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The Effect of Boundary Conditions and Concentrated Mass on The Performance and Vibration Results of Piezoelectric Bimorph Beam by Parallel and Series Layers

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Abstract: A variety of parameters influence the performance of piezoelectric sensors and actuators, such as support and concentrated mass. This paper presents a finite element formulation for piezoelectric structures and studies the effect of parameters on them. This method was developed based on the Bernoulli-Euler beam and the model is considered for use as a beam structure using the Variation Principle. The model was used for static and vibration analysis. The effects of support on the deflection of the piezoelectric beam were studied. Modal analysis was also carried out for the electromechanical coupling and uncoupling beams, and the effect of the concentrated mass was deduced. The finite element model was developed with FORTRAN programming Language and was implemented with MATLAB software. A comparison of the results between the analytical method, engineering software, and this program, showed acceptable accuracy.

Keywords: Actuator, Finite Element, Frequency, Piezoelectric Beam, Support

Biographical notes: Majid Jabbari is an assistant professor at the Department of Mechanical Engineering in Islamic Azad University, Khomeinishahr Branch. His current research interests are in the field of intelligent piezoelectric structures and computer-aided design and engineering. **Sofia Jabbari** is studying in the field of medical equipment at Isfahan Healthcare Center at University of Applied Sciences and Technology. His current research interest is the field of smart structures in medical.

Research paper

1 INTRODUCTION

The behavior of piezoelectric structures has been extracted by the coupling of electrical and mechanical parameters. In order to study this model, the finite element method can be applied. Finite element formulations for modeling piezoelectric structures have been used in many studies.

The first application of the finite element of the piezoelectric model was extracted by Allik and Hughes [1]. Zemcik [2] developed a piezoelectric shell element and implemented it with ANSYS software. Lazarus [3] presented a finite element model for the nonlinear vibrations of piezoelectric layered beams with application in NEMS. The element for the modeling of smart structures was studied by Kogl and Bucalem [4]. This element was then used by Lazarus [3]. Piefort and Preumont [5] used the Mindlin shell elements for piezoelectric materials. The response can respond in the low amplitude solution to harmonic excitation; Sebald et al. [6] suggested a method to excite the system to jump to the high amplitude solution for broadband piezoelectric energy harvesting. Erturk and Inman [7] investigated the dynamic response, including the chaotic response on high-energy orbits of the bistable Duffing oscillator with electromechanical coupling. Friswell et al [8] proposed a cantilever beam with a tip mass that is mounted vertically and excited in the transverse direction at its base. This device was highly non-linear with two potential wells for large tip masses when the beam was buckled. Bendigeri et al [9] developed finite elements for the dynamic analysis of a structure with the piezoelectric property. An eight nodded isoparametric three-dimensional hexahedral element was improved to model the coupled electro-mechanical behavior. In this work, the effects due to piezoelectric for the developed finite element are explained. Ghayour and Jabbari [10] presented the effect of support and concentrated mass on the performance of a piezoelectric beam actuator and frequencies through the finite element method. They also developed a new formulation for coupling beam elements on the numerical solution of the dynamic behavior of nonlinear piezoelectric beams [11].

Jabbari et al [12] studied the energy harvesting of a multilayer piezoelectric beam in resonance and off-resonance cases. They showed that the maximum value of electric power occurs at the optimum resistive load for the selected frequency value and the behavior of energy harvesting depends greatly on the excitation frequency.

Jabbari et al [13] presented the experimental and numerical results of the dynamic behaviour of a nonlinear piezoelectric beam. They showed the effects of the excitation velocity and the position of the concentrated mass on the output voltage.

The effect of strain nodes on the energy harvesting of the cantilever piezoelectric beam with the vibration mode excitation was presented by Jabbari [14]. This research showed that the resistive load affects the voltage and generated power and the optimum resistive load is considered for segmented and continuous electrodes, and then the power output is verified.

Jabbari and Ahmadi [15] studied the electric response of piezoelectric beam using the dynamic stiffness method. In this research, the dynamic stiffness matrix is developed for a two-segmented beam with a tip mass.

Hassannejad et al [16] presented the influence of non-circular cross-section shapes on the torsional vibration of a micro-rod. They demonstrated that the natural frequency of the micro-rod is completely affected by the shape and aspect ratio of the cross-section. These results can be useful in the micro-structure design stage.

Shameli et al [17] studied free torsional vibration analysis of nanorods with non-circular cross-sections**.** They showed that a small reduction can be observed in the natural frequencies by increasing cross-sectional dimensions.

The element developed within this paper is considered as beam. The node of the beam element has four degrees of freedom. The model presented was developed for piezoelectric actuators and sensors. The object of this research is to study model behavior under variable support and the concentrated mass effect on the frequency analysis of piezoelectric beam. These parameters influence the performance factor for energy harvesting. The results of the developed research are compared with the results of analytical and engineering software.

2 THE FINITE ELEMENT MODEL FOR THE PIEZOELECTRIC BEAM

The model of the beam is a piezoelectric bimorph which can be used as an actuator and sensor. The proposed element contains two nodes, and each node has two structural degrees of freedom (u, θ), and two electrical degrees of freedom φ and ψ ("Fig. 1"). The deflection function $u(x)$, and electrical potential φ across the beam length and thickness are evaluated by "Eq. (1)" [2].

$$u(x)=\left[N^{u}\right]\{\hat{u}\} \qquad \varphi=\left[N^{\varphi}\right]\{\hat{\varphi}\}$$
$$\{\hat{u}\}=\left\{u_{n} \quad \theta_{n} \quad u_{n+1} \quad \theta_{n+1}\right\}^{T}$$
$$\{\hat{\varphi}\}=\left\{\varphi_{n} \quad \psi_{n} \quad \varphi_{n+1} \quad \psi_{n+1}\right\}^{T}$$
$$\left[N^{u}\right]=[N^{1u}\ N^{1\theta}\ N^{2u}\ N^{2\theta}]$$
$$\left[N^{\varphi}\right]=[N^{1\varphi}\ N^{1\psi}\ N^{2\varphi}\ N^{2\psi}] \tag{1}$$

Where $\{\hat{u}\}$, is the displacement vector of the nodes, $\{\hat{\varphi}\}$, is the nodes potential, $[N^u]$ shows the shape functions of structural degrees of freedom (u, θ) and $[N^{\varphi}]$, shows the shape functions of electrical degrees of freedom (φ, ψ). The strain S, and the electric field vector $\{E\}$, can be expressed as "Eq. (2)".

$$S = \left[B^u\right]\{\hat{u}\} \quad \{E\} = -\left[B^{\varphi}\right]\{\hat{\varphi}\} \tag{2}$$

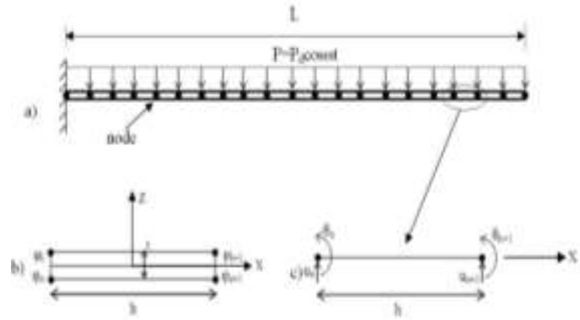

Fig. 1 (a): The finite element model of the piezoelectric beam, (b): Nodes potential degrees of freedom, and (c): Nodes displacement degrees of freedom.

$$\left[B^u\right] = z\frac{d^2}{dx^2}\left[N^u\right] = z\left[\frac{6}{h^2}\left(\frac{2x}{h}-1\right) \quad \frac{-2}{h}\left(\frac{3x}{h}-2\right) \quad \frac{6}{h^2}\left(-\frac{2x}{h}+1\right) \quad \frac{-2}{h}\left(\frac{3x}{h}-1\right)\right] \tag{3}$$

$$\left[B^{\varphi}\right] = -\nabla\left[N^{\varphi}\right] = -\begin{Bmatrix} \partial/\partial x \\ \partial/\partial z \end{Bmatrix}\left[N^{\varphi}\right] = \begin{bmatrix} \frac{1}{2h}+\frac{1}{2hH}z & \frac{1}{2h}-\frac{1}{2hH}z & -\frac{1}{2h}-\frac{1}{2hH}z & -\frac{1}{2h}+\frac{1}{2hH}z \\ \frac{1}{2H}+\frac{1}{2hH}x & \frac{1}{2t}-\frac{1}{2hH}x & -\frac{1}{2hH}x & \frac{1}{2hH}x \end{bmatrix} \tag{4}$$

Where $[B^u]$ and $[B^{\varphi}]$ are the shape function derivatives, H is the thickness and h is the element length. The motion Equations of a piezoelectric structure are obtained with the Hamilton principle. The finite element formulation can be derived with motion Equation. ("Eq. (5)", "Eq. (6)") [3].

$$[m]\{\ddot{u}^n\} + \left[K^{uu}\right]\{u^n\} + \left[K^{u\phi}\right]\{\phi^n\} = \int_V \left[B^u\right]^T \{T^r\}\, dV + \int_V \left[N^u\right]^T \{B\}\, dV + \int_S \left[N^u\right]^T \{t\}\, dS. \tag{5}$$

$$\left[K^{\phi u}\right]\{u^n\} + \left[K^{\phi\phi}\right]\{\phi^n\} = -\int_V \left[B^{\phi}\right]^T \{D^r\}\, dV - \int_V \left[N^{\phi}\right]^T q^v dV - \int_S \left[N^{\phi}\right]^T q^S dS. \tag{6}$$

$$\left[K^{uu}\right] = \int_V \left[B^u\right]^T \left[c^E\right]\left[B^u\right] dV$$
$$\left[K^{u\phi}\right] = \int_V \left[B^u\right]^T [e]\left[B^{\phi}\right] dV.$$
$$\left[K^{\phi u}\right] = \int_V \left[B^{\phi}\right]^T [e]\left[B^u\right] dV$$
$$\left[K^{\phi\phi}\right] = -\int_V \left[B^{\phi}\right]^T \left[\varepsilon^S\right]\left[B^{\phi}\right] dV \;.$$
$$[m] = \int_V \left[N^u\right]^T \rho\left[N^u\right] dV$$
$$\{T^r\} = \left[c^E\right]\{S^r\}$$
$$\{D^r\} = -[e]^T\{S^r\} + \{P^r\}$$

Where $\{S^r\}$, is residual strain, $\{P^r\}$ is residual polarization, $[c^E]$, is elasticity tensor under a constant electric field, $[e]$, is piezoelectric stress matrix, $\left[\varepsilon^S\right]$ is the dielectric matrix, $\{D^r\}$: residual displacement of electric, E is the electric field, $\{T^r\}$: residual stress tensor, $\{B\}$: body load, $\{t\}$: surface load, ρ: mass density and q^v, q^s: electric charges per volume and area. $[K^{u\varphi}]$ for the series connection of the piezoelectric layers is presented in "Eq. (7)".

$$\left[K^{u\varphi}\right] = 2eb\begin{bmatrix} \frac{H}{4h} & \frac{H}{4h} & \frac{H}{4h} & \frac{-H}{4h} \\ \frac{H}{4}+\frac{5H^2}{24h} & -\frac{H}{4}+\frac{5H^2}{24h} & \frac{-H^2}{24h} & \frac{-5H^2}{24h} \\ \frac{-H}{4h} & \frac{-H}{4h} & \frac{-H}{4h} & \frac{H}{4h} \\ \frac{-H}{2}+\frac{5H^2}{24h} & \frac{-5H^2}{24h} & \frac{H}{2}+\frac{H^2}{24h} & \frac{H}{4}+\frac{5H^2}{24h} \end{bmatrix} \tag{7}$$

$[K^{u\varphi}]$ for the series connection of the piezoelectric layers is presented in "Eq. (8)".

$$\left[K^{u\varphi}\right]=\frac{ebH^2}{3h}\begin{bmatrix}0&0&0&0\\1&-1&-1&1\\0&0&0&0\\-1&1&1&-1\end{bmatrix} \quad (8)$$

$$\left[K^{\phi\phi}\right]=-\int_V\left[B^{\phi}\right]^T\left[\varepsilon^s\right]\left[B^{\phi}\right]dV=2\frac{\varepsilon}{H}\begin{bmatrix}\frac{2H}{3h}+\frac{7h}{12H}&\frac{2H}{3h}+\frac{h}{12H}&\frac{-2H}{3h}-\frac{5h}{12H}&\frac{5H}{12h}+\frac{5h}{12H}\\\frac{2H}{3h}+\frac{h}{12H}&\frac{2H}{3h}+\frac{h}{6H}&\frac{-2H}{3h}-\frac{h}{12H}&\frac{-H}{12h}-\frac{h}{12H}\\\frac{-2H}{3h}-\frac{5h}{12H}&\frac{-2H}{3h}-\frac{h}{12H}&\frac{2H}{3h}+\frac{h}{6H}&\frac{H}{3h}-\frac{h}{6H}\\\frac{5H}{12h}+\frac{5h}{12H}&\frac{-H}{12h}-\frac{h}{12H}&\frac{H}{3h}-\frac{h}{6H}&\frac{2H}{3h}+\frac{h}{6H}\end{bmatrix} \text{……………...………………..}(9)$$

$$\left[M\right]=\int_V\left[N^u\right]^T\rho\left[N^u\right]dV \quad (10)$$

Where ε, is the permittivity factor.
For the finite element formulation, the electric potential is specified on the external boundary for the Dirichlet BC problem. The finite element model has been developed with FORTRAN programming Language.

3 SUPPORT EFFECT ON THE PERFORMANCE OF PIEZOELECTRIC BEAM ACTUATOR

The developed finite element program has been named Piezoelectric Beam Analysis (PIBA). The piezoelectric beam has two layers.

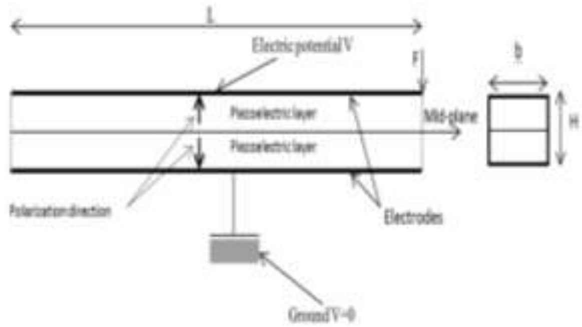

Fig. 2 Series connection of piezoelectric layers.

The parallel connection of the piezoelectric layers represents equal polarities (“Fig. 3”). In this study, the conditions of both series and parallel connections are used in the model.

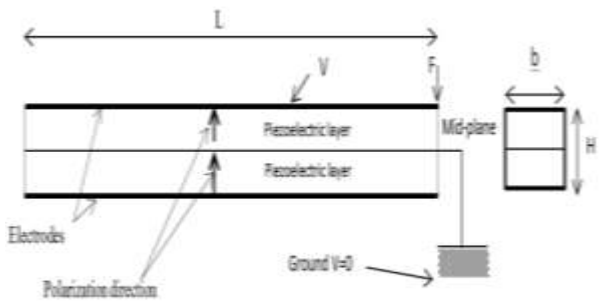

Fig. 3 Parallel connection of piezoelectric layers.

The series connection of the piezoelectric layers represents opposite polarities (“Fig. 2”).
The beam presented has L=100mm, H=0.5mm, b=1mm and is loaded by V=100V in the actuator. The piezoelectric layer is taken to be PZT-5H [18]. The properties of the piezoelectric material are shown in “Table 1”.

Table 1 The properties of the piezoelectric material

piezoelectric stress matrix ē [c/m^2]	dielectric permittivity matrix ϵ [F/m]	elasticity constant E[GPa]	Poisson ratio ν [-]	Mass density ρ[kg/m^3]
$\begin{bmatrix}0&0&0\\4.4*10^{-1}&4.4*10^{-1}&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&106.248*10^{-12}\end{bmatrix}$	200	0.3	7760

Figure 4 shows the deflection of the beam Using voltage V=100V (Voltage Driven Electrodes) for the Piezoelectric Beam clamped at one end. The Figure presents that the displacement of the piezoelectric beam has a linear behavior on the length of beam.

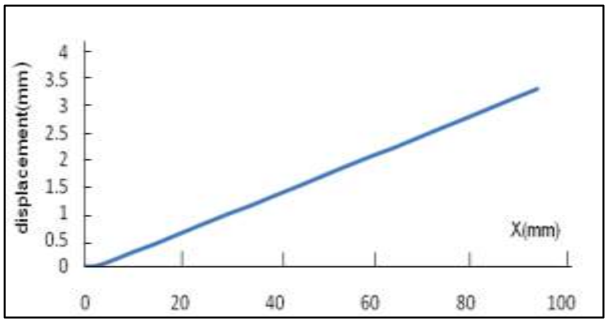

Fig. 4 Deflection of a Piezoelectric Beam Clamped at One End.

The deflection numerical results of the free end of the beam are compared to the analytical method and Ansys software results in "Table 2". The results show PIBA results have proper adjustment with Analytical results, also the deflection of beam in the parallel case is twice the deflection of beam in the series case.

Table 2 Deflection of the free end of the beam

	PIBA result[m]	Analytical result[m]	ANSYS result[m]
Series case	0.001755	0.00176	0.00132
Parallel case	0.00351	0.00352	0.00264

Figure 5 shows the deflection of the beam using voltage V=100V (Voltage Driven Electrodes) for the Piezoelectric Beam clamped at two ends. The result shows that the deflection of a piezoelectric beam clamped at two ends has harmonic behavior. The deflection direction of the beam changes in the middle of the beam. So, the middle of the beam deflection is zero. This result is important for the behavior of the actuator.

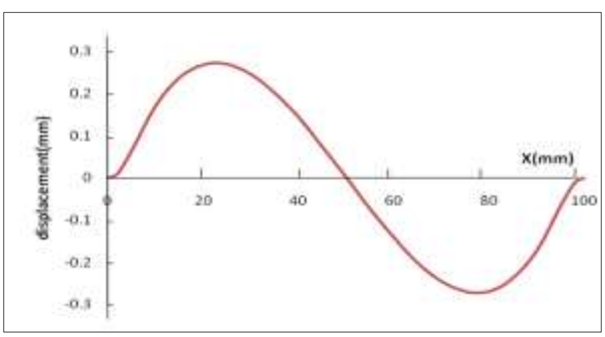

Fig. 5 Deflection of a piezoelectric beam clamped at two ends.

Figure 6 shows the deflection of the beam using voltage V=100V (Voltage Driven Electrodes) for the Piezoelectric Beam clamped at one end and pivoted at the other end. The result shows that the Deflection of piezoelectric beam clamped at one end and pivoted at the other end has harmonic behavior with semi-cycle. The maximum deflection is in x=40mm from Clamped support.

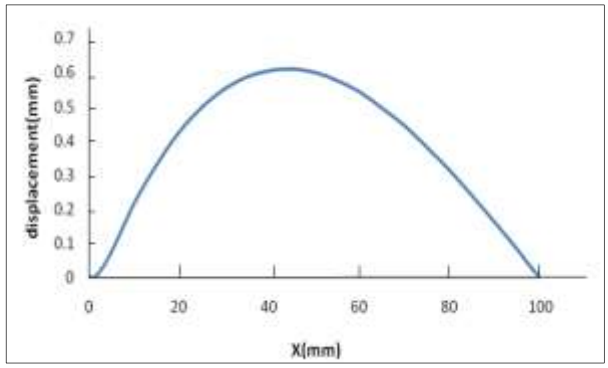

Fig. 6 Deflection of piezoelectric beam clamped at one end and pivoted at the other end.

Figure 7 shows the deflection of the beam using voltage V=100V (Voltage Driven Electrodes) for the Piezoelectric Beam clamped at one end and with vertical translation at the other end. The figure shows that the displacement of the piezoelectric beam has parabolic behavior on the length of the beam.

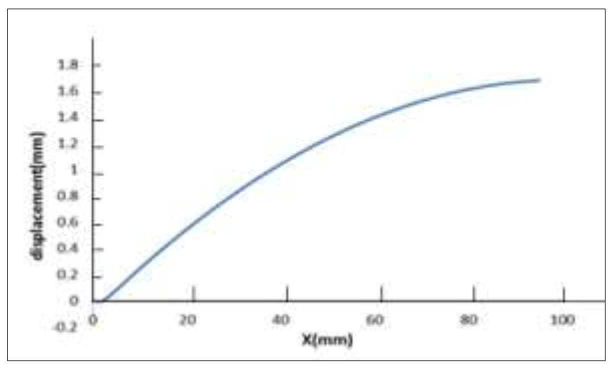

Fig. 7 Deflection of piezoelectric beam clamped at one end and with vertical translation at the other end.

4 EFFECTS OF CONCENTRATED MASS ON THE ANALYSIS OF PIEZOELECTRIC BEAM FREQUENCY

The stiffness matrix of the piezoelectric model depends on the electrical boundary conditions. The piezoelectric coupling parameter can affect Eigen frequencies. The concentrated mass and piezoelectric coupling influence the result. The structure presented in "Fig. 8" has been used for modal analysis. The results of electromechanical coupling and uncoupling and the concentrated mass effect (m=0.1gr) are presented in "Table 3". The results of the PIBA program are compared to the results of Ansys software. The piezoelectric beam element is not available in Ansys.

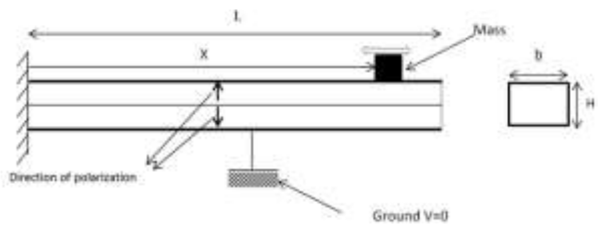

Fig. 8 Piezoelectric model and concentrated mass.

Table 3 The results of electromechanical coupling, uncoupling, and concentrated mass effect (m=0.1[gr])

Structure model	Mode 1[Hz]		Mode 2[Hz]		Mode3[Hz]		Mode4[Hz]	
	PIBA	Ansys	PIBA	Ansys	PIBA	Ansys	PIBA	Ansys
Beam without coupling and mass	69.3	69.91	437	440	1226	1246	2403	2483
Beam with coupling and without mass	60.5		375		1030		1429	
Beam without coupling and with mass (X=100mm)	34.4	34.67	327.5	329.3	1017	1032.6	2097	2163
Beam with coupling and mass (X=100mm)	30.5		287.2		831		1384	
Beam without coupling and with mass (X=70mm)	48.4	48.8	406.3	408.5	977.6	989.07	2264	2337
Beam with coupling and mass (X=70mm)	41.2		343.3		900.7		1186	
Beam without coupling and with mass (X=50mm)	59.5	60.05	298.8	300.3	1226	1245.7	1926	1976
Beam with coupling and mass (X=50mm)	51.7		279.7		869.3		1163	
Beam without coupling and with mass (X=20mm)	68.9	69.49	380.2	382.9	856.7	868.9	1882	1932
Beam with coupling and mass (X=20mm)	60.0		327.4		792.3		1331	

The effect of concentrated mass position on Eigen frequencies is shown in "Figs. 9-12" (mode 1- 4). Numerical results showed that the frequency of a piezoelectric beam is lower than that of an uncoupling beam. The reason for this problem is the coupling matrix of piezoelectric in the stiffness matrix of the beam.

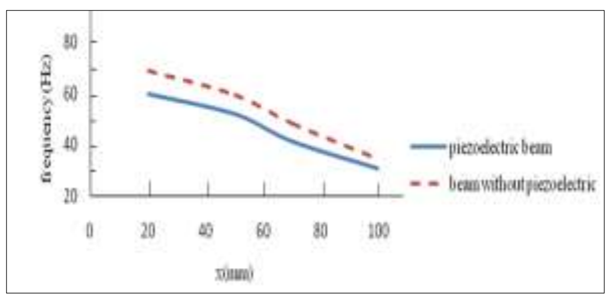

Fig. 9 The effect of position of concentrated mass on the first Eigen frequency.

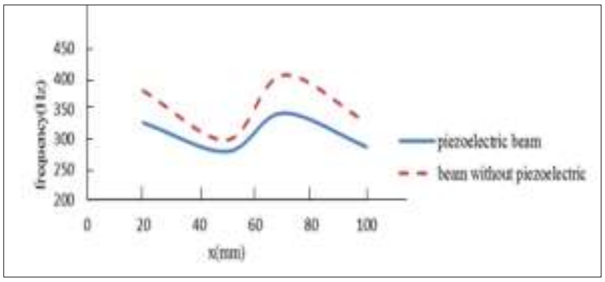

Fig. 10 The effect of position of concentrated mass on the second Eigen frequency.

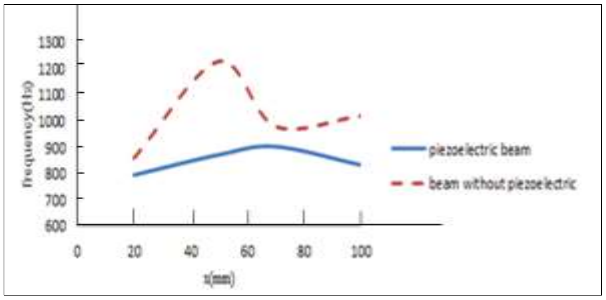

Fig. 8 The effect of position of concentrated mass on the third Eigen frequency.

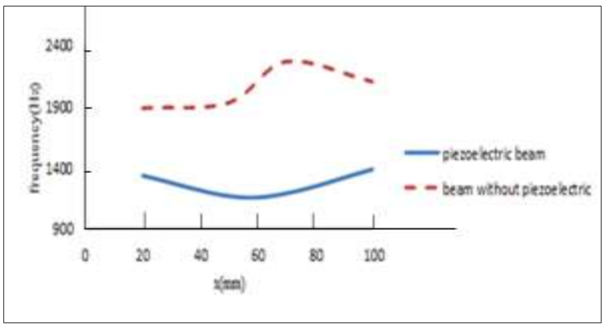

Fig. 8 The effect of position of concentrated mass on the fourth Eigen frequency.

As a second analysis, the position of the concentrated mass was set to vary. Frequency analysis was extended to examine concentrated mass. The results showed a change in Eigen frequency with the position of mass. Comparisons of the coupling and uncoupling beams indicate that the first and second Eigen frequencies change similarly for both models, but changes in the third and fourth Eigen frequencies are discordant for the two models. The reason for this problem is the positive and negative elements in the piezoelectric matrix.

5 CONCLUSIONS

A finite element program for analyzing the behavior of piezoelectric structures has been presented in this article. The beam element is based on the Euler-Bernoulli theory. The electrical boundary conditions of a closed circuit were considered in a static solution, and electric potentials were applied on the upper and lower beam faces. The model was loaded with electrical and mechanical forces and the results were compared to the analytical method and ANSYS software. The discrepancy was negligent.

The present work was used to study a piezoelectric beam under a variety of boundary conditions. The effects of support on the deflection of the piezoelectric beam would be considerable for actuator structures.

Modal analysis of the beam was also presented. This analysis was carried out for a coupling and uncoupling beam. Numerical results showed that the frequency of a piezoelectric beam is lower than that of an uncoupling beam.

As a second analysis, the position of the concentrated mass was set to vary. Frequency analysis was extended to examine concentrated mass. The results showed a change in Eigen frequency with the position of mass. Comparisons of the coupling and uncoupling beams indicate that the first and second Eigen frequencies change similarly for both models, but changes in the third and fourth Eigen frequencies are discordant for the two models. The reason for this problem is the positive and negative elements in the piezoelectric matrix.

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