{13}(\lambda)g_{42}(\lambda)\right] \tag{83}$$

$$C_3(\lambda)=\Delta^{-1}(\lambda)g_{21}(\lambda)\left[\left(\sum_{n=0}^{\infty}\beta_nM_n(\lambda a)+g_{13}(\lambda)\right)\frac{g_{41}(\lambda)}{g_{11}(\lambda)}-\sum_{n=0}^{\infty}\gamma_nM_n(\lambda b)+g_{43}(\lambda)\right] \tag{84}$$

where

$$\Delta(\lambda)=g_{12}(\lambda)g_{41}(\lambda)-g_{11}(\lambda)g_{42}(\lambda) \tag{85}$$

Substituting these functions  $C_2$  and  $C_3$  into Eqs. (78) and (80) we obtain

$$\begin{cases} \sum_{n=0}^{\infty}\int_0^{\infty}[\beta_nM_n(\lambda a)F_1(\lambda)+\gamma_nM_n(\lambda b)F_2(\lambda)]J_0(\lambda r)d\lambda=G_{3m}, 0\leq r\leq a \\ \sum_{n=0}^{\infty}\int_0^{\infty}[\beta_nM_n(\lambda a)F_3(\lambda)+\gamma_nM_n(\lambda b)F_4(\lambda)]J_0(\lambda r)d\lambda=G_{4m}, 0\leq r\leq b \end{cases} \tag{86}$$

We use the Gegenbauer’s formula (30) into Eq. (86), we get

$$\begin{cases} \sum_{n=0}^{\infty}\int_0^{\infty}[\beta_nM_n(\lambda a)F_1(\lambda)+\gamma_nM_n(\lambda b)F_2(\lambda)]X_m(\lambda a)d\lambda=G_{5m}\delta_{0m} \\ \sum_{n=0}^{\infty}\int_0^{\infty}[\beta_nM_n(\lambda a)F_3(\lambda)+\gamma_nM_n(\lambda b)F_4(\lambda)]X_m(\lambda b)d\lambda=G_{6m}\delta_{0m} \end{cases} \tag{87}$$

We obtain the following infinite system of simultaneous equations for obtaining the unknown coefficients  $\beta_n$  and  $\gamma_n$

$$\begin{cases} \sum_{n=0}^{\infty}\int_0^{\infty}[\beta_nB_{nm}+\gamma_nC_{nm}]=G_{5m}\delta_{0m} \\ \sum_{n=0}^{\infty}\int_0^{\infty}[\beta_nD_{nm}+\gamma_nE_{nm}]=G_{6m}\delta_{0m} \end{cases} \tag{88}$$

where

$$\begin{aligned} B_{nm} &= \int_0^{\infty} F_1(\lambda)M_n(\lambda a)X_m(\lambda a)d\lambda \\ C_{nm} &= \int_0^{\infty} F_2(\lambda)M_n(\lambda b)X_m(\lambda a)d\lambda \\ D_{nm} &= \int_0^{\infty} F_3(\lambda)M_n(\lambda a)X_m(\lambda b)d\lambda \\ E_{nm} &= \int_0^{\infty} F_4(\lambda)M_n(\lambda b)X_m(\lambda b)d\lambda \end{aligned} \tag{89}$$

We write the infinite system of the simultaneous Eq. (88) in dimensionless form, then

$$\begin{cases} \sum_{n=0}^{\infty} \int_0^{\infty} [\beta'_n B'_{nm} + \gamma'_n C'_{nm}] = G'_{5m} \delta_{0m} \\ \sum_{n=0}^{\infty} \int_0^{\infty} [\beta'_n D'_{nm} + \gamma'_n E'_{nm}] = G'_{6m} \delta_{0m} \end{cases} \quad (90)$$

where

$$\beta'_n = \frac{\beta_n}{b} \quad (91)$$

$$\gamma'_n = \frac{\gamma_n}{b} \quad (92)$$

and

$$\begin{aligned} B'_{nm} &= \int_0^{\infty} F'_1(t) M_n(\eta t) X_m(\eta t) dt \\ C'_{nm} &= \int_0^{\infty} F'_2(t) M_n(t) X_m(\eta t) dt \\ D'_{nm} &= \int_0^{\infty} F'_3(t) M_n(\eta t) X_m(t) dt \\ E'_{nm} &= \int_0^{\infty} F'_4(t) M_n(t) X_m(t) dt \end{aligned} \quad (93)$$

#### 2.4 Displacements and stresses on two layer boundaries

The displacement on the bottom of the layer is expressed by a following

$$(w_z^*)_{z=0} = \frac{(w_z)_{z=0}}{\varepsilon} = \sum_{n=0}^{\infty} \int_0^{\infty} [\beta_n M_n(\lambda a) F_1(\lambda) + \gamma_n M_n(\lambda b) F_2(\lambda)] J_0(\lambda a) d\lambda - G_{3m} \quad (94)$$

On the upper surface  $z=h$ , the components of the displacement can be calculated

$$(w_z^*)_{z=h} = \frac{(w_z)_{z=h}}{\varepsilon} = \sum_{n=0}^{\infty} \int_0^{\infty} [\beta_n M_n(\lambda a) F_3(\lambda) + \gamma_n M_n(\lambda b) F_4(\lambda)] J_0(\lambda a) d\lambda - G_{4m} \quad (95)$$

The normal stress on the upper surface  $z=0$  for  $r > a$  can be expressed as:

$$(\sigma_z^*)_{z=0} = \frac{(\sigma_z)_{z=0}}{\varepsilon} = \frac{2}{\pi} \sum_{n=0}^{\infty} \beta_n \frac{T_{2n+1}(r/a)}{r \sqrt{a^2 - r^2}} \quad (96)$$

whereas on  $z=h$  we obtain

$$(\sigma_z^*)_{z=h} = \frac{(\sigma_z)_{z=h}}{\varepsilon} = \frac{2}{\pi} \sum_{n=0}^{\infty} \gamma_n \frac{T_{2n+1}(r/b)}{r \sqrt{b^2 - r^2}} \quad (97)$$

The expression obtained for the magnitude of the total load  $P$  of the punch on the layer is using

$$P = -2\pi \int_0^{\infty} (\sigma_z)_{z=h} r dr = -4\varepsilon \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \gamma_n \tag{98}$$

The stress singularity factors corresponding to the studied problem are defined by

$$S_a = \lim_{r \rightarrow a^+} \sqrt{2\pi(r-a)} (\sigma_z^*)_{z=0} \tag{99}$$

$$S_b = \lim_{r \rightarrow b^-} \sqrt{2\pi(r-b)} (\sigma_z^*)_{z=h} \tag{100}$$

Substituting Eqs. (96) and (97) into Eqs. (99), (100). We obtain the simple expression for the stress singularity factors as following

$$S_a = \frac{2}{\eta\sqrt{\pi}} \sum_{n=0}^{\infty} \beta'_n \tag{101}$$

$$S_b = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \gamma'_n \tag{102}$$

### 2.5 Numerical results and discussions

We solve the infinite set of simultaneous Eq. (90) to determine the unknown coefficients  $\beta'_n$  and  $\gamma'_n$ . Replacing the integrals (93) by

$$\begin{aligned} B'_{nm} &\approx \int_0^{t_0} F_1'(\delta t) M_n(\eta t) X_m(\eta t) dt + \int_{t_0}^{\infty} M_n(\eta t) X_m(\eta t) dt \\ C'_{nm} &\approx \int_0^{\infty} F_2'(\delta t) M_n(t) X_m(\eta t) dt \\ D'_{nm} &\approx \int_0^{\infty} F_3'(\delta t) M_n(\eta t) X_m(t) dt \\ E'_{nm} &\approx \int_0^{\infty} F_4'(\delta t) M_n(t) X_m(t) dt + \int_{t_0}^{\infty} M_n(t) X_m(t) dt \end{aligned} \tag{103}$$

As the second and third one converge rapidly whereas  $F_1'$  and  $F_4'$  tend to one at the infinity. The first term on the right hand side of Eq.(103) are integrated numerically by means of Simpson's rule, here, we choose 1000 subintervals and  $t_0=1500$ . The second term is integrated by using the approximate form of the Bessel functions. For large values of  $M_n(t) X_n(t)$  are equivalent to

$$\frac{4}{\pi^2 t^2} \left[ \sin(t) + \frac{(-1)^m}{2} (1 - \cos(2t)) \right] \tag{104}$$

Then, we obtain

$$\int_{t_0}^{\infty} M_n(t) X_n(t) dt \approx \frac{4}{\pi^2 t^2} \left[ \frac{\sin t_0}{t_0} - \text{si}(t_0) + \frac{(-1)^m}{2} \left[ \frac{1 - \cos(2t_0)}{t_0} - 2\text{si}t_0 \right] \right] \tag{105}$$

We choose a steel medium with the parameters shown in Table 3. The thermoelastic coefficients  $\beta'_n$  and  $\gamma'_n$  are shown in the following Tables 4-5 with different values of the layer thickness and the radii  $a$  and  $b$ .

**Table 3**  
Thermal and elastic constants for steel medium.

Temperature (°C)	Coefficient of linear thermal expansion ( $K^{-1}$ )	Poisson ratio ( $\nu$ )
100	$18.8 \times 10^{-6}$	0.29

**Table 4**  
Values of thermoelastic coefficients for  $a/b=0.75$  and various values of  $h/b$ .

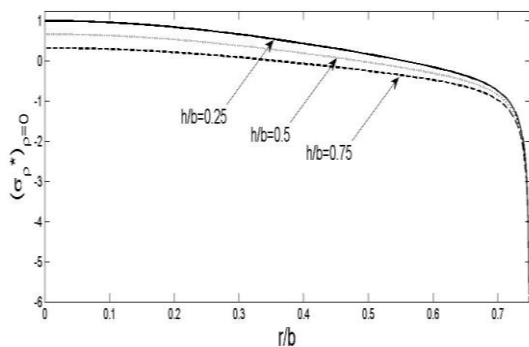
$h/b=0.75$		
$n$	$\beta'_n$	$\gamma'_n$
0	0.037063906323658	0.019588180160849
1	-0.049186362118059	-0.021122729190494
2	0.014005932182722	0.003799932721199
3	-0.000921252780973	-0.000722124317185
4	0.000217822572496	-0.000225587133000
5	0.000104048373888	-0.000129184115299
6	0.000091012514144	-0.000114178946520
7	0.000080598766410	-0.000046097698105
8	0.000067903981412	-0.000055491857328
9	0.000047736467711	0.00000630440299
$h/b=1$		
$n$	$\beta'_n$	$\gamma'_n$
0	0.045364049531263	0.022565770573150
1	-0.042990237267195	-0.014755288555510
2	0.004672496413822	-0.002005426282923
3	-0.000401933474951	-0.001302986863204
4	-0.000042044101045	-0.000839069755529
5	0.000137480440466	-0.000406889047939
6	0.000223299398252	-0.000336483735700
7	0.000254957691014	-0.000138335497595
8	0.000242823222587	-0.000156550725376
9	0.000182609042888	-0.000001362455372
$h/b=1.25$		
$n$	$\beta'_n$	$\gamma'_n$
0	-0.016640424709399	-0.001300588087711
1	-0.015966485330693	-0.001422192925306
2	0.019912931290127	0.001912951387677
3	-0.000407696772340	0.000003423421174
4	0.000609981363588	0.000170537389376
5	-0.000046594450001	0.000071675210362
6	-0.000288234692434	0.000059398101645
7	-0.000390574704145	0.000024530927899
8	-0.000395266378638	0.000027345467725
9	-0.000305969403527	0.000000386708857



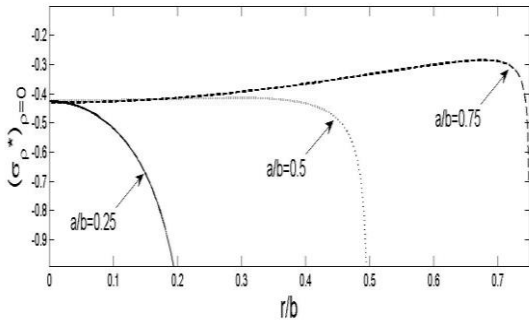
**Table 5**  
Values of thermoelastic coefficients for  $h/b=1.25$  and various values of  $a/b$ .

$a/b=0.5$		
$n$	$\beta'_n$	$\gamma'_n$
0	-0.014860403570175	-0.001788792054877
1	0.007798887911232	0.000402854905408
2	0.001912041704800	0.000777891728993
3	0.000511212249070	0.000082990403248
4	0.000147998786974	0.000098249324534
5	-0.000029331923887	0.000050645427679
6	-0.000113624048535	0.000036533087666
7	-0.000148079358976	0.000019591022413
8	-0.000148208706544	0.000015787688326
9	-0.000114310619160	0.000004227381694
$a/b=0.75$		
$n$	$\beta'_n$	$\gamma'_n$
0	-0.016640424709399	-0.001300588087711
1	-0.015966485330693	-0.001422192925306
2	0.019912931290127	0.001912951387677
3	-0.000407696772340	0.000003423421174
4	0.000609981363588	0.000170537389376
5	-0.000046594450001	0.000071675210362
6	-0.000288234692434	0.000059398101645
7	-0.000390574704145	0.000024530927899
8	-0.000395266378638	0.000027345467725
9	-0.000305969403527	0.000000386708857
$a/b=1$		
$n$	$\beta'_n$	$\gamma'_n$
0	-0.023605662338271	0.006587218140131
1	-0.073818764065537	-0.007207988112731
2	0.075230786150529	0.000917152879540
3	-0.009843668855371	0.000022116365662
4	0.002421516314454	-0.000035634470479
5	-0.000254285212655	-0.000015270475110
6	-0.000717611458449	-0.000024500325864
7	-0.000950907186823	-0.000004441384152
8	-0.000953250812152	-0.000014040435172
9	-0.000734375936788	0.000005200083905

Figs. 5-6 show the variation of the nondimensional normal stress with different values for  $h/b$  and  $a/b$ , respectively. The distribution gets it at maximum values at the centre of the rigid base. The stress has an infinite value  $r/b=a/b$ . It decreases with increasing the layer thickness and decreasing the radius of the rigid base.

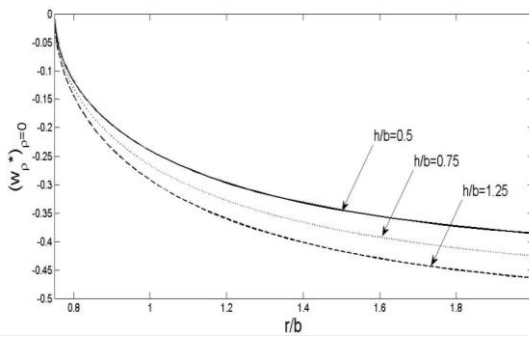


**Fig.5**  
The variation of  $(\sigma^*_\rho)_{\rho=0}$  for  $a/b=0.75$  and various values of  $h/b$ .

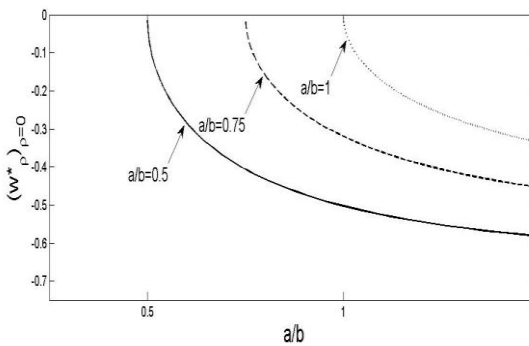


**Fig.6**  
The variation of  $(\sigma_\rho^*)_{\rho=0}$  for  $h/b=1.5$  and various values of  $a/b$ .

The distribution of the nondimensional axial displacement at the edge of the rigid base is given in Fig. 7 with various values of  $h/b$ . It is noted that the value are decreasing with increasing the layer thickness and decreasing the rigid base radius. Graphically they are illustrated in Fig. 8.

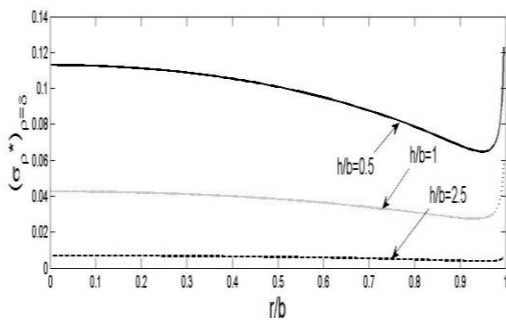


**Fig.7**  
The variation of  $(w_p^*)_{\rho=0}$  for  $a/b=0.75$  and various values of  $h/b$ .

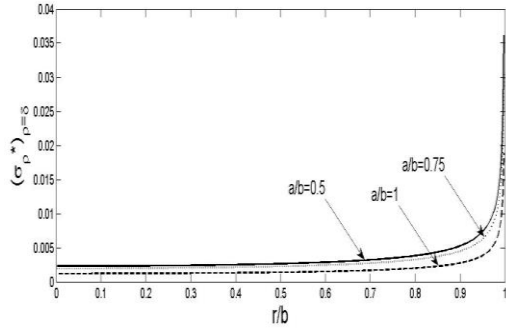


**Fig.8**  
The variation of  $(w_p^*)_{\rho=0}$  for  $h/b=1.25$  with and various values of  $a/b$ .

The nondimensional normal stress in upper surface can be seen from Figs. 9-10. It is shown for various values for  $h/b$  and  $a/b$ , respectively. The distribution gets its maximum values with the centre of the punch. It decreases with increasing the thickness of the elastic layer and the radius of the rigid base.

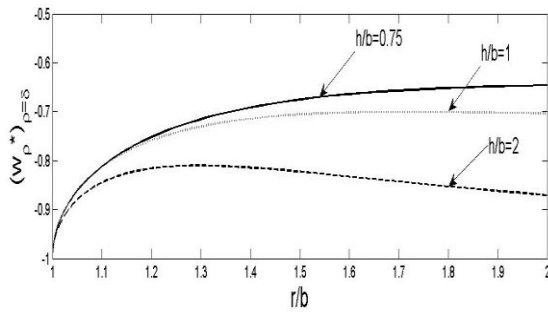


**Fig.9**  
The variation of  $(\sigma_\rho^*)_{\rho=\delta}$  for  $a/b=0.75$  and various values of  $h/b$ .

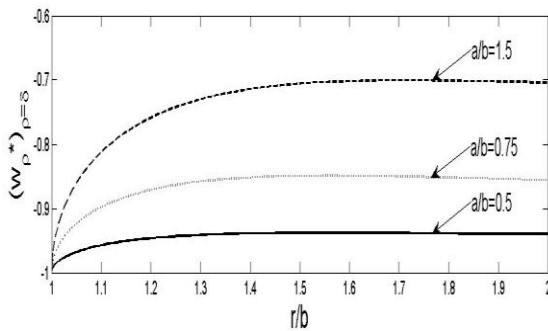


**Fig.10**  
The variation of  $(\sigma_{\rho}^*)_{\rho=\delta}$  for  $h/b=1.5$  and various values of  $a/b$ .

The distribution of the nondimensional displacement at the edge of the punch is shown from Figs. 11-12 with  $h/b$ ,  $a/b$ . It increases with increasing the layer thickness and decreasing the rigid base radius.

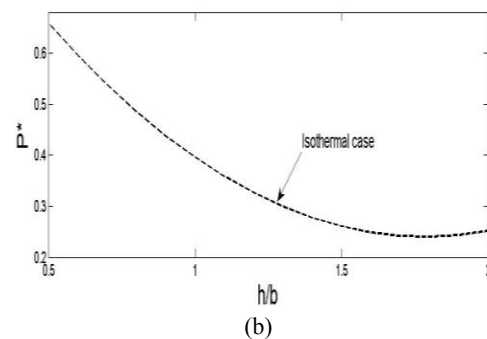
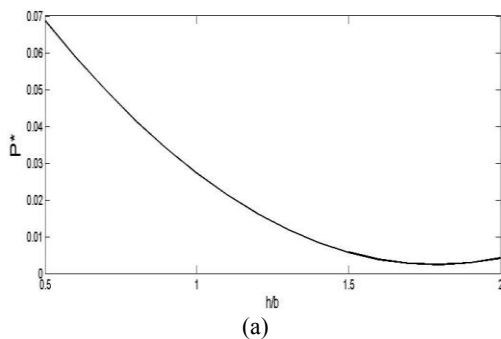


**Fig.11**  
The variation of  $(w_{\rho}^*)_{\rho=\delta}$  for  $a/b=0.75$  and various values of  $h/b$ .

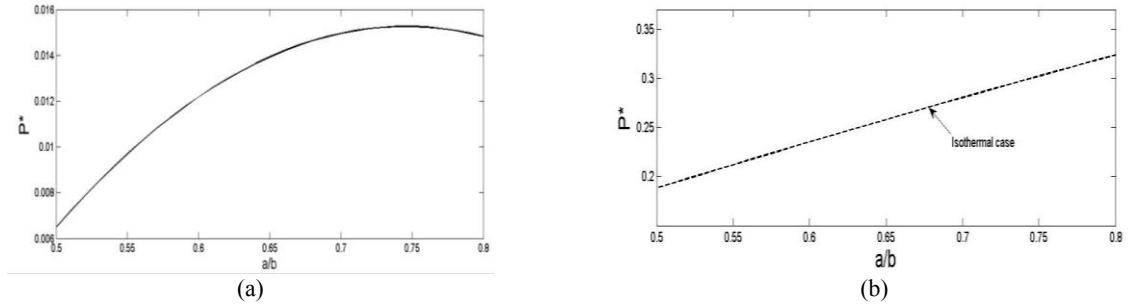


**Fig.12**  
The variation of  $(w_{\rho}^*)_{\rho=\delta}$  for  $h/b=1.5$  and various values of  $a/b$ .

The variations of the total load  $P^* = \frac{P}{4\epsilon}$  applied to the punch with the layer thickness and rigid base are mentioned in Figs. 13-15. It is noted that the value of  $P^*$  changes with the layer thickness and the rigid base radius.

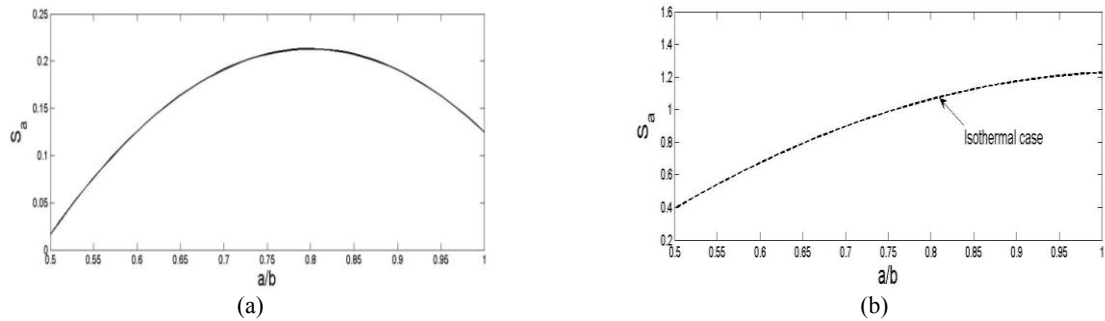


**Fig.13**  
The variation of  $P^*$  for  $a/b=0.75$  and various values of  $h/b$ .

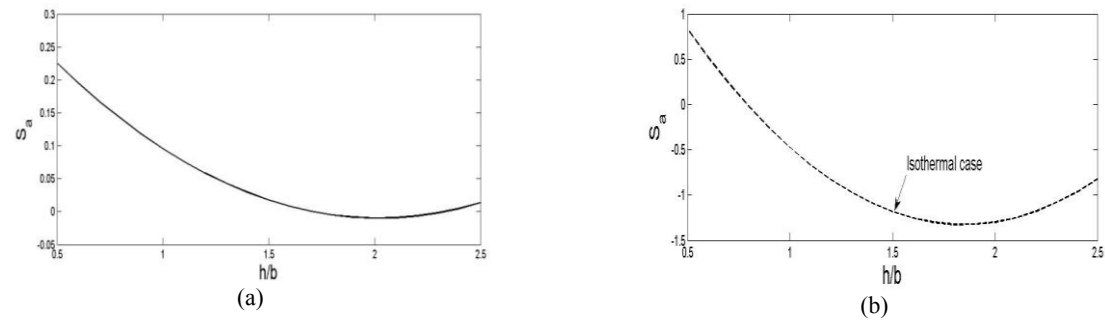


**Fig.14**  
 The variation of  $P^*$  for  $h/b=1.5$  and various values of  $a/b$ .

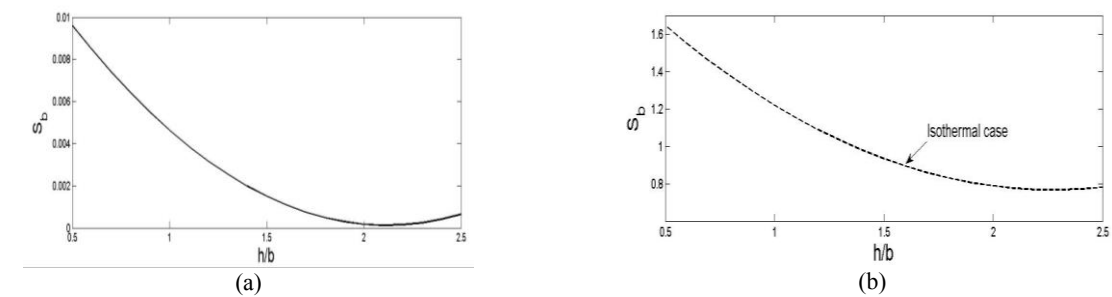
The variation of the stress singularity factors corresponding to the studied problem is graphically illustrated in Figs. 15-18. The stress singularity factors  $S_a$  and  $S_b$ , give a large value with decreasing the layer thickness and increasing with the rigid base radius.



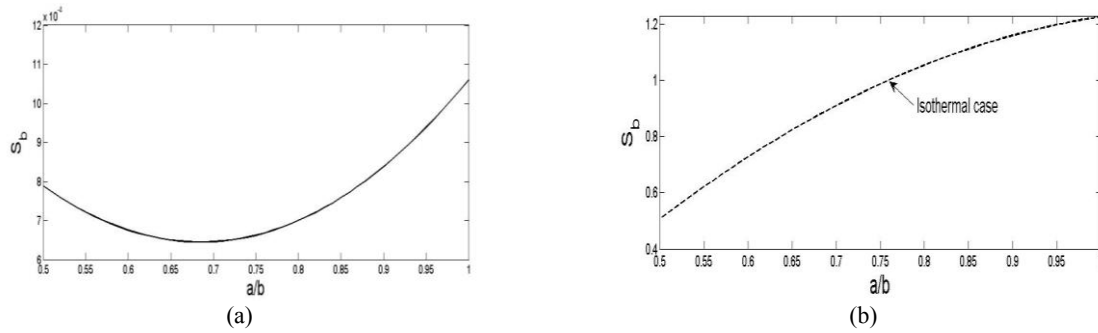
**Fig.15**  
 The variation of  $S_a$  for  $h/b=1.5$  and various values of  $a/b$ .



**Fig.16**  
 The variation of  $S_a$  for  $a/b=0.75$  and various values of  $h/b$ .



**Fig.17**  
 The variation of  $S_b$  for  $a/b=0.75$  and various values of  $h/b$ .



**Fig.18**  
The variation of  $S_b$  for  $h/b=1.5$  and various values of  $a/b$ .

### 3 CONCLUSIONS

In the present paper, we studied a mixed boundary value problem corresponding to a thermoelastic layer. An analytical solution was obtained through an infinite system of simultaneous equations using the Gegenbauer formula. The coefficient can calculate of the series.

The obtained results are summarized as follows:

- Analytical solution based upon the integral Hankel transforms for contact problem have been developed and utilized.
- The analytical solution was obtained using the thermal and the thermoelastic coefficients. An infinite algebraic system has been solved with different values of the elastic layer thickness and the rigid base radius.
- The numerical results revealed the effects of the layer thickness and the radius of the punch with the rigid base on the temperature, the displacement, the normal stress, the load and the stress singularity factors.

The graphs obtained are analyzed as follows:

- The temperature distribution gets its maximum values at the center of the punch, whereas the temperature increases with decreasing the layer thickness and increasing the rigid base radius. It is noted that it becomes infinite in the rigid base edge where it decreases at different distances from it. The variation of the flux in the plane  $z/b=0$  becomes infinite in the rigid base edge. It decreases at different distances from it.
- The variation of the nondimensional normal stress  $(\sigma_{\rho}^*)_{\rho=0}$  for  $h/b$  and  $a/b$  gets its maximum values at the centre of the rigid base. The stress has an infinite value  $r/b=a/b$ . It decreases with increasing layer thickness and decreasing the radius of the rigid base.
- It is noted that, the distribution of the nondimensional displacement  $(w_{\rho}^*)_{\rho=\delta}$  at the edge of the rigid base with various values of  $h/b$  decreases with increasing the layer thickness and decreasing the rigid base radius.
- The nondimensional normal stress in the plane  $z=h/b$  gets its maximum values at the centre of the punch. It decreases with increasing of the elastic layer thickness and the rigid base radius.
- An opposite behaviour is remarked for the distribution of the displacement at the edge of the punch.
- The variations of the total load  $P^*$  applied to the punch with the layer thickness and rigid base as also mentioned. It is noted that the value  $P^*$  changes with the layer thickness and rigid base radius.
- The variation of the stress singularity factors of the problem is graphically illustrated. The stress singularity factors  $S_a$  and  $S_b$ , give a large value with decreasing the layer thickness and increasing with the rigid base radius.
- The graphical result illustrates the effects of the layer thickness and the punch radius with the rigid base on the applied load and the stress singularity factors.
- The obtained graphs for the isothermal case are incomplete agreement with those given by Kebli et al. [12].

## APPENDIX A

The Hankel transform of the displacement components vector  $U(z)$ ,  $W(z)$  and the temperature  $T_H(z)$  are defined as:

$$\begin{aligned} U(z) &= \int_0^{\infty} ru(r, z) J_0(\lambda r) dr \\ W(z) &= \int_0^{\infty} rw(r, z) J_0(\lambda r) dr \\ T_H(z) &= \int_0^{\infty} rT(r, z) J_1(\lambda r) dr \end{aligned} \quad (\text{A.1})$$

The functions  $g_{ij}(\lambda)$  in the system of dual integral Eqs. (78), (79), (80) and (81) are given by

$$\begin{aligned} g_{11}(\lambda) &= 2\nu + (1-2\nu)(2\lambda h + e^{2\lambda h}) \\ g_{12}(\lambda) &= \frac{1}{\lambda} \left[ (1-2\nu)(2(1-\nu)(-1+e^{2\lambda h}) + \lambda h e^{2\lambda h}) + \lambda h(3-2\nu(5-4\nu)) - \lambda \right] \\ g_{13}(\lambda) &= -\frac{\alpha_0 T_0 (1-2\nu)}{2} \int_0^{\infty} \left\{ \frac{b}{\lambda} J_1(\lambda b) (-1+2\lambda h + e^{2\lambda h}) e^{-\lambda h} - \sum_{n=0}^{\infty} a_n M_n(\lambda a) [-1+(1-2\lambda h)e^{-2\lambda h}] \right\} \frac{J_0(\lambda r) d\lambda}{\lambda(1+e^{-2\lambda h})} \\ g_{21}(\lambda) &= \frac{-1+e^{2\lambda h}}{h} \\ g_{22}(\lambda) &= 1+e^{2\lambda h} + \frac{2(1-\nu)(-1+e^{2\lambda h})}{\lambda h} \\ g_{23}(\lambda) &= \frac{\alpha_0 T_0}{2h} \int_0^{\infty} \left\{ \frac{b(-1+e^{2\lambda h})e^{-\lambda h}}{\lambda} J_1(\lambda b) - \sum_{n=0}^{\infty} a_n M_n(\lambda a) \left( \frac{(2(1-\nu)+\lambda h)e^{-2\lambda h} + \lambda h}{2(1-\nu)} - 1 \right) \right\} \frac{J_0(\lambda r) d\lambda}{\lambda(1+e^{-2\lambda h})} \\ g_{41}(\lambda) &= (1-2\nu) \left[ -e^{-\lambda h} + (1+2\lambda h)e^{\lambda h} \right] \\ g_{42}(\lambda) &= \frac{1}{\lambda} \left[ ((1-2\nu)(1+\lambda h)(2(1-\nu)+\lambda h) + 2(-1+\nu(3-\nu))e^{-\lambda h} + \lambda h(\lambda h(1-2\nu)+4\nu(-1+\nu)+1))e^{\lambda h} \right] \\ g_{43}(\lambda) &= \frac{\alpha_0 T_0 (1-2\nu)}{2} \int_0^{\infty} \left\{ \frac{b}{\lambda} J_1(\lambda b) (-e^{-\lambda h} + (1+2\lambda h)e^{\lambda h}) - \sum_{n=0}^{\infty} a_n M_n(\lambda a) (-1-2\lambda h + e^{-2\lambda h}) \right\} \frac{J_0(\lambda r) d\lambda}{\lambda(1+e^{-2\lambda h})e^{\lambda h}} \\ g_{31}(\lambda) &= \frac{2(1-\nu)(1-e^{2\lambda h})}{h} e^{-\lambda h} \\ g_{32}(\lambda) &= \frac{1}{\lambda h} \left[ 4(-1+\nu)^2 e^{-\lambda h} + ((\lambda h)^2 - (2(1-\nu)+\lambda h)^2) e^{\lambda h} \right] \end{aligned} \quad (\text{A.2})$$

and

$$G_{2m} = -\frac{\alpha_0 T_0}{h} \int_0^{\infty} \frac{1}{\lambda} \left\{ \frac{b J_1(\lambda b)}{\lambda} \left[ \frac{2(1-\nu)e^{-2\lambda h} - (2(1-\nu) - \lambda h)}{2} + \frac{((1-\nu) - \lambda h)e^{-2\lambda h} - (1-\nu)e^{-4\lambda h}}{(1+e^{-2\lambda h})} \right] \right\} J_0(\lambda r) d\lambda$$

The functions given in the system Eq. (86) are expressed by

$$\begin{aligned}
 F_1(\lambda) &= \Delta^{-1}(\lambda) [g_{22}(\lambda)g_{41}(\lambda) - g_{21}(\lambda)g_{42}(\lambda)] \\
 F_2(\lambda) &= \Delta^{-1}(\lambda) [g_{12}(\lambda)g_{21}(\lambda) - g_{11}(\lambda)g_{22}(\lambda)] \\
 F_3(\lambda) &= \Delta^{-1}(\lambda) [g_{32}(\lambda)g_{41}(\lambda) - g_{31}(\lambda)g_{42}(\lambda)] \\
 F_4(\lambda) &= \Delta^{-1}(\lambda) [g_{31}(\lambda)g_{12}(\lambda) - g_{32}(\lambda)g_{11}(\lambda)]
 \end{aligned}
 \tag{A.3}$$

and

$$\begin{aligned}
 G_{3m} &= \int_0^\infty \Delta^{-1}(\lambda) \{g_{12}(\lambda) [g_{21}(\lambda)g_{43}(\lambda) + g_{13}(\lambda)g_{42}(\lambda)] - g_{22}(\lambda)F_5(\lambda) - g_{23}(\lambda)\} J_0(\lambda r) d\lambda \\
 G_{4m} &= \int_0^\infty \Delta^{-1}(\lambda) \{g_{31}(\lambda) [g_{12}(\lambda)g_{43}(\lambda) + g_{13}(\lambda)g_{42}(\lambda)] - g_{32}(\lambda)F_5(\lambda)\} J_0(\lambda r) d\lambda + G_{2m} \\
 F_5(\lambda) &= \left\{ g_{11}(\lambda)g_{43}(\lambda) + g_{13}(\lambda)g_{41}(\lambda) - \sum_{n=0}^\infty a_n M_n(\lambda a) \frac{((1-\nu) - \lambda h)e^{-\lambda h} - (1-\nu)e^{-3\lambda h}}{(1+e^{-2\lambda h})} \right\} J_0(\lambda r) d\lambda - 1
 \end{aligned}$$

The functions given in the system Eq. (87) are defined

$$\begin{aligned}
 G_{5m} &= \int_0^\infty \Delta^{-1}(\lambda) \{g_{12}(\lambda) [g_{21}(\lambda)g_{43}(\lambda) + g_{13}(\lambda)g_{42}(\lambda)] - g_{22}(\lambda)F_5'(\lambda) - g_{23}(\lambda)\} X_m(\lambda a) d\lambda \\
 F_5'(\lambda) &= \left\{ g_{11}(\lambda)g_{43}(\lambda) + g_{13}(\lambda)g_{41}(\lambda) - \sum_{n=0}^\infty a_n M_n(\lambda a) \frac{((1-\nu) - \lambda h)e^{-\lambda h} - (1-\nu)e^{-3\lambda h}}{(1+e^{-2\lambda h})} \right\} X_m(\lambda a) d\lambda - 1 \\
 G_{6m} &= \int_0^\infty \Delta^{-1}(\lambda) \{g_{31}(\lambda) [g_{12}(\lambda)g_{43}(\lambda) + g_{13}(\lambda)g_{42}(\lambda)] - g_{32}(\lambda)F_5''(\lambda)\} X_m(\lambda b) d\lambda + G_{2m}' \\
 F_5''(\lambda) &= \left\{ g_{11}(\lambda)g_{43}(\lambda) + g_{13}(\lambda)g_{41}(\lambda) - \sum_{n=0}^\infty a_n M_n(\lambda a) \frac{((1-\nu) - \lambda h)e^{-\lambda h} - (1-\nu)e^{-3\lambda h}}{(1+e^{-2\lambda h})} \right\} X_m(\lambda b) d\lambda - 1 \\
 G_{2m}' &= -\frac{\alpha_0 T_0}{h} \int_0^\infty \frac{1}{\lambda} \left\{ \frac{bJ_1(\lambda b)}{\lambda} \left[ \frac{2(1-\nu)e^{-2\lambda h} - (2(1-\nu) - \lambda h)}{2} + \frac{((1-\nu) - \lambda h)e^{-2\lambda h} - (1-\nu)e^{-4\lambda h}}{(1+e^{-2\lambda h})} \right] \right\} X_m(\lambda b) d\lambda
 \end{aligned}
 \tag{A.4}$$

and

$$\begin{aligned}
 G_{5m}' &= \int_0^\infty \Delta^{-1}(t) \{g_{12}(t) [g_{21}(t)g_{43}(t) + g_{13}(t)g_{42}(t)] - g_{22}(t)F_5'(t) - g_{23}(t)\} X_m(\eta t) dt \\
 F_5'(t) &= \left\{ g_{11}(t)g_{43}(t) + g_{13}(t)g_{41}(t) - \sum_{n=0}^\infty a_n M_n(\eta t) \frac{((1-\nu) - \delta t)e^{-\delta t} - (1-\nu)e^{-3\delta t}}{(1+e^{-2\delta t})} \right\} X_m(\eta t) dt - 1 \\
 G_{6m}' &= \int_0^\infty \Delta^{-1}(t) \{g_{31}(t) [g_{12}(t)g_{43}(t) + g_{13}(t)g_{42}(t)] - g_{32}(t)F_5''(t)\} X_m(t) dt + G_{2m}'' \\
 F_5''(t) &= \left\{ g_{11}(t)g_{43}(t) + g_{13}(t)g_{41}(t) - \sum_{n=0}^\infty a_n M_n(\eta t) \frac{((1-\nu) - \lambda h)e^{-\delta t} - (1-\nu)e^{-3\delta t}}{(1+e^{-2\delta t})} \right\} X_m(t) dt - 1 \\
 G_{2m}'' &= -\frac{\alpha_0 T}{\delta} \int_0^\infty \left\{ \frac{2(1-\nu)e^{-2\delta t} - (2(1-\nu) - \delta t)}{2} + \frac{((1-\nu) - \delta t)e^{-2\delta t} - (1-\nu)e^{-4\delta t}}{(1+e^{-2\delta t})} \right\} \frac{J_1(t)}{t^2} X_m(t) dt
 \end{aligned}
 \tag{A.5}$$

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