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# Investigation of Magnitude and Position of Maximum von Mises Stress in The Cylindrical Contact Problems

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**Abstract:** In the analysis of contact mechanics problems, determination of stress field in mechanical elements is essential. Between the stress components the von Mises stress is more important, because it is used in the investigation of yield criteria and fatigue fracture of elements. The aim of this study is to present formulas for determining the magnitude and position of maximum von Mises stress. For this purpose, the effect of various material properties, element geometries and loading conditions on these two parameters are investigated. By applying Hertzian contact stress and von Mises relations, the magnitude and position of maximum von Mises stress are determined. The von Mises stress is assumed to be a function of material properties, geometry of the element and loading conditions and finally two formulas are presented for the calculation of the magnitude and position of maximum von Mises stress. The results of these presented formulas are in close agreement with the literature. The error is less than 1% for depth prediction and less than 6% for stress value prediction, which confirms the accuracy of the presented formulas.

**Keywords:** Contact Mechanics, Depth Prediction, Hertzian Stresses, Maximum Von Mises Stress

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Research paper

## 1 INTRODUCTION

In many mechanical machines, there are elements that have contact together. These parts can be stationary or moving. When these elements are loaded, significant compression stresses are created in the contact area that call contact stresses or Hertzian stresses. Gears, rolling bearings, railroad and wheels, clutches, brakes, screws and riveted joints are from the most important contacting mechanical element [1]. In the case that elements are statically loaded and have not relative motion, it is essential to determine the stress components, such as maximum principle, shear and von Mises stress, to an investigation yield of elements. Using maximum distortion energy criterion needs to know the maximum magnitude of von Mises stress in the elements [2]. In the case that elements have relative motion because of the moving contact area, contact stresses will be cyclic. In the presence of cyclic loading, fatigue phenomena occur and fatigue fracture may appear. When an element is subjected to cyclic contact loading, after a certain number of cycles, cracks initiate. Due to effects of initial crack position on the fatigue life [3], size and shape of spalls, determination of the exact position of crack initiation is very important.

The subject of contact mechanics may be said to have started in 1882 [4] with the publication by Heinrich Hertz of his classic paper on the contact of elastic solids. Developments in the theory did not appear in the literature until the beginning of this century, stimulated by engineering developments on the railways, in marine reduction gears and in the rolling contact bearing industry. Glodez et al. [5-8] have studied surface pitting and fatigue life of gears. They have determined the magnitude and position of maximum von Mises stress and put an initial crack at the location of maximum von Mises stress. In many studies about contact fatigue behavior of gears, the maximum von Mises stress and its location were used [9-15]. Guan et al. [16] analyzed the crack propagation in bearing steel with a non-metallic inclusion. They calculated von Mises stress distribution in the micro-domain to determine the crack initiation. Heirani and Farhangdoost [17] calculated the position of maximum von Mises stress in three gear samples by Abaqus software and compared the results with experimental spalling depth. Sun et al. [18] have determined that in ball bearing contact problem the maximum von Mises stress is approximately equal to $0.621p_0$, where $p_0$ is the maximum contact pressure. Kohn and Silva [19] have investigated the contact stresses between a rail and a wheel. They showed that the peak of von Mises stress in cylindrical contact occurred at a depth of $0.73a$, where $a$ is half-width of contact area.

Direct determination of von Mises stress equation requires a replacement of complicated Hertzian contact stress equations in the von Mises equation, that it is not the proper approach because of complication and heavy equations. On the other hand, the magnitude and position of maximum von Mises stress are very important, but knowing other magnitude of von Mises stress in various depths under the contact area is not necessary. In this paper, due to the importance of knowing the magnitude and position of maximum von Mises stress in both static and cyclic loading condition, two formulas are presented to calculate the magnitude and position of maximum von Mises stress as functions of material properties, geometry of elements and loading conditions. So far, no formula has been presented in the literature to predict the position and the value of the maximum von Mises stress. The existence of these formulas eliminates the need for expensive and time-consuming simulations.

## 2 NUMERICAL PROCEDURES

In this paper, contact mechanics problem of two cylindrical parts is studied. Figure 1 illustrates two cylinders with length l and diameters $d_1$ and $d_2$, which are pressured together with force F.

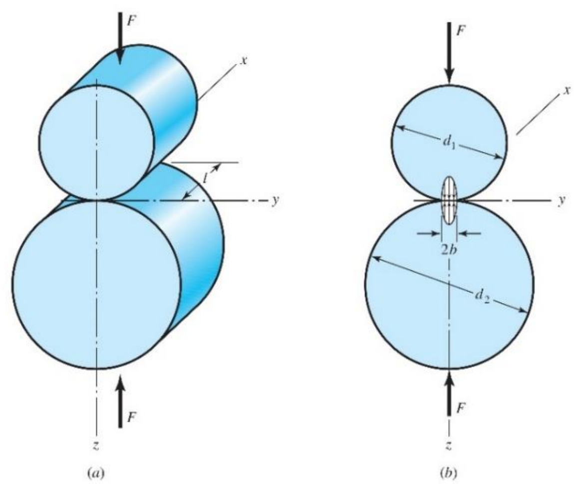


**Fig. 1** (a): Two right circular cylinders held in contact by forces F uniformly distributed along cylinder length l, and (b): Contact stress has an elliptical distribution across the contact zone width 2b [20].

The coordinate system is indicated in this Figure. The contact area is a narrow rectangle of width 2b and length l, and the pressure distribution is elliptical. The half-width b is given by the "Eq. (1)" and the maximum pressure is obtained by "Eq. (2)" [20]:

$$b = \sqrt{\frac{2F}{\pi l}\frac{(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2}{1/d_1 + 1/d_2}} \tag{1}$$

$$p_{max} = \frac{2F}{\pi b l} \tag{2}$$

Hertzian contact stress equations have been used for analytical investigation of maximum von Mises stress. These equations determine the stress field in the contact area of two cylinders. The stress state along the z axis is given by "Eq. (3)":

$$\begin{aligned} \sigma_x &= -2\upsilon p_0\left(\sqrt{1+\frac{z^2}{b^2}} - \frac{z}{b}\right) \\ \sigma_y &= -p_0\left(\frac{1+2\frac{z^2}{b^2}}{\sqrt{1+\frac{z^2}{b^2}}} - 2\frac{z}{b}\right) \\ \sigma_z &= \frac{-p_0}{\sqrt{1+\frac{z^2}{b^2}}} \end{aligned} \tag{3}$$

Due to contact stress components, presented in "Eq. (3)", there are principles stress components so by using von Mises stress definition, "Eq. (4)" can be presented:

$$\sigma_{mises} = \left[\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2}{2}\right]^{1/2} \tag{4}$$

According to above equations, it is seen that direct determination of von Mises stress equation requires a replacement complicated of "Eq. (3) in Eq. (4)", that it is not a suitable approach because of complication and heavy equations. On the other hand, with respect to section 1, the magnitude and position of maximum von Mises stress are very important, but knowing other von Mises stress values at various depths below contact area is not always necessary. Therefore, in this study, a Matlab code has been used to calculate Mises stresses at various depths and represents the magnitude and depth of maximum von Mises stress.

## 3 RESULTS AND DISCUSSION

In below results of the Matlab code for various material properties, element geometries and loading conditions are presented. Stress components of $\sigma_x$, $\sigma_y$, $\sigma_z$ and $\sigma_{mises}$ as functions of depth from contact area are illustrated in "Fig. 2". These results are in good agreement with data presented in "Fig. 3" [20] and this good agreement confirms the accuracy of the applied method in this paper.

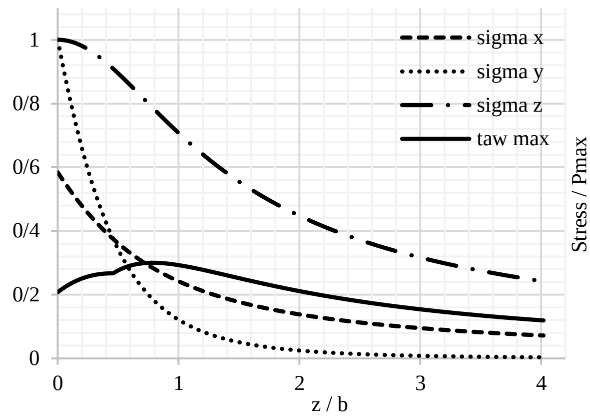


**Fig. 2** Stress components as functions of depth below the contact area.

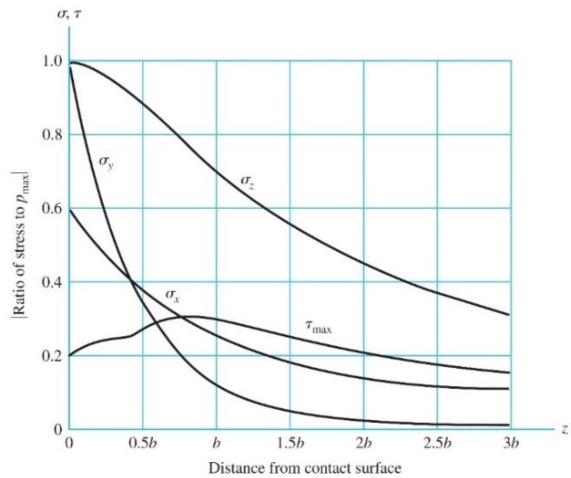


**Fig. 3** Stress components as functions of depth below the contact area [20].

In this section the effect of effective parameters on stress field, such as material properties, element geometry and loading conditions, on the magnitude and position of maximum von Mises stress is investigated. Variations of the magnitude and position of maximum von Mises stress, with respect to maximum contact pressure, are illustrated in "Fig. 4 and Fig. 5", respectively, for the material properties of E=201.8 GPa, ν=0.3 and the equivalent radius of curvature $R^*$=6 mm. As shown in these diagrams, the magnitude and position of maximum von Mises stress are linear functions of maximum contact pressure. Due to this relation between the magnitude and position of maximum von Mises stress and the maximum contact pressure, in the next steps to examine the effects of equivalent curvature radius and material properties, the ratio of maximum von Mises stress to the maximum contact pressure, $mises_{max}/p_{max}$, and the ratio of depth of maximum von Mises stress to the maximum contact pressure, $depth/p_{max}$, will be used.

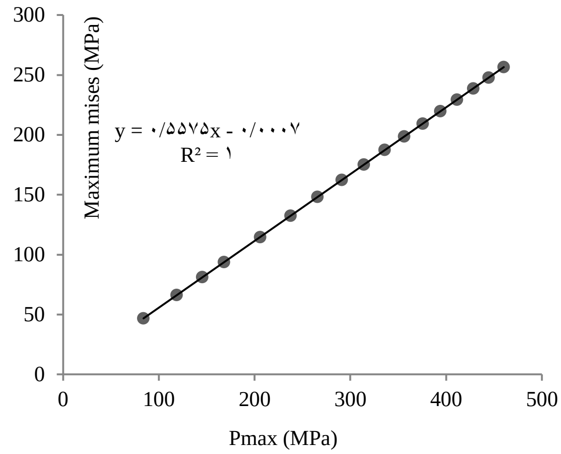


**Fig. 4** Variation of the maximum von Mises stress vs. the maximum contact pressure.

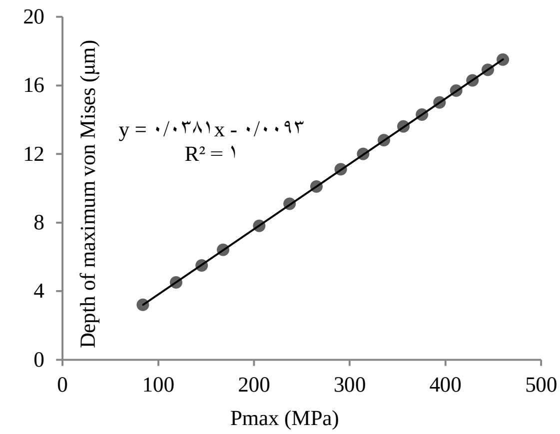


**Fig. 5** Variation of the depth of maximum von Mises stress vs. the maximum contact pressure.

Variations of the von $Mises_{max}/p_{max}$ and $depth/p_{max}$, with respect to elastic modulus, are illustrated in “Fig. 6 and Fig. 7”, respectively, for constant equivalent radius of curvature and Poisson’s ratio. As shown in “Fig. 6”, the maximum von Mises stress is independent of elastic modulus. According to “Fig. 7”, the depth of the maximum von Mises stress decreases by increasing elastic modulus. $Depth/p_{max}$ is a quadratic function of elastic modulus.

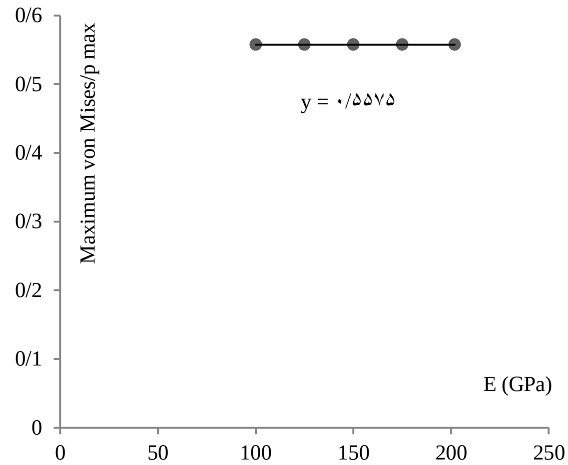


**Fig. 6** The ratio of maximum von Mises stress to maximum contact pressure vs. elastic modulus.

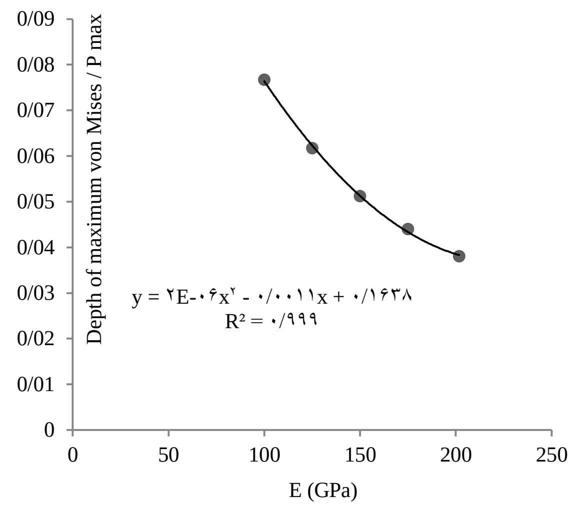


**Fig. 7** The ratio of depth of maximum von Mises stress to maximum contact pressure vs. elastic modulus.

Variations of the von $Mises_{max}/p_{max}$ and $depth/p_{max}$, with respect to the equivalent radius of curvature, are illustrated in “Fig. 8 and Fig. 9”, respectively, for constant elastic modulus and Poisson’s ratio. As shown in “Fig. 8”, the maximum Mises stress is independent of the equivalent radius of curvature, but according to “Fig. 9”, the depth of maximum von Mises stress increases by increasing the equivalent radius of curvature linearly, so it is a linear function of the equivalent radius of curvature.

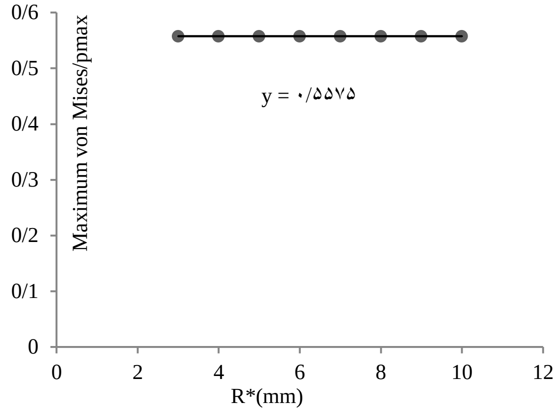


**Fig. 8** The ratio of the maximum von Mises stress to the maximum contact pressure vs. equivalent curvature radius.

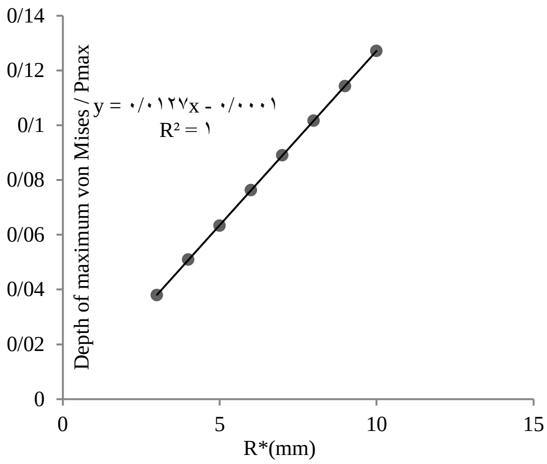


**Fig. 9** The ratio of the depth of maximum von Mises stress to the maximum contact pressure vs. equivalent curvature radius.

The last effective parameter on the stress field is the Poisson's ratio. Variations of the von $Mises_{max}/p_{max}$ and depth/$p_{max}$, with respect to the Poisson's ratio, are illustrated in "Fig. 10 and Fig. 11", respectively, for constant elastic modulus and equivalent radius of curvature. As shown in these diagrams, the magnitude and depth of maximum von Mises stress are dependent to Poisson's ratio, and the magnitude and depth of maximum von Mises stress decrease and increase with increasing Poisson's ratio, respectively. According to "Fig. 10 and Fig. 11", in the range of typical common values for Poisson's ratio of metals, $0.2 < \nu < 0.3$, a relation between the magnitude and depth of maximum von Mises stress can be approximated as linear functions with acceptable accuracy or can be considered as quadratic polynomials with better precision.

According to the results obtained in this study, it can be seen that the magnitude of maximum von Mises stress in cylindrical contacting elements is a function of the maximum contact pressure and Poisson's ratio, but is independent to the equivalent curvature radius and elastic modulus of the material. The depth of maximum von Mises stress is a function of all effective parameters on the stress field.

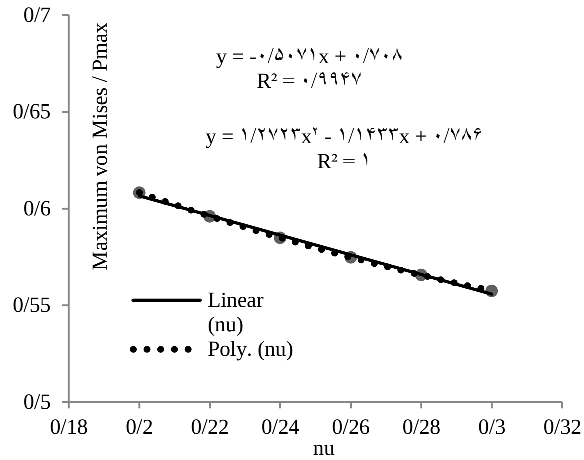


**Fig. 10** The ratio of maximum von Mises stress to maximum contact pressure vs. Poisson's ratio.

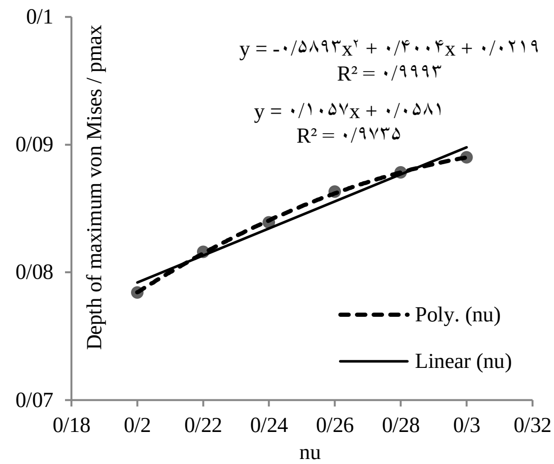


**Fig. 11** The ratio of the depth of maximum von Mises stress to maximum contact pressure vs. Poisson's ratio.

According to the results shown in "Figs. 4, 6, 8 and 10", a simple formula, "Eq. (5)", can be presented for calculation of the maximum von Mises stress:

$$\sigma_{mises,max} = p_{max}(0.704 - 0.488\nu) \quad , R^2 = 1 \tag{5}$$

Where $\sigma_{mises,max}$, $p_{max}$ and $\nu$ are the magnitude of maximum von Mises stress, maximum contact pressure and Poisson's ratio, respectively. Depth of maximum von Mises stress is a function of four parameters: maximum contact pressure, equivalent curvature radius, elastic modulus and Poisson's ratio of (See "Eq. (6)"):

$$y_{max} = -6.742 \times 10^{-6} p_{max} R(\upsilon^2 - 0.619\upsilon)(E^2 - 493.352E + 78499.492) - 0.071 \quad , R^2 = 1 \tag{6}$$

Where $y_{max}$ is the depth of maximum von Mises stress in $\mu m$, $p_{max}$ is the maximum contact pressure in MPa, E is the elastic modulus in GPa, R is the equivalent radius of curvature in mm and ν is Poisson's ratio. The magnitude and depth of maximum von Mises stress in some contact condit

ion are calculated by the presented formulas and compared with the results of literature in "Table 1 and Table 2", respectively.

Data show a close agreement between the results of presented formulas and literature. Therefore, these two formulas can be used instead of complicated and time-consuming simulations or finite element programs to calculate the magnitude and depth of maximum von Mises stress in cylindrical contact problems.

**Table 1** Comparison between maximum von Mises stress results from presented formula and literature

| Maximum contact pressure (MPa) | Poisson's ratio | Maximum von Mises stress (MPa) | | Error (%) |
|---|---|---|---|---|
| | | literature | Presented formula | |
| 1000 | 0.3 | 562.00 [6] | 557.60 | 0.78 |
| 1200 | 0.3 | 680.00 [6] | 669.12 | 1.6 |
| 1400 | 0.3 | 795.00 [6] | 780.64 | 1.8 |
| 1550 | 0.3 | 882.00 [6] | 864.28 | 2 |
| 1700 | 0.3 | 968.00 [6] | 947.92 | 2.07 |
| 1550 | 0.3 | 863.35 [7] | 864.28 | 0.11 |
| 1336 | 0.3 | 728.00 [16] | 744.95 | 2.33 |
| 1000 | 0.3 | 588.00 [21] | 557.60 | 5.17 |

**Table 2** Comparison between depth of maximum von Mises stress results from presented formula and literature

| Maximum contact pressure (MPa) | ν | E (GPa) | $R^*$ (mm) | Depth of maximum von Mises stress (μm) | | Error (%) |
|---|---|---|---|---|---|---|
| | | | | literature | Presented formula | |
| 1000 | 0.3 | 206 | 06 | 74.0 [6] | 074.7 | 0.90 |
| 1200 | 0.3 | 206 | 08 | 119 [6] | 120.0 | 0.42 |
| 1400 | 0.3 | 206 | 10 | 173 [6] | 174.0 | 0.76 |
| 1550 | 0.3 | 206 | 14 | 268 [6] | 270.0 | 0.83 |
| 1700 | 0.3 | 206 | 20 | 421 [6] | 423.0 | 0.58 |
| 1550 | 0.3 | 206 | 10 | 192 [10] | 193.0 | 0.52 |

## 4 CONCLUSIONS

In analysis of elements subjected to static contact loading, the maximum distortion energy criterion is a common criterion in mechanical design. Using maximum distortion energy criterion requires knowing the maximum magnitude of von Mises stress in the elements. In the case that elements have relative motion, because of the moving contact area, contact stresses will be cyclic. In the presence of cyclic loading, fatigue phenomena occur and fatigue fracture may appear. When an element is subjected to cyclic contact loading, after a certain cycle, cracks initiate. Due to effect of initial crack position on the fatigue life, size and shape of spalls, determination of the exact position of crack initiation is very important.

In this work, Hertzian contact stress relations are used and a Matlab code is provided to calculate the magnitude of von Mises stress in various depths under the contact area. Then the largest value in every case is selected. The stress field in contact problems is affected by some of the parameters: material properties, element geometry and loading conditions. For contact problems of cylindrical elements, frictionless condition and linear elastic materials, four parameters have effect on stress field: maximum contact pressure, equivalent curvature radius, elastic modulus and Poisson's ratio of materials. The Effect of various values of these four parameters is investigated.

The magnitude of maximum von Mises stress is affected by maximum contact pressure and Poisson's ratio, but the depth of maximum von Mises stress is a function of every four parameters. Two formulas are presented to calculate the magnitude and depth of maximum von Mises stress.

## REFERENCES


[1] Boresi, A. P., Schmidt, R. J., and Sidebottom, O. M., Advanced Mechanics of Materials, 5th ed, John Wiley and Sons, New York, USA, 1993, Chap. 18.

[2] Beer, F. P., Johnston, E. R., DeWolf, J. T., and Mazurek, D. F., Mechanics of Materials, 7th ed, McGraw-Hill, New York, USA, 2014, Chap. 7.

[3] Heirani, H., Farhangdoost, K., Effect of Friction Coefficient and Maximum Contact Pressure on The Spalling Depth of Gear Teeth Flank, Proceedings of the 7th International Conference on Fracture Fatigue and Wear, 2018, pp. 63-68, DOI: 10.1007/978-981-13-0411-8_6.

[4] Johnson, K. L., Contact Mechanics, Cambridge University Press, Cambridge, UK, 1985, Chap. 4.

[5] Glodez, S., Winter, H., and Stuwe, H. P., A Fracture Mechanics Model for The Wear of Gear Flanks by

Pitting, Wear, Vol. 208, No. 1-2, 1997, pp. 177-183, DOI: 10.1016/S0043-1648(97)00008-2.

[6] Glodez, S., Ren, Z., Modelling of Crack Growth Under Cyclic Contact Loading, Theoretical and Applied Fracture Mechanics, Vol. 30, No. 2, 1998, pp. 159-173, DOI: 10.1016/S0167-8442(98)00053-6.

[7] Glodez, S., Ren, Z., and Flasker, J., Simulation of Surface Pitting Due to Contact Loading, International Journal for Numerical Method in Engineering, Vol. 43, No. 1, 1998, pp. 33-50, DOI: 10.1002/(SICI)10970207(19980915)43:1<33::AID-NME410>3.0.CO;2-Z.

[8] Glodez, S., Abersek, B., Flasker, J., and Ren, Z., Evaluation of the Service Life of Gears in Regard to Surface Pitting, Engineering Fracture Mechanics, Vol. 71, No. 4-6, 2004, pp. 429-438, DOI: 10.1016/S0013-7944(03)00049-3.

[9] Aslantas, K., Tasgetiren, S., A Study of Spur Gear Pitting Formation and Life Prediction, Wear, Vol. 257, No. 11, 2004, pp. 1167-1175, DOI: 10.1016/j.wear.2004.08.005.

[10] Fajdiga, G., Glodez, S., and Kramar, J., Pitting Formation Due to Surface and Subsurface Initiated Fatigue Crack Growth in Contacting Mechanical Elements, Wear, Vol. 262, No. 9-10, 2007, pp. 1217-1224, DOI: 10.1016/j.wear.2006.11.016.

[11] Francois, H., Gilbert, H., Damien, H., Mikael, G., and Thomas, B., CDM Approach Applied to Fatigue Crack Propagation on Airframe Structural Alloys, Procedia Engineering , Vol. 2, No. 1, 2010, pp. 1403-1412, DOI: 10.1016/j.proeng.2010.03.152.

[12] Wei, P., Zhou, H., Liu, H., Zhu, C., Wang, and W., Deng, G., Modeling of Contact Fatigue Damage Behavior of A Wind Turbine Carburized Gear Considering Its Mechanical Properties and Microstructure Gradients, International Journal of Mechanical Sciences, Vol. 156, 2019, pp. 283-296, DOI: 10.1016/j.ijmecsci.2019.04.004.

[13] Liu, H., Wang, W., Zhu, C., Jiang, C., Wu, W., and Parker, R. G., A Microstructure Sensitive Contact Fatigue Model of A Carburized Gear, Wear, Vol. 436, 2019, pp. 203035, DOI: 10.1016/j.wear.2019.203035.

[14] He, H., Liu, H., Zhu, C., and Tang, J., Study on the Gear Fatigue Behavior Considering the Effect of Residual Stress Based on The Continuum Damage Approach, Engineering Failure Analysis, Vol. 104, 2019, pp. 531-544, DOI: 10.1016/j.engfailanal.2019.06.027.

[15] Li, W., Deng, S., and Liu, B., Experimental Study on The Influence of Different Carburized Layer Depth on Gear Contact Fatigue Strength, Engineering Failure Analysis, Vol. 107, 2020, pp. 104225, DOI: 10.1016/j.engfailanal.2019.104225.

[16] Guan, J., Wang, L., Zhang, C., and Ma, X., Effects of Non-Metallic Inclusions on The Crack Propagation in Bearing Steel, Tribology International, Vol. 106, 2017, pp. 123-131, DOI: 10.1016/j.triboint.2016.10.030.

[17] Heirani, H., Farhangdoost, K., Predicting Depth and Path of Subsurface Crack Propagation at Gear Tooth Flank Under Cyclic Contact Loading, Journal of Solid Mechanics, Vol. 9, No. 3, 2017, pp. 587-598, DOI: 20.1001.1.20083505.2017.9.3.10.9.

[18] Sun, P., An, Z., and Jiang, W., Analysis of Non-Hertz Contact Stress and Bearing Capacity on Meshing Pairs in a Real-Time Non-Clearance Precision Ball Transmission, Journal of the Brazilian Society of Mechanical Sciences and Engineering, Vol. 40, No. 6, 2018, pp. 1-11, DOI: 10.1007/s40430-018-1197-2.

[19] Kohn, A. O., Silva, F. A., A Numerical and Analytical Study of The Stress Field Generated by The Contact Between a Rail and a Wheel, SN Applied Sciences, Vol. 2, No. 7, 2020, pp. 1-8, DOI: 10.1007/s42452-020-3044-1.

[20] Budynas, R., Nisbett, K., Shigley's Mechanical Engineering Design, 10th ed, McGraw-Hill, New York, USA, 2015, Chap. 3.

[21] Qin, W. J., Guan, C. Y., An Investigation of Contact Stresses and Crack Initiation in Spur Gears Based on Finite Element Dynamics Analysis, International Journal of Mechanical Sciences, Vol. 83, 2014, pp. 96-103, DOI: 10.1016/j.ijmecsci.2014.03.035.