The Impact of Channel Estimation on the Number of Active User with Rayleigh Fading Broadcast Channels

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ABSTRACT:
It has already been shown that in rate-constrained broadcast channels, under the assumption of independent Rayleigh fading channels for different receivers, the user capacity (i.e. the maximum number of users that can be activated simultaneously) scales with $\ln(P \ln n)/R_{\min}$ where P is transmit power, $R_{\min}$ represents the minimum rate required for each receiver to be activated, and n denotes the total number of receivers in the system. However, to achieve the aforementioned result, it is assumed that channel state information (CSI) is perfectly known to the receivers. In practical situations, the receivers do not have access to the true CSI and they only know estimated channels. In this paper, the effects of channel estimation is analyzed on the user capacity of rate-constrained broadcast channels. In particular, the Minimum Mean Square Error (MMSE) channel estimation scheme is considered and the effects of this estimation method on the user capacity is investigated. Under the assumption of independent Rayleigh fading channels for different receivers, it is shown that the user capacity scales with $\ln(\hat{\sigma}^2 P \ln n)/R_{\min}$ where $\hat{\sigma}^2$ denotes the variance of estimated channels and is determined by the MMSE channel estimation algorithm. As the received signal model is linear with respect to the fading channels, it is shown that the user capacity scaling law is unchanged and the difference is only a constant factor depending on the channel estimation scheme.

KEYWORDS:
Active users, channel estimation, scaling laws, broadcast channels, fading channels, minimum-rate constraint, power allocation.

I. Introduction

In a dynamic environment, the channel states are time-varying. In the theoretical analysis of wireless communication systems, it is usually assumed that receivers perfectly know the CSI; however, in reality, only estimated channels are available to receivers and transmitters. Depending on the channel estimator used in a communication system, statistical properties of the estimated channels will change compared to the true channels. Some statistical distributions being commonly used to model fading in wireless communication systems are Rayleigh, Rician, and Nakagami distributions [1, Sec. 5.1]. In many applications, it is of interest to analyze the effect of channel estimation on performance of a wireless communication system.

The basic idea is to adapt power allocation to the variations of the channel states. The transmission rate for a receiver is increased when its channel state becomes better; therefore, higher rates can be achieved using less power. This raises the issue of the trade-off between ergodic capacity and outage capacity, for which, extensive studies have been given in [2]–[4] in the context of broadcast channels.In this paper, a rate-constrained broadcast channel is considered and an opportunistic power allocation scheme with a minimum rate constraint $R_{\min} > 0$ is utilized. Since for a fixed $R_{\min}$, in a time-varying fading environment, it may not be always possible for all receivers to achieve this minimum rate simultaneously, a scheme has been proposed in [5] to maximize the number of active receivers, for each of which, such a minimum rate can be supported, while allocating no power to the other inactive receivers. As the number of supportable active receivers depends on the specific channel states, the asymptotic behavior should be analyzed when the total number of receivers n is large. In [5], under the assumption of independent Rayleigh fading channels for different receivers with unit noise variance, it is shown that the maximum number of active receivers is very close to $\ln(P \ln n)/R_{\min}$ with probability approaching 1, and $R_{\min}$ is in the unit of nats. In this paper, the effects of channel estimation is analyzed on the user capacity of rate-constrained broadcast channels. In particular, the Minimum Mean Square Error (MMSE) channel estimation scheme is considered and the effects of this estimation method on the user capacity is investigated. Under the assumption of independent Rayleigh fading channels for different receivers, it is shown that the user capacity scales with $\ln(\hat{\sigma}^2 P \ln n)/R_{\min}$

where $\hat{\sigma}^2$ denotes the variance of estimated channels and is determined by the MMSE channel estimation algorithm.

II. Channel Model

Consider a broadcast channel with one transmitter and n receivers with the following channel model in the time block $t = 1, 2, \ldots, T$:

$$Y_i(t) = g_i X(t) + Z_i(t), \quad i = 1, 2, \ldots, n, \tag{1}$$

where $X(t) \in \mathbb{C}$ is the signal sent by the transmitter, and $Y_i(t) \in \mathbb{C}$ is the signal received by receiver i. The noise $Z_i(t) \in \mathbb{C}$, $i = 1, \ldots, n$, $t = 1, \ldots, T$ are assumed to be i.i.d. complex Gaussian distributed according to $\mathcal{CN}(0, 1)$. The channel gains $g_i \in \mathbb{C}$, $i = 1, \ldots, n$ are assumed to be constant during this time block.

Equivalently, the model (1) can be written as

$$Y_i'(t) = X(t) + Z_i(t)/g_i, \quad i = 1, 2, \ldots, n \tag{2}$$

where the noise $Z_i(t)/g_i$ is still complex Gaussian distributed, but with variance $1/|g_i|^2$.

III. MMSE Channel Estimation

Coherent demodulation requires the complex channel tap $g_i = |g_i|e^{j\theta_i}$; $i = 1, \ldots, n$, to be available via perfect channel estimation. In practice, g_i is estimated from pilot symbols extracted from the pilot tone transmitting simultaneously with the signal (IS-95). The pilot symbol (i.e. s_p) is known but noisy and hence one wishes to have the best estimate. The time between the pilot symbols is much smaller than the channel coherence time T_c to reduce channel estimation error (slow fading). In this case the channel taps can be considered constant during a symbol, that is, $g_i(t) = g_i, \quad t \leq T_s$.

It is well-known that the MMSE estimated channel is given by [6, Sec. 10.3]

$$\hat{g} = a^* Y \tag{3}$$

$$a = \frac{\mathbf{E}\left\{|g|^2\right\} s_p}{\mathbf{E}\left\{|g|^2\right\} |s_p|^2 + \sigma_Z^2} \tag{4}$$

or equivalently, with respect to the Signal-to-Noise Ratio (SNR), we have

$$a = \frac{SNR_p}{SNR_p + 1}\left(\frac{s_p}{|s_p|^2}\right) \tag{5}$$

where σ_Z^2 is the noise variance and $SNR_p = |s_p|^2/\sigma_Z^2$ is the SNR of the pilot symbol transmitted. The estimate gets more accurate as the SNR increases. Hence, it can be seen that the distribution of $\hat{g}$ is complex Gaussian with zero mean and variance $\hat{\sigma}^2 = |a|^2(\sigma_h^2 |s_p|^2 + 1)$. where σ_h^2 denotes to actual channel gain variance. That is,

$$\hat{g} \sim \mathcal{CN}\left(0, \hat{\sigma}^2\right) \tag{6}$$

Note that because of linearity of the observation model, other classical estimators such as Maximum Likelihood (ML) estimators result in complex Gaussian random variable.

IV. Power Allocation

Let $N_i = 1/|\hat{g}_i|^2$. Without loss of generality, assume that $N_1 \leq N_2 \leq \cdots \leq N_n$. It is well known [7, Sec.14.6] that the broadcast channel (2) is stochastically degraded, and the capacity region is given by

$$R_i < \ln\left(1 + \frac{P_i}{\sum_{j=1}^{i-1} P_j + N_i}\right), \quad i = 1, \ldots, n \tag{7}$$

where R_i is the achievable rate for receiver i, to which, the power $P_i \geq 0$ is allocated by the transmitter under the total transmit power constraint: $\sum_{i=1}^{n} P_i = P$.

Different rates can be achieved by different power allocations in (7). As shown by Lemma 2.1 in [5], in order to maximize the total throughput, all power should be allocated to receiver 1, which has the maximum channel gain $|\hat{g}_1|$, or the minimum equivalent noise variance N_1. Hence, the following power allocation scheme is proposed in [5]

$$\begin{cases} \max\{m\} & (8) \\ \ln\left(1 + \frac{P_1}{N_1}\right) \geq R_{\min} & (9) \\ \ln\left(1 + \frac{P_i}{\sum_{j=1}^{i-1} P_j + N_i}\right) = R_{\min}; \quad 2 \leq i \leq m & (10) \\ \sum_{i=1}^{m} P_i = P & (11) \end{cases}$$

where $R_{\min} > 0$ (in nats) is a minimum rate constraint for all active receivers. In [5], a simple recursive algorithm is also proposed to solve the aforementioned optimization problem (8)-(11). Obviously, with fixed P and $R_{\min}$, the maximum number of active receivers completely depends on the equivalent noise variance $N_i = 1/|\hat{g}_i|^2$, . When the estimated channel gains $\hat{g}_i$ obey some statistical distribution, asymptotic behavior of the maximum m can be determined when the total number of receivers n becomes large. It is of interest to analyze how the user capacity of broadcast channels obtained asymptotically by Theorem 2.1 in [5] changes as the estimated channel gains are substituted for the true gains. The following Theorem shows the effect of MMSE channel estimation on the user capacity scaling law.

Theorem 4.1: Assume the estimated channel gains are distributed by (6) for different receivers in a broadcast channel. For any arbitrary $\epsilon > 0$, the maximum number of active receivers m determined by (8)-(11) is bounded as

$$\mathbb{P}(\lfloor \nu(n) - \epsilon \rfloor \leq m \leq \nu(n) + \epsilon) \to 1, \quad \text{as } n \to \infty, \tag{12}$$

where, $\lfloor x \rfloor$ denotes the maximum integer no greater than x, n is the total number of receivers, and

$$\nu(n) = \ln(\hat{\sigma}^2 P \ln n)/R_{\min}. \tag{13}$$

Proof: Consider the broadcast channel (1), with the independent gains $\hat{g}_i \sim \mathcal{CN}\left(0, \hat{\sigma}^2\right)$ for $i = 1, \ldots, n$. Then, $|\hat{g}_i| \sim Rayleigh\left(\sqrt{\hat{\sigma}^2/2}\right)$ and $|\hat{g}_i|^2 \sim \Gamma\left(1, \hat{\sigma}^2\right)$ for $i = 1, \ldots, n$. The Gamma cumulative distribution function is given by

$$\mathbf{F}(x; k, \theta) = \frac{\gamma(k, x/\theta)}{\Gamma(k)}$$

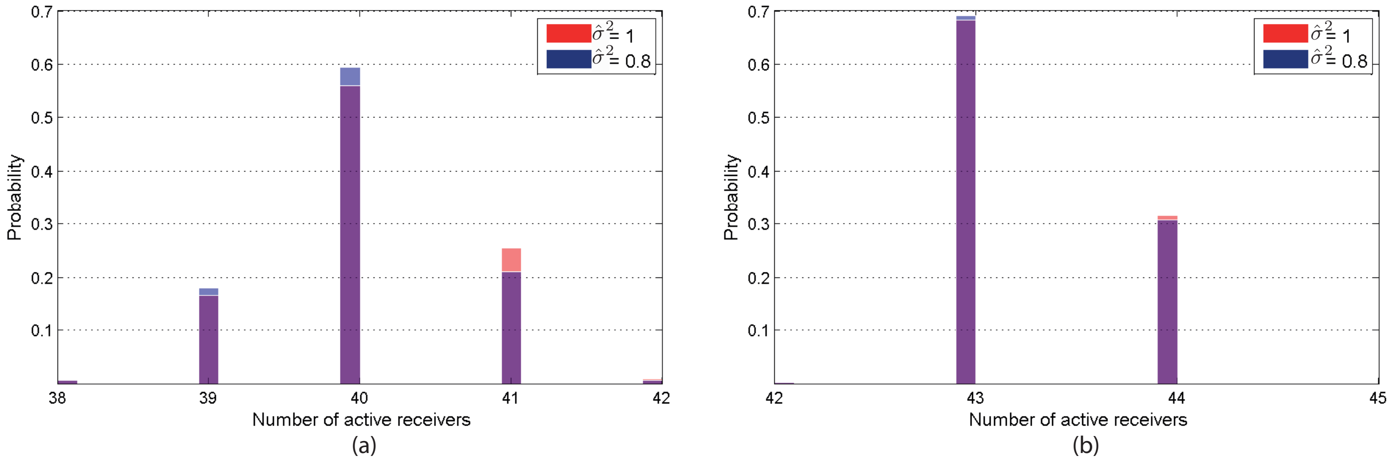

Fig. 1. The histogram of the number of active receivers for $R_{\min} = 25$ Kbps, SNR$= 40$ dB, and (a) $n = 50$, $\nu_{old}(n) = 42.29$, $\nu(n) = 41.40$ and (b) $n = 1000$, $\nu_{old}(n) = 44.57$, $\nu(n) = 43.67$.

where $\gamma(.,.)$ is the lower incomplete gamma function. Hence, for any $y > 0$,

$$\begin{aligned}\mathbf{F}(y) &= \mathbb{P}(N_i < y) = \mathbb{P}(1/|\hat{g}_i|^2 < y) = \mathbb{P}\left(|\hat{g}_i|^2 > 1/y\right) \\ &= e^{-\frac{1}{\hat{\sigma}^2 y}} \end{aligned} \tag{14}$$

It can be seen that MMSE channel estimation slightly changes the distribution function of $|g_i|^2$; $i = 1, \ldots, n$, from Exp(1) in [5] to $\Gamma\left(1, \hat{\sigma}^2\right)$ in this paper. The rest of the proof is basically similar to the proof of Theorem 2.1 in [5] with some modifications due to the Gamma distribution function.

For any fixed $N_0 > 0$, we can characterize the number of "good" channels with the equivalent noise variance N_i less than N_0 as the following. Let $p_0 = \mathbf{F}(N_0) = e^{-\frac{1}{\hat{\sigma}^2 N_0}}$. Then, with probability p_0, a channel is good. Consider a Bernoulli sequence:

$$x_i = \begin{cases} 1, & \text{with probability } p_0 \\ 0, & \text{with probability } 1 - p_0 \end{cases}$$

for $i = 1, 2, \ldots, n$. Then, the number of good channels has the same distribution as $X = \sum_{i=1}^{n} x_i$, which satisfies the binomial distribution $B(n, p_0)$.

For any integer $m \geq 1$, obviously,

$$\mathbb{P}(X \leq m - 1) = \sum_{j=0}^{m-1} \binom{n}{j} p_0^j (1 - p_0)^{n-j}$$

which, however, is not easy to analyze. If $m - 1 \leq np_0$, we can use the Chernoff inequality [8, page 70]:

$$\mathbb{P}(X \leq m - 1) \leq \exp\left(-\frac{1}{2p_0}\frac{(np_0 - m + 1)^2}{n}\right).$$

Hence,

$$\mathbb{P}(X \geq m) \geq 1 - \exp\left(-\frac{1}{2p_0}\frac{(np_0 - m + 1)^2}{n}\right). \tag{15}$$

Now, consider the following power allocations for the m best receivers:

$$P_i = \frac{c}{\alpha^{m-i}}, \text{ for } i = 1, \ldots, m,$$

where $\alpha = e^{R_{\min}} > 1$, and $c = (1 - 1/\alpha)P$. It is easy to check that the total power constraint is satisfied.

$$\sum_{i=1}^{m} \frac{c}{\alpha^{m-i}} = c\frac{1 - (1/\alpha)^m}{1 - 1/\alpha} \leq c\frac{1}{1 - 1/\alpha} = P.$$

If $\max_{1 \leq i \leq m} N_i \leq P/\alpha^m$, the following lower bound is achieved for the SINR's at all these m receivers. That is, for $i = 1$,

$$\frac{P_1}{N_1} \geq \frac{c/\alpha^{m-1}}{P/\alpha^m} = \alpha - 1,$$

and for any $i = 2, \ldots, m$,

$$\begin{aligned}\frac{P_i}{\sum_{j=1}^{i-1} P_j + N_i} &\geq \frac{c/\alpha^{m-i}}{\sum_{j=1}^{i-1} c/\alpha^{m-j} + P/\alpha^m} \\ &= \frac{1/\alpha^{m-i}}{\frac{(1/\alpha)^{m-i+1} - (1/\alpha)^m}{1 - 1/\alpha} + \frac{(1/\alpha)^m}{1 - 1/\alpha}} = \alpha - 1.\end{aligned}$$

Then, obviously, the minimum rate constraint is satisfied for all these m receivers, since

$$\ln\left(1 + (\alpha - 1)\right) = \ln\alpha = R_{\min}.$$

Next, we show that for any $\epsilon > 0$, if $m \leq \nu(n) - \epsilon$, $\max_{1 \leq i \leq m} N_i \leq P/\alpha^m$ holds with probability approaching one as n tends to infinity. Let $N_0 = P/\alpha^m$. Then,

$$\begin{aligned}p_0 = \mathbf{F}(N_0) = \exp\left(-\frac{\alpha^m}{\hat{\sigma}^2 P}\right) &\geq \exp\left(-\frac{\alpha^{\nu(n)-\epsilon}}{\hat{\sigma}^2 P}\right) \\ &= \exp\left(-\alpha^{-\epsilon}\ln n\right) = n^{-\lambda},\end{aligned}$$

where $\lambda = \alpha^{-\epsilon} < 1$. Then it is obvious that as $n \to \infty$,

$$\frac{1}{2p_0}\frac{(np_0 - m + 1)^2}{n} \sim \frac{n^2 p_0^2}{2np_0} = \frac{np_0}{2} \geq \frac{n^{1-\lambda}}{2} \to \infty. \tag{16}$$

Hence, by (15), the probability of $\max_{1 \leq i \leq m} N_i \leq \alpha^m P$ approaches 1 as $n \to \infty$.

Therefore, we proved that as $n \to \infty$, with probability approaching 1, there are at least $m = \lfloor \nu(n) - \epsilon \rfloor$ good channels with $N_i \leq P/\alpha^m$, for which the minimum rate constraint is satisfied.

Next, we prove the upper bound, i.e., $m \leq \nu(n) + \epsilon$ holds with probability approaching 1. First, we show that for any $\delta > 0$, for sufficiently large m, the best receiver should have the equivalent noise variance $N_1 \leq P_\delta/\alpha^m$, with $P_\delta := P + \delta$. Otherwise, if $\min_{1 \leq i \leq n} N_i > P_\delta/\alpha^m$, according to the minimum rate constraint,

$$\frac{P_i}{\sum_{j=1}^{i-1} P_j + N_i} \geq \alpha - 1. \quad \text{for} i = 1, 2, \ldots, m,$$

Hence, we have

$$P_1 \geq (\alpha - 1)N_1 > (\alpha - 1)P_\delta/\alpha^m,$$

and inductively, for $i = 2, \ldots, m$,

$$\begin{aligned} P_i &\geq (\alpha - 1)\left(\textstyle\sum_{j=1}^{i-1} P_j + N_i\right) \\ &> (\alpha - 1)\left(\textstyle\sum_{j=1}^{i-1} (\alpha - 1)P_\delta/\alpha^{m-j+1} + P_\delta/\alpha^m\right) \\ &= (\alpha - 1)P_\delta/\alpha^{m-i+1}, \end{aligned}$$

which violates the total power constraint since

$$\sum_{i=1}^{m} P_i > \sum_{i=1}^{m} (\alpha - 1)P_\delta/\alpha^{m-i+1} = (1 - 1/\alpha^m)P_\delta > P$$

for sufficiently large m.

Therefore, to show that

$$\mathbb{P}(m \leq \nu(n) + \epsilon) \to 1,$$

or

$$\mathbb{P}(m > \nu(n) + \epsilon) \to 0,$$

we only need to show that

$$\mathbb{P}(N_1 \leq P_\delta/\alpha^{\nu(n)+\epsilon}) \to 0.$$

Let $p_1 = \mathbf{F}(P_\delta/\alpha^{\nu(n)+\epsilon})$. Then, $(1 - p_1)^n$ is the probability that all the receivers have equivalent noise variance greater than $P_\delta/\alpha^{\nu(n)+\epsilon}$. Hence,

$$\mathbb{P}(N_1 \leq P_\delta/\alpha^{\nu(n)+\epsilon}) = 1 - (1 - p_1)^n, \qquad (17)$$

which tends to 0 if and only if

$$\left(1 - \exp\left(-\frac{\alpha^{\nu(n)+\epsilon}}{\hat{\sigma}^2 P_\delta}\right)\right)^n \to 1. \qquad (18)$$

Since

$$\left(1 - \exp\left(-\frac{\alpha^{\nu(n)+\epsilon}}{\hat{\sigma}^2 P_\delta}\right)\right)^{\exp\left(\frac{\alpha^{\nu(n)+\epsilon}}{\hat{\sigma}^2 P_\delta}\right)} \to e^{-1},$$

(18) holds if

$$n \cdot \exp\left(-\frac{\alpha^{\nu(n)+\epsilon}}{\hat{\sigma}^2 P_\delta}\right) = n \cdot \exp\left(-\frac{P\alpha^\epsilon \ln n}{P + \delta}\right) \to 0, \qquad (19)$$

which holds by choosing $\delta < (\alpha^\epsilon - 1)P$. ∎

Remark 4.1: Theorem 4.1 states that the number of active receivers is close to $\nu(n)$ with high probability. Comparing $\nu(n)$ obtained by Theorem 4.1 and the one presented in [5], it can be seen that the difference is only a constant factor depending on the channel estimation algorithm. Indeed, as the signal model in (1) is linear, the distribution of estimated channels given by (6) is still Gaussian and only the variance has been changed.

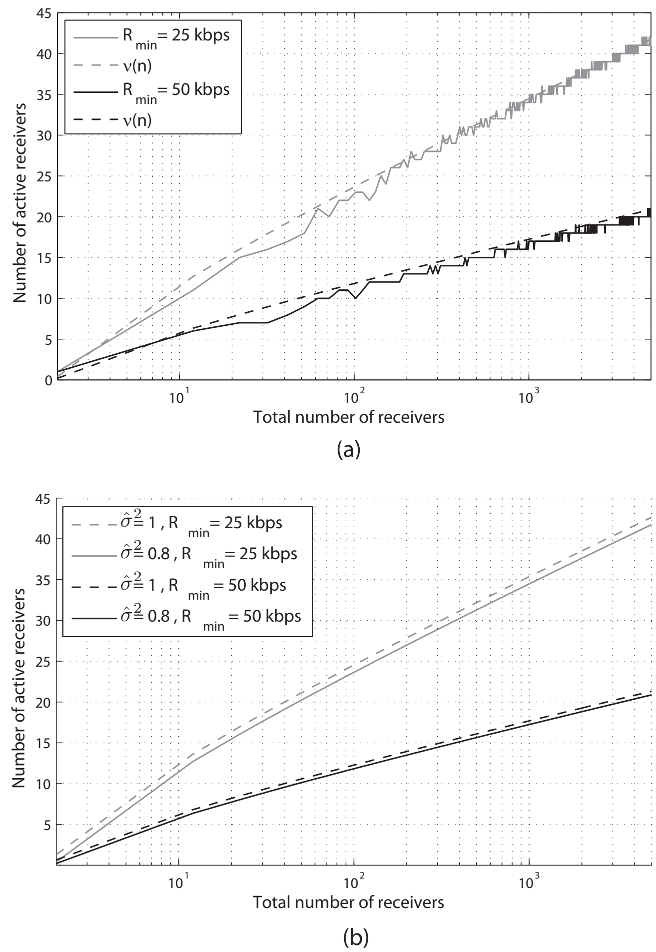

Fig. 2. The optimal number of active receivers versus the total number of users for $R_{\min} = 25, 50$ Kbps and Linearly increasing transmit power (i.e. $P = n$). (a) experimental and theoretical results with $\hat{\sigma}^2 = 0.8$ and (b) theoretical bounds $\hat{\sigma}^2 = 0.8, 1$.

V. Simulations

Consider the broadcast channel model in (1) with bandwidth of $B = 100$ KHz. Figure 1 indicates the histogram of the number of active users for $R_{min} = 25$ Kbps and $SNR = 40$ dB. In Figure 1.a, the total number of receivers equals 50 and in Figure 1.b, the total number of users equals 1000. The number of active users given by Theorem 4.1 (i.e. $\nu(n)$) and Theorem 2.1 in [5] (i.e. $\nu_{old}(n)$) are also shown in Figure 1 for different values of $\hat{\sigma}^2$. Note that the purple color shows overlapping of two histograms. It can clearly be seen that $\nu(n)$ presented by Theorem 4.1 provides better estimates of the number of active users that $\nu_{old}(n)$ obtained in [5]. Hence, considering the channel estimation effect results in more accurate scaling laws.

Figure 2 shows the optimal number of active users versus the total number of users for $R_{min} = 25, 50$ Kbps and linearly increasing transmit power $P = n$. The value of $\nu(n)$ given by (13) is also indicated in Figure 2. As shown in Figure 2, the number of active users is almost doubled as $R_{\min}$ is halved. In Figure 2.a, both numerical results and the theoretical bound given by (13) are shown for $\hat{\sigma}^2 = 0.8$. Note that the curves drawn in Figure 2.a is a single realization, not the Monte-Carlo average. Figure 2.b only indicates the theoretical bound for $\hat{\sigma}^2 = 0.8, 1$ and illustrates the effect of $\hat{\sigma}^2$ and R_{min} on the scaling law.

VI. Conclusion

In [5], under the assumption of independent Rayleigh fading channels for different receivers with unit noise variance, it is shown that the maximum number of active receivers is very close to $\ln(P \ln n)/R_{\min}$ with probability approaching 1, and $R_{\min}$ is in the unit of nats. In this paper, the effects of channel estimation is analyzed on the user capacity of rate-constrained broadcast channels. In particular, the MMSE channel estimation scheme is considered and the effects of this estimation method on the user capacity is investigated. Under the assumption of independent Rayleigh fading channels for different receivers, it is shown that the user capacity scales with $\ln(\hat{\sigma}^2 P \ln n)/R_{\min}$ .It can clearly be seen that $\nu(n)$ presented by Theorem 4.1 provides better estimates of the number of active users that $\nu_{old}(n)$ obtained in [5]. Hence, considering the channel estimation effect results in more accurate scaling laws.

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