Vibration Analysis of Rectangular Kirchhoff Nano-Plate using Modified Couple Stress Theory and Navier Solution Method

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Abstract: In this study, the characteristics of rectangular Kirchhoff nano-plate vibrations are investigated using a modified couple stress theory. To consider the effects of small-scale, the modified couple stress theory proposed by Young (2002) is used as it has only one length scale parameter. In modified couple stress theory, the strain energy density is a function of the components of the strain tensor, curvature tensor, stress tensor, and symmetric part of the couple stress tensor. After obtaining the strain energy, external work, and kinetic energy equation and inserting them in the Hamilton principle, the main and auxiliary equations of nano-plate are obtained. Then, by applying the boundary and force conditions in the governing equations, the vibrations of the rectangular Kirschhof nano-plate with the thickness are investigated with simple support around. The solution method used in this study is the Navier method and the effects of material length scale, length and thickness of the nanoplate on the vibration are investigated and the results are presented and discussed in details.

Keywords: Kirchhoff Plate, Navier Solution Method, Modified Couple Stress Theory, Rectangular Nano-Plate, Vibrations

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1 INTRODUCTION

Atomic and molecular scale testing is the safest method for the small-scale study of materials. In this method, the nanostructure is studied in real dimensions. In this method, to determine the mechanical properties of nanostructures, Atomic Force Microscopy (AFM) is used to apply different mechanical loads on nano-plates and measure the response. The main problems in this method are the difficulty of controlling the test conditions at this scale, as well as the high economic costs and time-consuming nature of the method. Therefore, this method is only used to validate other simple and low-cost methods. Atomic simulation is another solution for evaluating structures on a small scale. In this method, the behavior of atoms and molecules is examined by considering the effect of intermolecular and interatomic on their motion, which ultimately involves the deformation of the whole body. Using this method has a very high computational cost when the problem has a large deformation or a scale larger than one or more atoms. Therefore, its use is limited to problems with small deformations.

Due to the limitations of the above methods for evaluating nanostructures, researchers have sought simpler solutions to evaluate nanostructures. Modeling small-scale structures using continuum mechanics is another solution in the study of these materials. There are a variety of size-dependent continuum theories that consider the effects of size, including micromorphic theory, microstructure theory, micropolar theory, Cosserat theory, nonlocal theory, modified couple stress theory, and strain gradient elasticity theory. These are developed from classical field theories that consider size effects.

In this paper a rectangular Kirchhoff nano-plate model is developed for the vibration analysis of a graphene nanoplate based on a modified couple stress theory and the results are presented with new figures and tables.

2 MODIFIED COUPLE STRESS THEORY

Young et al. (2002) [1] using a couple of stress theories proposed by Toupin [2], Mindlin and Tiersten [3], Koiter [4], and Mindlin [5] in 1964 proposed a modified couple stress model that has only one material length scale parameter to illustrate the effect of size, while the classical couple stress theory has two material length scale parameters.

In the modified couple stress theory, the strain energy density in three-dimensional vertical coordinates for an object bounded by volume V and surface Ω is expressed as follows [6]:

$$U = \frac{1}{2}\int_V \left(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij}\right)dV \quad i,j = 1,2,3 \tag{1}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

$$\chi_{ij} = \frac{1}{2}\left(\theta_{i,j} + \theta_{j,i}\right) \tag{3}$$

Where, χ_{ij} and ε_{ij} are the symmetric parts of the curvature tensor and the strain tensor, respectively. Also, θ_i and u_i are defined as the displacement vector and the rotational vector, respectively:

$$\theta = \frac{1}{2} Curl\, \boldsymbol{u} \tag{4}$$

Where, σ_{ij} and m_{ij} are the stress tensor and the deviatoric part of the couple stress tensor, respectively, which are defined as follows:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{5}$$

$$m_{i,j} = 2\mu\, l^2\chi_{ij} \tag{6}$$

Where, λ and μ are the lame constants, δ_{ij} is the Kronecker delta, and l is the material length scale parameter. From "Eqs. (3) and (6)" , it can be seen that χij and mij are symmetric.

3 KIRCHHOFF PLATE MODEL

The displacement equations for the Kirchhoff plate are defined as follows:

$$u_1(x,y,z,t) = -z\frac{\partial w(x,y,t)}{\partial x}$$
$$u_2(x,y,x,t) = -z\frac{\partial w(x,y,t)}{\partial y} \tag{7}$$
$$u_3(x,y,z) = w(x,y)$$

Where, w is the amount of displacement of the center point of the plane along the z-axes. The symmetric part of the curvature tensor and the strain and stress tensors and the rotational vector for the Kirchhoff plate model is as follows:

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2} \tag{8}$$

$$\varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2} \tag{9}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = -z\frac{\partial^2 w}{\partial x \partial y} \tag{10}$$

$$\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{zx} = \varepsilon_{yz} = \varepsilon_{zy} = 0 \quad (11)$$

$$\theta_x = \frac{\partial w}{\partial y} \quad (12)$$

$$\theta_y = -\frac{\partial w}{\partial x} \quad (13)$$

$$\theta_z = 0 \quad (14)$$

$$x_{xx} = \frac{\partial^2 w}{\partial x\, \partial y} \quad (15)$$

$$x_{yy} = -\frac{\partial^2 w}{\partial x\, \partial y} \quad (16)$$

$$x_{xy} = x_{yx} = \frac{1}{2}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right) \quad (17)$$

$$x_{xz} = x_{zx} = x_{yz} = x_{zy} = x_{zz} = 0 \quad (18)$$

$$\sigma_{xx} = -(\lambda + 2\mu)\left(z\frac{\partial^2 w}{\partial x^2}\right) - \lambda\left(z\frac{\partial^2 w}{\partial y^2}\right) \quad (19)$$

$$\sigma_{yy} = -\lambda\left(z\frac{\partial^2 w}{\partial x^2}\right) - (\lambda + 2\mu)\left(z\frac{\partial^2 w}{\partial y^2}\right) \quad (20)$$

$$\sigma_{zz} = -\lambda z\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \quad (21)$$

$$\sigma_{yx} = \sigma_{xy} = -2\mu\left(z\frac{\partial^2 w}{\partial x \partial y}\right) \quad (22)$$

$$\sigma_{xz} = \sigma_{zx} = \sigma_{yz} = \sigma_{zy} = 0 \quad (23)$$

The variation of strain energy is expressed as follows:

$$\delta U = \int_V (\sigma_{xx}\, \delta\, \varepsilon_{xx} + \sigma_{yy}\, \delta\varepsilon_{yy} + 2\sigma_{xy}\, \delta\, \varepsilon_{xy} + 2\sigma_{xz}\, \delta\, \varepsilon_{xz} + 2\sigma_{yz}\, \delta\, \varepsilon_{yz} + m_{xx}\, \delta\, x_{xx} + m_{yy}\, \delta x_{yy} + m_{zz}\, \delta x_{zz} + 2m_{xy}\, \delta x_{xy} + 2m_{xz}\, \delta x_{xz} + 2m_{yz}\, \delta\, x_{yz}) dV \quad (24)$$

For simplification, the coefficients of the variables can be named from F1 to F3 according to “Eq. (25)” and obtained separately.

$$\delta U = \int_V (F_1 \delta w,_{xx} + F_2\, \delta w,_{yy} + F_3\, \delta w,_{xy})dV \quad (25)$$

Where:

$$+ F_1 = \frac{\partial^2 w}{\partial x^2}[(\lambda + 2\mu)z^2 + \mu l^2] \; \frac{\partial^2 w}{\partial y^2}(\lambda z^2 - \mu l^2) \quad (26)$$

$$+ F_2 = \frac{\partial^2 w}{\partial y^2}[(\lambda + 2\mu)z^2 + \mu l^2] \; \frac{\partial^2 w}{\partial x^2}(\lambda z^2 - \mu l^2) \quad (27)$$

$$F_3 = \frac{\partial^2 w}{\partial x \partial y}(4\mu z^2 + 4\mu l^2) \quad (28)$$

4 EQUATION OF VIRTUAL WORK OBTAINED BY EXTERNAL FORCE [7]

Virtual work performed by an external force consists of three parts:

1. Virtual work performed by force body on V= Ω× (- h⁄2, h⁄2).
2. Virtual work performed by surface traction at upper and lower levels (Ω).
3. Virtual work done by surface traction on lateral surfaces S= Γ× (- h⁄2, h⁄2), where Ω is the middle plate of the sheet and Γ is the middle environment of the sheet.

If (f_x, f_y, f_z) are the body forces, (c_x, c_y, c_z) are the body couples, (q_x, q_y, q_z) are the forces acting on the Ω plane, (t_x, t_y, t_z) are the Cauchy's tractions and (S_x, S_y, S_z) are surface couples, the Variations of the virtual work is expressed as:

$$\delta w = -\Big[\int_\Omega (f_x\delta u + f_y\delta V + f_z\delta w + q_x\delta u + q_y\delta V + q_z\delta w + c_x\delta\theta_x + c_y\, \delta\theta_y + c_z\delta\theta_z)\, dx\, dy + \int_\Gamma (t_x\delta u + t_y\delta V + t_z\delta w + s_x\theta_x + s_y\delta\theta_y + s_z\delta) \quad (29)$$

Given that in this study only external force qz was applied, virtual work is as follows:

$$\delta w = \int_0^a \int_0^b q(x,y)\delta w(x,y) dx\, dy \quad (30)$$

Kinetic energy variations are described as follows:

$$\delta T = \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(\dot{u}_1\delta\dot{u}_1 + \dot{u}_2\delta\dot{u}_2 + \dot{u}_3\delta\dot{u}_3) dA\, dz = \int_A \left[\rho h\dot{w}\delta\dot{w} + \frac{\rho h^3}{12}\left(\frac{\partial\dot{w}}{\partial x}\frac{\partial\delta\dot{w}}{\partial x} + \frac{\partial\dot{w}}{\partial y}\frac{\partial\delta\dot{w}}{\partial y}\right)\right] dA \quad (31)$$

Where, ρ is the density. Using the Hamilton principle [8]:

$$\int_0^T \big(\delta T - (\delta U - \delta w)\big) dt = 0 \tag{32}$$

Where, T is the kinetic energy, U is the strain energy, and W is the external energy.

5 THE FINAL EQUATION OF THE PLANE WITH THE APPLICATION OF EXTERNAL FORCE

Using the Hamilton principle, the main equation is obtained as follows:

$$\left[\int_{-h/2}^{h/2} \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial x \partial y}\right) dz\right] = q(x,y) + \rho h\, \ddot{w} - \frac{\rho h^3}{12} \nabla^2 \ddot{w} \tag{33}$$

6 OBTAINING KIRCHHOFF'S PLATE EQUATIONS IN THE MOST GENERAL CASE

Considering the following values:

$$A_1 = (\lambda + 2\mu) I_2 + \mu l^2 h \tag{34}$$

$$I_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} Z^i \; dz \tag{35}$$

The general equations of the Kirchhoff plate will be obtained as follows:

$$A_1 \frac{\partial^4 w}{\partial x^4} + A_1 \frac{\partial^4 w}{\partial y^4} + 2A_1 \frac{\partial^4 w}{\partial x^2 \partial y^2} = q(x,y) + \rho h \frac{\partial^2 w}{\partial t^2} - \frac{\rho h^3}{12}\left(\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2}\right) \tag{36}$$

7 NAVIER SOLUTION METHOD

The Navier solution method can be used for rectangular plates with simply supported boundary conditions at all edges. Because the boundary conditions are automatically satisfied in this method, the unknown functions of the middle surface of the plate are expressed as dual trigonometric series [7], [9]:

$$W(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \alpha x \sin \beta y \, e^{i\omega t} \tag{37}$$

The force can also be calculated from the following relations:

$$q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \sin \alpha x \sin \beta y \, e^{i\omega t} \tag{38}$$

$$Q_{mn} = \frac{4}{ab} \int_0^a \int_0^b q(x,y) sin\alpha x \sin \beta y \, dx \, dy \tag{39}$$

$$Q_{mn} = \begin{cases} q_0 & ; \textit{For sinusoidal force} \\ \frac{16 q_0}{mn\pi^2} & ; \textit{For uniform force} \\ \frac{4Q_0}{ab} & ; \textit{For point force in the plane center} \end{cases} \tag{40}$$

Where:

$$\alpha = \frac{\pi m}{a} \;, \; \beta = \frac{\pi n}{b} \;, \; i = \sqrt{-1} \tag{41}$$

The simply supported boundary conditions are also satisfied by the Navier method according to the following equations:

$$x = 0 \,,\, x = a \begin{cases} w(0,y) = w(a,y) = \sum\sum w_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y = 0 \\ \varphi_y(0,y) = \varphi_y(a,y) = \sum\sum y_{mn} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y = 0 \end{cases} \tag{42}$$

$$y = 0 \,,\, y = b \begin{cases} w(x,0) = w(x,b) = \sum\sum w_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y = 0 \\ \varphi_x(x,0) = \varphi_x(x,b) = \sum\sum X_{mn} \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y = 0 \end{cases} \tag{43}$$

8 OBTAINING THE MATRIX OF KIRCHHOFF PLATE EQUATIONS

After solving the equations using the Navier method and naming the coefficients of the variables of the equations, we have:

$$w_{mn}(A_1\alpha^4 + A_1\beta^4 + 2A_1\alpha^2\beta^2) = Q_{mn} - \rho h w_{mn}\, \omega^2 - \frac{\rho h^3}{12} w_{mn} \alpha^2 \,\omega^2 - \frac{\rho h^3}{12} w_{mn}\, \beta^2 \omega^2 \tag{44}$$

The general matrix of Kirchhoff plate equations with auxiliary equations is obtained as follows:

$$([N_1] - \omega^2 [K_1])\, [w_{mn}] = [Q_{mn}] \tag{45}$$

Where:

$$N_1 = A_1\alpha^4 + A_1\beta^4 + 2A_1\alpha^2\beta^2 \tag{46}$$

$$K_1 = -\rho h - \frac{\rho h^3}{12}\alpha^2 - \frac{\rho h^3}{12}\beta^2 \tag{47}$$

The material of the plate is considered to be different materials such as epoxy, graphene, copper, etc. In this study, the plate material was considered as graphene. A single-layer graphene plate has the following properties [8]:

$$E = 1.06TPa, \nu = 0.25\,, h = 0.34nm, \rho = 2250\, {}^{kg}/_{m^3}$$

Also, the relationship between E and μ, and ν can be written as follows:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad , \mu = \frac{E}{2(1+\nu)} \qquad (48)$$

Where, E is Young's Modulus, and μ and λ are the Lame coefficients [10]. Also, the amount of force is considered q = 1N⁄m ^ 2.

9 RESULTS & DISCUSSIONS

The calculation program is written in MATLAB software and the results are obtained using this program. All boundary conditions are considered as a simple support.

According to "Figs 1 to 4" , the frequencies of different modes (ω_{11}-ω_{12}-ω_{21}-ω_{22}) of the Kirchhoff nano-plate decrease with increasing length to nano-plate thickness ratio. Also, when the effect of the size parameter is not considered (classical theory), the frequency is the lowest. Besides, as the effect of size increases, the frequency increases as well. It is noteworthy that the frequency is the lowest for the first mode and increases for the next modes.

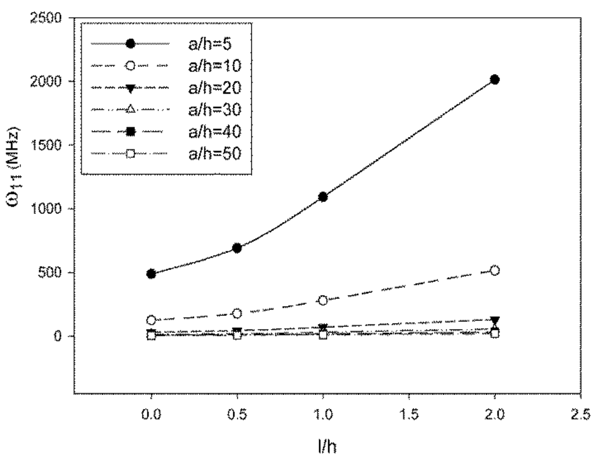

Fig. 1 Comparison of mode frequencies (ω_{11}) for different ratios of length to thickness and ratios of length parameter to thickness of the plate for Kirchhoff nano-plate (a/b = 1).

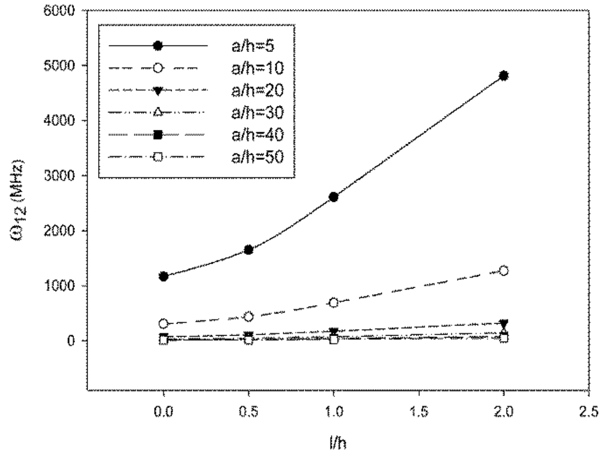

Fig. 2 Comparison of mode frequencies (ω_{12}) for different ratios of length to thickness and ratios of length parameter to thickness of the plate for Kirchhoff nano-plate (a/b = 1).

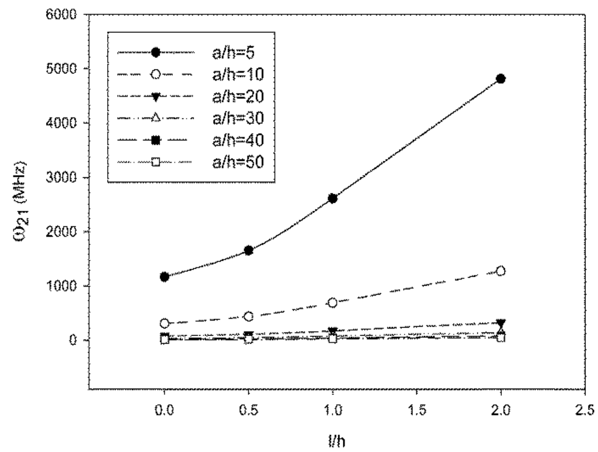

Fig. 3 Comparison of mode frequencies (ω_{21}) for different ratios of length to thickness and ratios of length parameter to thickness of the plate for Kirchhoff nano-plate (a/b = 1).

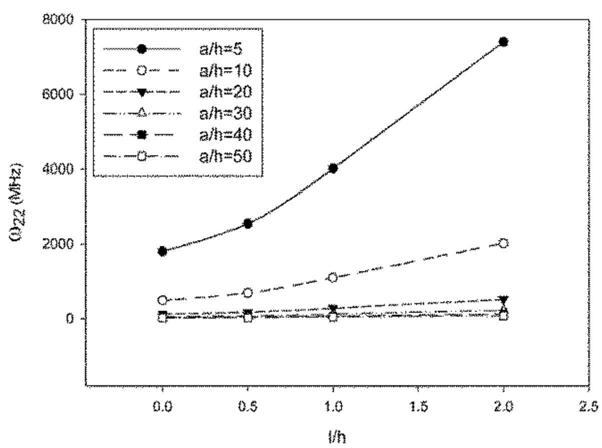

Fig. 4 Comparison of mode frequencies (ω_{22}) for different ratios of length to thickness and ratios of length parameter to thickness of the plate for Kirchhoff nano-plate (a/b = 1).

Figure 5 shows that with increasing the ratio of the length scale parameter to the nano-plate thickness, the frequencies of different modes of Kirchhoff nano-plate increase as well.

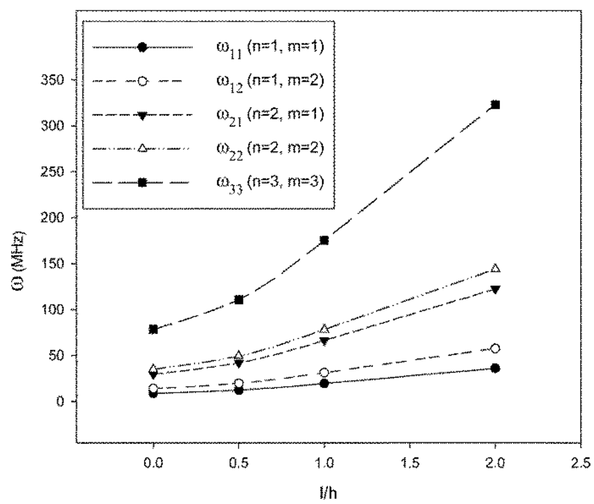

Fig. 5 Comparison of the different frequencies of modes for different ratios of length parameter to thickness for Kirchhoff nano-plate (a/b = 2 and a/h=30).

Comparing "Fig. 6" and "Figs. 1 to 4" shows that the frequency increases when the length to width ratio of the plate is halved.

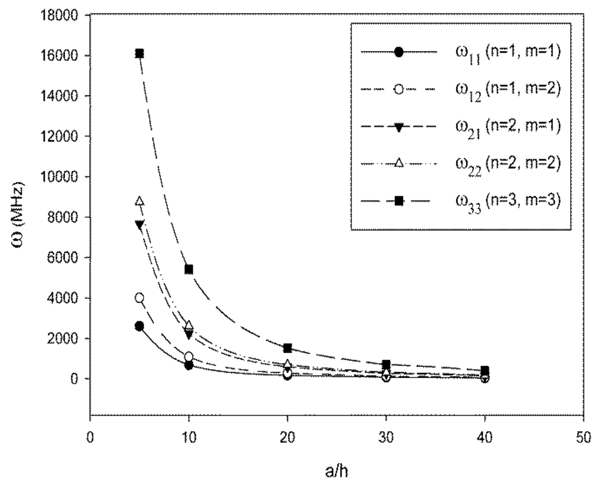

Fig. 6 Comparison of the different frequencies of modes for different ratios of length to thickness for Kirchhoff nano-plate (a/b = 0.5, a/h=1).

"Tables 1 to 4" show the frequencies of different (ω_{11}-ω_{12}-ω_{21}-ω_{22}) for different nano-plates. According to the tables, the frequency is the highest for the Kirchhoff nano-plate and the lowest for the Mindlin nano-plate if the length scale parameter is not considered.

Table 1 Comparison of the first mode frequencies (ω_{11}) for length-to-thickness ratio and different ratios length parameter to thickness of different nano-plates (a/b = 1)

l/h	a/h				
	10	20	30	40	50
	Kirchhoff plate				
0	124.98838	31.43847	13.98857	7.87172	5.03883
0.5	176.76026	44.46071	19.78283	11.13229	7.12598
1	279.48251	70.29855	31.27940	17.60169	11.26717
2	515.34028	129.62412	57.67637	32.45591	20.77563
	Mindlin plate				
0	121.5505	31.2102	13.9429	7.8572	5.0329
0.5	249.5297	64.2570	28.7266	16.1924	10.3732
1	436.5378	115.4757	51.9052	29.3145	18.7965
2	722.2379	215.5686	99.1252	56.4382	36.3253
	Third-order shear deformation plate				
0	121.6342	31.2161	13.9441	7.8576	5.0330
0.5	173.8911	44.2699	19.7447	11.1202	7.1210
1	276.5826	70.1049	31.2407	17.5894	11.2621
2	511.3107	129.3539	57.6223	32.4387	20.7686

Table 2 Comparison of the mode frequencies (ω_{12}) for length-to-thickness ratio and different length parameter to thickness ratios of different nano-plates (a/b = 1)

a/h	l/h			
	0	0.5	1	2
	Kirchhoff plate			
10	308.7461	436.6329	690.3772	1272.9926
20	78.3559	110.8119	175.2090	323.0695
30	34.9237	49.3895	78.0917	143.9940
40	19.6641	27.8093	43.9704	81.0774
50	12.5909	17.8062	28.1540	51.9135
	Mindlin plate			
10	289.5156	592.8023	988.5087	1223.2879
20	76.9722	158.2169	280.4153	492.9660
30	34.6425	71.3140	128.0217	237.9174
40	19.5743	40.3193	72.7219	137.8488
50	12.5539	025.8663	46.7575	89.4593
	Third-order shear deformation plate			
10	289.9375	420.6563	674.3836	1250.9755
20	77.0069	109.6563	174.0385	321.4395
30	34.6497	49.1546	77.8533	143.6613
40	19.5766	27.7342	43.8941	80.9709
50	12.5548	17.7753	28.1226	51.8696

Table 3 Comparison of the mode frequencies (ω_{21}) for length-to-thickness ratio and different length parameter to thickness ratios of different nano-plates (a/b = 1)

a/h	l/h			
	0	0.5	1	2
	Kirchhoff plate			
10	308.7461	436.6329	690.3772	1272.9926
20	78.3559	110.8119	175.2090	323.0695
30	34.9237	49.3895	78.0917	143.9940
40	19.6641	27.8093	43.9704	81.0774
50	12.5909	17.8062	28.1540	51.9135
	Mindlin plate			
10	289.5156	592.8023	988.5087	1223.2879
20	76.9722	158.2169	280.4153	492.9660
30	34.6425	71.3140	128.0217	237.9174
40	19.5743	40.3193	72.7219	137.8488
50	12.5539	25.8663	46.7575	89.4593
	Third-order shear deformation plate			
10	289.9375	420.6563	674.3836	1250.9755
20	77.0069	109.6563	174.0385	321.4395
30	34.6497	49.1546	77.8533	143.6613

40	19.5766	27.7342	43.8941	80.9709
50	12.5548	17.7753	28.1226	51.8696

Table 4 Comparison of the mode frequencies (ω_{22}) for length-to-thickness ratio and different length parameter to thickness ratios of different nano-plates (a/b = 1)

a/h	l/h			
	0	0.5	1	2
	Kirchhoff plate			
10	488.2420	690.4785	1091.7424	2013.0735
20	124.9884	176.7603	279.4825	515.3403
30	55.8018	78.9157	124.7766	230.0767
40	31.4385	44.4607	70.2985	129.6241
50	20.1355	28.4759	45.0243	83.0207
	Mindlin plate			
10	443.6884	908.3644	1444.5250	1088.1654
20	121.5505	249.5297	436.5378	722.2379
30	55.0918	113.3246	202.1703	365.8010
40	31.2102	64.2570	115.4757	215.5686
50	20.0412	41.2804	74.4444	141.0270
	Third-order shear deformation plate			
10	444.5855	653.6982	1055.3211	1963.4588
20	121.6342	173.8911	276.5826	511.3107
30	55.1098	78.3225	124.1752	229.2384
40	31.2161	44.2699	70.1049	129.3539
50	20.0437	28.3971	44.9443	82.9090

10 CONCLUSIONS

In this study, the vibrations of the Kirchhoff nano-plate are investigated using the modified couple stress theory. Based on the obtained results, the frequencies of the different modes of Kirchhoff nano-plate decrease with increasing the length to thickness ratio of the nano-plate. Also, the frequency is the lowest, when the effect of the size parameter is not considered (classical theory). Meanwhile, with increasing the effect of size, the frequency increases as well. The frequency is the lowest for the first mode and increases for the next modes. Furthermore, frequency increases when the length to width ratio of the plate is halved.

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