Nonlinear Vibration Analysis of FG Nano-Beams in Thermal Environment and Resting on Nonlinear Foundation based on Nonlocal and Strain-Inertia Gradient Theory

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Abstract: In present research, nonlinear vibration of functionally graded nano-beams subjected to uniform temperature rise and resting on nonlinear foundation is comprehensively studied. The elastic center can be defined to remove stretching and bending couplings caused by the FG material variation. The small-size effect, playing essential role in the dynamical behavior of nano-beams, is considered here applying strain-inertia gradient and non-local elasticity theory. The governing partial differential equations have been derived based on the Euler-Bernoulli beam theory utilizing the von Karman strain-displacement relations. Subsequently, using the Galerkin method, the governing equations is reduced to a nonlinear ordinary differential equation. The closed form analytical solution of the nonlinear natural frequency is then established using the homotopy analysis method. Finally, the effects of different parameters such as length, nonlinear elastic foundation parameter, thermal loading, non-local parameter and gradient parameters are comprehensively investigated on the FG nano-beams vibration using the homotopy analysis method. As the main results, it is observed that by increasing the non-local parameter, the frequency ratio for strain-inertia gradient theory has an increasing trend while it has decreasing trend for non-local elasticity theory. Also, the nonlinear natural frequencies obtained using strain-inertia gradient theory are greater than the results of non-local elasticity and classical theory.

Keywords: FG Nano-beam, Homotopy Analysis Method, Nonlinear Foundation, Strain-Inertia Gradient Theory

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1 INTRODUCTION

Advances made in industry, have led engineers to exploit materials with better properties. Functionally graded (FG) materials are composite materials with heterogeneous fine structures. The mechanical properties of FG materials continuously change from one level to the other. This can be achieved by gradual change of volume fraction as a function of position along the thickness of the ingredients. In most cases, the FG materials are made of combination of metal and ceramic to have metal strength as well as resistance to high temperature of environment and yet, to eliminate the share surface problems [1-3].
While nano-beams have been extensively used in many applications of nano-sized devices and systems, the studies of nano-structures using the non-local elasticity theory have been an extensive section of research in recent years [4-8]. The equations of motion of non-local nonlinear free vibration of functionally graded nano-beams with simply supported and simple-clamped boundary conditions are solved using multiple scale method [9]. An exact solution for the nonlinear forced vibration analysis of nano-beams made of FG materials subjected to the thermal environment including the effect of surface stress employed the classical beam theory is furthermore presented [10]. Transient analysis of a three-layer microbeam subjected to electric potential [11], Also, vibration and bending analyses of the piezomagnetic three-layer nano-beams based on sinusoidal shear deformation theory are investigated [12-14].
Size-dependent electro-elastic analysis of a sandwich microbeam based on higher-order sinusoidal shear deformation theory [15] and vibration bending analysis of a sandwich microbeam with two integrated piezo-magnetic face-sheets are investigated [16]. Wave propagation analysis of a functionally graded magneto-electro-elastic nano-beam rest on Visco-Pasternak foundation [17] and nonlinear vibration analysis of sandwich nano-beam with FG-CNTRCs face-sheets in electro-thermal environment are investigated [18]. The Homotopy Analysis Method is widely used to investigate the nonlinear vibration behavior of beams and it is shown that Homotopy Analysis Method is a very strong semi-analytical method for vibration analysis of structures [19-21]. Gradient elasticity theories provide extensions of the classical equations of elasticity with additional higher-order spatial derivatives of strains, stresses and/or accelerations.
Askes and Aifantis [22] discussed various formats of gradient elasticity and their performance in static and dynamic applications. New numerical results showed the removal of singularities in statics and dynamics, as well as the size-dependent mechanical response predicted by gradient elasticity. The carbon nano-tubes are also widely studied in the literature using the stress gradient and strain-inertia gradient elasticity theories [23] and stress, strain and combined strain-inertia gradient elasticity theories [24].
In the present research, the nonlinear vibration of a FG nano-beam resting on nonlinear elastic foundation subjected to thermal load is investigated based on the strain-inertia gradient elasticity and also non-local elasticity theory. The different behaviors of the two theories would be comprehensively examined. The Homotopy Analysis Method is also utilized to solve the governing equations of FG nano-beams. Finally, the parametric study on nano-beam lengths, linear and nonlinear foundation stiffness, temperature rise and gradient index would be presented considering the small scale effects on the frequency ratios of FG nano-beams.

2 GOVERNING EQUATION

2.1. Stress Gradient (Non-local) Elasticity Theory

In the non-local (nl) theory of elasticity, the points undergo translational motion as in the classical case, but the stress at a point depends on the strain in a region near that point [25]. The non-local constitutive behavior of Hookean solids can be represented by the following differential constitutive relation [26-27]:

$$(1-\mu^2\nabla^2)\sigma^{nl}=\sigma^{l} \quad (1)$$

Where μ is the non-local parameter, σ^{l} is the macroscopic or local stress tensor at a point and σ^{nl} is the non-local stress tensor.
An FG nano-beam with length L, thickness h and width b rests on nonlinear Winkler-Pasternak foundation is shown in “Fig. 1”.

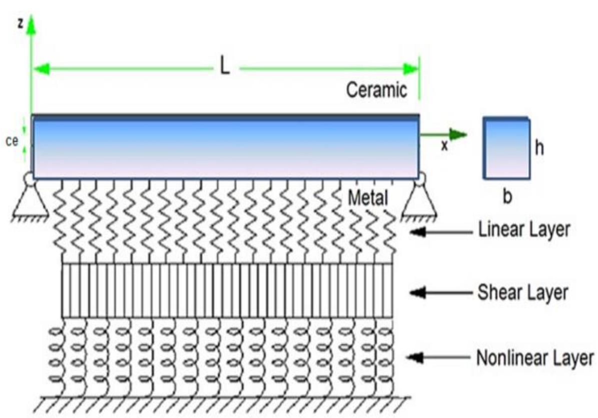


Fig. 1 Geometry of a FG nano-beam.

Since the FG nano-beam is generally composed of two different materials at the top and the bottom surfaces, the power law distribution for the effective material properties dictates the material variation profile through the thickness of the small-scale FG beam, as follow [17]:

$$\mathrm{P}(z_m) = (\mathrm{P}_U - \mathrm{P}_L)\left(\frac{z_m}{h} + \frac{1}{2}\right)^P + \mathrm{P}_L \quad (2)$$

Where P_U and P_L are the material properties at the upper and lower surface of the FG nano-beam, respectively. And a gradient index P determines the variation profile of material properties across the FG nano-beam thickness.

It is worth noting that the geometric center (z_m) is taken as references for the previous effective material properties. The elastic center can be defined to remove stretching and bending couplings caused by the FG material variation. For a FG nano-beam, the neutral axis passes through the elastic center of the Young elastic moduli field of longitudinal fibers. The relation between the elastic and geometric center is determined as [28]:

$$z = z_m + ce\ ,ce = \frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} z_m E(z_m) dz_m}{\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z_m) dz_m} = \frac{h(E_U - E_L)P}{2(2+P)(E_U + PE_L)} \quad (3)$$

Where ce is the position of elastic center. If the elastic center is taken as a reference, the Poisson's ratio $\nu(z)$, elastic modulus $E(z)$, mass density $\rho(z)$ and thermal, expansion coefficients $\alpha(z)$ are assumed to vary continuously along the z direction and can be expressed as:

$$E(z) = (E_c - E_m)(\frac{z}{h} + \frac{1}{2})^P + E_m \quad (4)$$

$$\rho(z) = (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^P + \rho_m \quad (5)$$

$$\alpha(z) = (\alpha_c - \alpha_m)\left(\frac{z}{h} + \frac{1}{2}\right)^P + \alpha_m \quad (6)$$

$$\nu(z) = (\nu_c - \nu_m)\left(\frac{z}{h} + \frac{1}{2}\right)^P + \nu_m \quad (7)$$

Where the subscripts c and m refer to the ceramic and metal phases, respectively. Using the classic Euler-Bernoulli beam theory, the displacement field at any point of the nano-beam can be written as:

$$u_x(x,z,t) = u(x,t) - z\frac{\partial w(x,t)}{\partial x} \quad (8)$$

$$u_y(x,z,t) = 0 \quad (9)$$

$$u_z(x,z,t) = w(x,t) \quad (10)$$

Where $u(x,t)$ and $w(x,t)$ are the displacement components of the mid-plane at time t. In accordance, the von Karman type nonlinear strain-displacement relation may be shown to be [29]:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}(\frac{\partial u_z}{\partial x})^2 = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \quad (11)$$

So for the beam in thermal environment the non-zero component of the stress tensor can be obtained as [30]:

$$\sigma_{xx} = \frac{E(z)}{1-\nu(z)^2}\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) - \frac{E(z)\alpha(z)}{1-\nu(z)}\Delta T \quad (12)$$

Where $E(z)$ is the Young modulus and $\nu(z)$ is Poisson's ratio and $\alpha(z)$ is thermal expansion coefficient and $\Delta T = T - T_\circ$ where T is the temperature distributed through the FG nano-beam and $T_\circ$ is the reference temperature. Using Hamilton's principle and minimizing strain energy and kinetic energy [31], the nonlinear equations of motion of the nano-beam can be derived as:

$$\frac{\partial N_{xx}}{\partial x} = I_1\frac{\partial^2 u}{\partial t^2} \quad (13)$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}\right) + f = I_1\frac{\partial^2 w}{\partial t^2} \quad (14)$$

Where N_{xx} and M_{xx} are the local force and bending moment resultants, respectively, given by:

$$N_{xx} = \int_A \sigma_{xx} dA = bA_1\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) - N_{th} \quad (15)$$

$$M_{xx} = \int_A z\sigma_{xx} dA = -bC_1\left(\frac{\partial^2 w}{\partial x^2}\right) - M_{th} \quad (16)$$

Moreover, the resultant thermal force and moment can be described as [32]:

$$N_{th} = b\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(z).\alpha(z)}{1-\nu}\Delta T dz \quad ,M_{th} = b\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(z).\alpha(z)}{1-\nu}\Delta T z dz \quad (17)$$

In the present study, ΔT is constant along the beam thickness and the initial uniform temperature ($T_0 =$

$27^{℃}$) is considered a stress free state. It is noteworthy that the resultant thermal force and moment N_{th} and M_{th} are constant. Also the cross section parameters $\{A_1, C_1\}$andI_1 introduced in preceding "Eqs. (13-16)" are defined as:

$$\{A_1, C_1\} = b\int_{-\frac{h}{2}-ce}^{\frac{h}{2}-ce} E(z)\{1, z, z^2\}\, dz \quad , \quad I_1 = b\int_{-\frac{h}{2}-ce}^{\frac{h}{2}-ce} \rho(z)\, dz \tag{18}$$

Moreover, the driving force f in "Eq. (14)" could be shown to be:

$$f = F(x,t) - k_L w + k_S\nabla^2 w - k_{NL}w^3 \tag{19}$$

Where $F(x,t)$ is lateral distributed loading and k_L, k_S are Winkler and Pasternak elastic foundation coefficients, respectively and k_{NL} is the nonlinear elastic foundation coefficient [33]. The non-local form of the equations of motions, "Eqs. (13) and (14)", can be shown to be as follows [9]:

$$\frac{\partial N_{xx}^{nl}}{\partial x} = I_1\frac{\partial^2 u}{\partial t^2} \tag{20}$$

$$\frac{\partial^2 M_{xx}^{nl}}{\partial x^2} + \frac{\partial}{\partial x}\left(N_{xx}^{nl}\frac{\partial w}{\partial x}\right) + f = I_1\frac{\partial^2 w}{\partial t^2} \tag{21}$$

Where the superscript nl denote the non-local elasticity theory. It is also noteworthy that the effect of longitudinal or inplane deformation and inertia on the large amplitude flexural vibrations of slender beams and thin plates is negligible [34]. If the axial inertia is neglected, "Eq. (20)" results in:

$$N_{xx}^{nl} = N_{NL} - N_{th} = constant \tag{22}$$

Where N_{NL} is the nonlinear in-plane force that is caused by stretching and bending of the neutral axis. Then the non-local force and bending moment resultants can be obtained by imposing the operator of $(1-\mu^2\nabla^2)$ on the left-hand side of "Eqs. (15) and (17)" [25], using Eqs. (20-22) it may be shown that:

$$N_{xx}^{nl} = bA_1\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) - N_{th} \tag{23}$$

$$M_{xx}^{nl} = \mu^2\left(-N_{xx}^{nl}\frac{\partial^2 w}{\partial x^2} - f + I_1\frac{\partial^2 w}{\partial t^2}\right) - bC_1\left(\frac{\partial^2 w}{\partial x^2}\right) - M_{th} \tag{24}$$

For nano-beams with immovable ends (i.e. u=0 and w= 0, at x = 0 and L), integrating "Eq. (23)" with respect to x while using "Eq. (22)" leads to:

$$N_{NL} = \frac{bA_1}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \tag{25}$$

Finally, substitution of "Eqs. (23) and (24)" into "Eq. (21)" would result in the governing equation of the non-local elastic nonlinear forced vibration for the Euler–Bernoulli functionally graded nano-beam resting on nonlinear foundation in thermal environment as:

$$D^{**}\frac{\partial^4 w}{\partial x^4} + (N_{th} - N_{NL})\frac{\partial^2 w}{\partial x^2} + I_1\frac{\partial^2 w}{\partial t^2} - \mu^2 I_1\frac{\partial^4 w}{\partial x^2\partial t^2} + \frac{\partial^2 M_{th}}{\partial x^2} = f - \mu^2\frac{\partial^2 f}{\partial x^2} \tag{26}$$

Where:

$$D^{**} = bC_1 + \mu^2\left[\frac{bA_1}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx - b\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(z).\alpha(z)}{1-\nu}\Delta T dz\right]$$

2.2. Strain-inertia Gradient Theory

In another famous theory developed by Mindlin [35], the strain energy is considered as a function of the first and second-order gradient of strain tensor. This theory was then reformulated and renamed strain gradient theory by Fleck and Hutchinson [36]. The strain gradient theory has been successfully applied to analyze static and dynamic mechanical behavior of micro and nano-structures [17]. Also the strain gradient formula combined with inertia gradient was introduced by Askes and Aifantis [17] and is described as:

$$\sigma = E(1 + l_s^2\nabla^2)\varepsilon + \rho l_d^2\ddot{\varepsilon} \tag{27}$$

Where l_s and l_d are the length scale parameters related to strain gradients and inertia gradients, respectively.
It is also worth noting that the nonlinear equation of motions, "Eqs. (13) and (14)", can be used in strain-inertia gradient (SIG) form as [37]:

$$\frac{\partial N_{xx}^{sig}}{\partial x} = I_1\frac{\partial^2 u}{\partial t^2} \tag{28}$$

$$\frac{\partial^2 M_{xx}^{sig}}{\partial x^2} + \frac{\partial}{\partial x}\left(N_{xx}^{sig}\frac{\partial w}{\partial x}\right) + f = I_1\frac{\partial^2 w}{\partial t^2} \tag{29}$$

Where the superscript *sig* denotes the strain-inertia gradient theory. The axial stress at a generic point of a nano-beam based on the strain-inertia gradient theory can be formulated as:

$$\sigma_{xx} = E(1 + l_s^2 \nabla^2)\varepsilon_{xx} + \rho l_d^2 \ddot{\varepsilon}_{xx} - \frac{E(z)\alpha(z)}{1-\nu(z)}\Delta T \quad (30)$$

For nano-beam with immovable ends the axial strain can be determined by the von Karman strain as:

$$\varepsilon_{xx} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \quad (31)$$

Similar to the procedure of determining the force and moment resultants in non-local elasticity theory, the force and moment resultants could be obtained considering gradient parameters as:

$$N_{xx}^{sig} = \int_A \sigma_{xx} dA = bA_1\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + l_s^2\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}\right]\right] + bD_1 l_d^2\left[\left(\frac{\partial^2 w}{\partial x \partial t}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x \partial t^2}\right] - bE_1 l_d^2 \frac{\partial^4 w}{\partial x^2 \partial t^2} - N_{th} \quad (32)$$

$$M_{xx}^{sig} = \int_A z\sigma_{xx} dA = -bC_1\left[\frac{\partial^2 w}{\partial x^2} + l_s^2\frac{\partial^4 w}{\partial x^4}\right] - bF_1 l_d^2\frac{\partial^4 w}{\partial x^2 \partial t^2} - M_{th} \quad (33)$$

Where the parameters $\{D_1, E_1, F_1\}$ introduced in "Eqs. (32) and (33)" are the generalized mass moment of inertia and defined as:

$$\{D_1, E_1, F_1\} = \int_{-\frac{h}{2}-ce}^{\frac{h}{2}-ce} \rho(z)\{1, z, z^2\}\, dz \quad (34)$$

Finally, by substituting "Eqs. (32) and (33)" into "Eq. (29)", the governing equation of motion for forced nonlinear vibration of FG nano-beams resting on nonlinear elastic foundation and subjected to thermal load using strain-inertia gradient theory can be written as:

$$\begin{aligned} & bC_1\left[\frac{\partial^4 w}{\partial x^4} + l_s^2\frac{\partial^6 w}{\partial x^6}\right] - bA_1\left[\frac{3}{2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + l_s^2\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^3 + 4\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3} + \frac{\partial^4 w}{\partial x^4}\left(\frac{\partial w}{\partial x}\right)^2\right]\right] - bD_1 l_d^2 \\ & \times\left[\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial x \partial t}\right)^2 + 2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x \partial t^2} + 2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial t}\frac{\partial^3 w}{\partial x^2 \partial t} + \frac{\partial^4 w}{\partial x^2 \partial t^2}\left(\frac{\partial w}{\partial x}\right)^2\right] \\ & + bF_1 l_d^2\frac{\partial^6 w}{\partial x^4 \partial t^2} + bE_1 l_d^2\left[2\left(\frac{\partial^3 w}{\partial x^2 \partial t}\right)^2 + \frac{\partial^2 w}{\partial x^2}\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^3 w}{\partial x \partial t^2}\frac{\partial^3 w}{\partial x^3} + 2\frac{\partial^2 w}{\partial x \partial t}\frac{\partial^4 w}{\partial x^3 \partial t}\right] \\ & + N_{th}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 M_{th}}{\partial x^2} + I_1\frac{\partial^2 w}{\partial t^2} = f \end{aligned} \quad (35)$$

3 NONLINEAR FREE VIBRATION

In general, M_{th} is a function of x *and* z coordinate and time t. If it is assumed that the temperature only varies in the thickness direction, then $\partial^2 M_{th}/\partial x^2 = 0$ [30]. The governing equations for nonlinear free vibration analysis can be obtained by setting $(x,t) = 0$. Furthermore, to reduce the nonlinear equation of free vibration of FG nano-beam based on non-local elasticity "Eq. (26)", strain-inertia gradient theory "Eq. (35)" into a time-varying set of ordinary differential equations, the Galerkin method would be employed here. To this end, the displacement function is supposed to have the separable form of:

$$w(x,t) = \phi(x)q(t) \quad (36)$$

Where $q(t)$ is a time base function to be determined later and $\phi(x)$ is the linear spatial mode shape. Considering the simply supported boundary conditions, an appropriate approximation for spatial base function $\phi(x)$ can be expressed as [38]:

$$\phi_n(x) = Sin(\alpha x) \quad (37)$$

Where $\alpha = \pi/l$ .The governing equation of the time base function in free vibration of FG nano-beam based on non-local elasticity could be determined by substituting “Eq.(36) into Eq.(26)” subsequently multiplying by the linear spatial mode shape and integrating along the FG nano-beam length as [39]:

$$\ddot{q} + \gamma_1 q + \gamma_3 q^3 = 0 \tag{38}$$

Where to summarize the mathematical formulation, the coefficients $\gamma_1, \gamma_2 and\ \gamma_3$ are introduced as:

$$\begin{aligned}
\gamma_1 &= \left(D^* a_3 + N_{th} a_2 + K_L(a_1 - \mu^2 a_2)\right)a^* - K_S(a_2 - \mu^2 a_3)a^* \\
\gamma_3 &= -\frac{bA_1 a_5(a_2 - \mu^2 a_3)}{2la^*} + \frac{K_{NL} l\left(a_6 - 3\mu^2(2a_7 + a_8)\right)}{la^*} \\
\{a_1, a_2, a_3, a_5, a_6, a_7, a_8\} &= \int_0^l \left\{\phi^2, \phi\frac{d^2\phi}{dx^2}, \phi\frac{d^4\phi}{dx^4}, (\frac{d\phi}{dx})^2, \phi^4, \phi^2(\frac{d\phi}{dx})^2, \phi^3\frac{d^2}{dx}\right.
\end{aligned} \tag{39}$$

$$a^* = I_1(a_1 - \mu^2 a_2)$$

It is also worth noting that ignoring the effect of elastic foundation and thermal loading in “Eq. (38) and (39)” would result in the same governing equation for non-local nonlinear free vibration of FG nano-beams as determined by Nazemnezhad et al. [9]. The initial conditions for “Eq. (38)” would be assumed to be:

$$\begin{cases} q(0) = a_0 \\ \dot{q}(0) = 0 \end{cases} \tag{40}$$

Where a_0 is maximum amplitude corresponding to the time base function $q(t)$. By using Galerkin method and substituting “Eq. (36)” into “Eq. (35)”, the governing equation of the time base function in free vibration of FG nano-beam based on strain-inertia gradient elasticity could be determined as:

$$\ddot{q} + \beta_1 q + \beta_2 q\dot{q}^2 + \beta_3 q^2\ddot{q} + \beta_4 q\ddot{q} + \beta_5\dot{q}^2 + \beta_6 q^3 = 0 \tag{41}$$

Where again to summarize the mathematical formulation, the coefficients $\beta_1, \beta_2, \ldots, \beta_6$ are introduced as:

$$\begin{aligned}
\beta_1 &= (bC_1(a_3 + l_s^2 a_{12}) + N_{th} a_2 + K_L a_1 - K_S a_2)/a^{**} \\
\beta_2 &= -3bD_1 l_d^2 a_9/a^{**} \\
\beta_3 &= -3bD_1 l_d^2 a_9/a^{**} \\
\beta_4 &= -bE_1 l_d^2(a_{16} + a_{17})/a^{**} \\
\beta_5 &= -2bE_1 l_d^2(a_{16} + a_{17})/a^{**} \\
\beta_6 &= (bA_1[-1.5a_9 - l_s^2(a_{10} + 4a_{11} + a_{13})] + K_{NL} a_6)/a^{**}
\end{aligned}$$

$$a^{**} = I_1 a_1 + bF_1 l_d^2 a_3$$

$$\{a_9, a_{10}, a_{11}, a_{12}\} = \int_0^l \left\{\phi\frac{d^2\phi}{dx^2}\left(\frac{d\phi}{dx}\right)^2, \phi\left(\frac{d^2\phi}{dx^2}\right)^3, \phi\frac{d\phi}{dx}.\frac{d^2\phi}{dx^2}.\frac{d^3\phi}{dx^3}, \phi\frac{d^6\phi}{dx^6}\right\} dx$$

$$\{a_{13}, a_{16}, a_{17}\} = \int_0^l \left\{\phi\frac{d^4\phi}{dx^4}\left(\frac{d\phi}{dx}\right)^2, \phi\left(\frac{d^2\phi}{dx^2}\right)^2, \phi\frac{d\phi}{dx}.\frac{d^3\phi}{dx^3}\right\} dx$$

With the same initial conditions according to “Eq. (40)”.

4 SOLUTION METHOD

4.1. Homotopic Analysis Method

Homotopic Analysis Method (HAM) is a general analytic method for solving the non-linear differential equations that successfully results in convergent series solutions of strongly nonlinear problems [40]. The HAM transforms a non-linear differential equation into an infinite number of linear differential equations with embedding an auxiliary parameter *p* that typically ranges from zero to one. As *p* increases from 0 to 1, the solution varies from the initial guess to the exact solution [40]. To illustrate the basic ideas of the HAM, consider the following non-linear differential equation:

$$N[q(t)] = 0 \tag{42}$$

Where N is a nonlinear operator and *t* denotes time as the independent variable and *q* (*t*) is an unknown variable. The homotopy function is constructed as follows [40]:

$$\overline{H}(\varphi; p, \hbar, H(t)) = (1-p)L[\varphi(t;p) - q_0(t)] - p\hbar H(t)N[\varphi(t;p)] \tag{43}$$

Where φ is a function of *t* and *p* and also ħ and *H*(*t*) are non-zero auxiliary parameter and non-zero auxiliary function, respectively. The parameter L denotes an auxiliary linear operator. As *p* increases from 0 to 1, the $\varphi(t;p)$ varies from the initial approximation to the exact solution. So the zero-order deformation is constructed as [40]:

$$(1-p)L[\varphi(t;p) - q_0(t)] = p\hbar H(t)N[\varphi(t;p)] \tag{44}$$

With the following initial conditions corresponding to initial conditions of “Eq. (42)”:

$$\varphi(0;p) = a_0 \quad , \quad \frac{d\varphi(0;p)}{dt} = 0 \tag{45}$$

The higher order approximations of the solution can be obtained by calculating the *m*-order (*m*>1) deformation equation as [40]:

$$L[q_m - \chi q_{m-1}] = \hbar H(t) R_m(q_{m-1}, \omega_{m-1}) \quad (46)$$

Where the χ and $R_m(q_{m-1}, \omega_{m-1})$ are defined as follows [40]:

$$R_m(q_{m-1}, \omega_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(t;p), \omega(p)]}{\partial p^{m-1}} |p = 0 \quad (47)$$

$$\chi = \begin{cases} 0 & m \leq 1 \\ 1 & m \geq 2 \end{cases} \quad (48)$$

Subjected to the following homogeneous initial conditions:

$$q_m(0) = \dot{q_m}(0) = 0 \quad (49)$$

Since the initial conditions of “Eq. (45)” are already imposed to the zero-order deformation.

4.2. Application of the HAM

The governing equation for non-local nonlinear free vibration of FG nano-beam resting on elastic nonlinear foundation may be transformed to the following equation using the change of variable $\tau = \omega t$:

$$\omega^2 \ddot{q} + \gamma_1 q + \gamma_3 q^3 = 0 \quad (50)$$

Where the dot denotes the $d/d\tau$. In order to solve the “Eq. (50)” through HAM, the first conjecture of the problem solution, which satisfies initial conditions “Eq. (49)”, can be stated as follow:

$$q_0(\tau) = a_0 \cos(\tau) \ , \ q_0(0) = a_0 \ , \dot{q}_0(0) = 0 \quad (51)$$

Where $a_0 = w_{max}/r$ is the maximum dimensionless amplitude or amplitude ratio. Linear and nonlinear operator can also be expressed as follow:

$$L[q(\tau,p)] = \omega_0^2 \left[\frac{\partial^2 q(\tau,p)}{\partial \tau^2} + q(\tau,p)\right] \quad (52)$$

$$N[q(\tau,p)] = \omega^2 \frac{\partial^2 q(\tau,p)}{\partial \tau^2} + \gamma_1 q + \gamma_3 q^3 \quad (53)$$

The first-order deformation equation which gives the first-order approximation of the $q(\tau)$ can be written as:

$$L[q_1(\tau)] = \hbar H(t) N[q(\tau,p), \omega]_{q=0} \quad (54)$$

$$q_1(0) = 0 \ , \ \frac{\partial q_1(0)}{\partial \tau} = 0 \quad (55)$$

The auxiliary function $H(\tau)$ and the auxiliary parameter ħ which adjust convergence region and rate of approximate solution must be chosen in such a way that the solution of “Eq. (54)” could be expressed by a set of base functions [40]. While assuming ħ = -1 and $H(\tau) = 1$ can satisfy this constraint [16], as a result the Eq. (54) utilizing the $q_0(\tau)$as “Eq. (51)”, The time response of the first-order transformation equation and the equivalent terms for nonlinear natural frequencies would be determined as follows:

$$q(\tau) \approx a_0 \cos(\tau) - \frac{\gamma_3 a_0^3}{32\omega_0^2}[\cos(\tau) - \cos(3\tau)] \quad (56)$$

$$\omega_n \approx \omega_0 + \frac{\gamma_3 a_0^2}{128\omega_0^3}[2(\omega_0^2 - \gamma_1) - 3\gamma_3 a_0^2] \quad (57)$$

Where $\omega_0 = (\gamma_1 + \frac{3}{4}\gamma_3 a_0^2)^{0.5}$. It is worth noting that $\sqrt{\gamma_1}$ is the linear natural frequency, therefore according to ω_0 the nonlinear frequencies are greater than linear frequencies. Finally, according to HAM, first-order approximation of natural frequencies and time response for nonlinear vibration of FG nano-beam based on strain-inertia gradient theory utilizing “Eq. (41) and Eq. (51)” and preventing so-called secular term in time response, the coefficients of the term $\cos(\tau)$ are set to zero, are expressed as:

$$\omega_0 = \sqrt{\frac{4\beta_1 + 3\beta_7 a_0^2}{4 + (3\beta_3 - \beta_2)a_0^2}} \quad (58)$$

$$q(\tau) \approx q_0(\tau) + q_1(\tau) = a_0 \cos(\tau) + \frac{a_0^2}{96\omega_0^2}\Big[-48\omega_0^2(\beta_5 - \beta_4) + \big[32\big(\omega_0^2(2\beta_5 - \beta_4)\big) + 3\omega_0^2(\beta_3 + \beta_2) - 3\beta_6\big]\cos(\tau) - 16\omega_0^2(\beta_5 + \beta_4))\cos(2\tau) + 3a_0[\beta_6 - \omega_0^2(\beta_3 + \beta_2)]\cos(3\tau)\Big] \quad (59)$$

The exact solution for “Eq. (38)” so-called duffing equation, can be expressed as [43]:

$$\omega_n = \frac{\pi\sqrt{\alpha_1 + \alpha_3 a_0^2}}{2K} \quad (60)$$

Where K is defined by using complete elliptic integral of the first kind as [44]:

$$K = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1+m\times sin^2(x)}} dx \qquad (61)$$

Where $m = \alpha_3 a_0^2/(2(\alpha_1 + \alpha_3 a_0^2))$.

5 RESULT AND DISCUSSION

5.1 Comparison Study

A comparative study for evaluation of classical nonlinear frequency to classical linear frequency ratios (ω_{NL}/ω_L) between the present second-order homotopy analysis solution, the exact solution, the multiple scale method[9] and the Ritz–Galerkin method [45] is carried out in “Table 1” for simply supported isotropic beams with $L = 2\ m$ and $h = 0.1\ m$.

It is observed that the present results agree well with those given by using exact solution and Singh et al. [45] using Ritz–Galerkin method. It should be noted here that as the amplitude ratio increases, the difference between present solution and the multiple scale solutions used in Ref. [9] compare with exact solution becomes more. Therefore, the HAM has more accuracy than multiple scale method. It can be observed from Table 1 that the HA method has high accuracy and high adapted with exact solution.

Table 1 Comparison of frequency ratio (ω_{NL}/ω_L) for a simply supported isotropic beam

$\frac{a_0}{r}$	Present			Ref.[9]	Ref.[45]	Exact (Eq.60)
	ω_{NL}	ω_L	ω_{NL}/ω			
1	10.7496	9.8696	1.08916	1.0937	1.0897	1.08916
2	13.0063	9.8696	1.31781	1.3750	1.3229	1.31778
3	16.0469	9.8696	1.62589	1.8438	1.6394	1.62568
4	19.5079	9.8696	1.97656	-	-	1.97602

It is noted that $r = \sqrt{I/A}$ is gyration radius. “Table 2” shows a comparison between non-local linear and nonlinear frequency ratios of FG nano-beams for various amplitude ratios, gradient indices, non-local parameter values and nano-beam length. It can be seen that the results from the present study are comparable to the results of Nazemnezhad et al. [9].

Table 2 Comparison of linear and nonlinear frequency ratios for a simply supported FG nano-beam

$\frac{a_0}{r}$	P	L,nm	Present μ^2, nm^2		Ref.[9] μ^2, nm^2	
			2	4	2	4
0 (Linear)	3	10	0.91386	0.84673	0.9139	0.8467
		20	0.97620	0.95403	0.9762	0.9540
		30	0.98921	0.97876	0.9892	0.9788
1(Nonlinear)	1	10	0.9282	0.8729	0.9247	0.8659
		20	0.9801	0.9615	0.9792	0.9599
		30	0.9909	0.9822	0.9906	0.9815
	2	10	0.9267	0.8702	0.9221	0.8607
		20	0.9797	0.9608	0.9786	0.9585
		30	0.9908	0.9819	0.9903	0.9809

5.2. Benchmark Results

To demonstrate the small scale effects based on non-local and SIG elasticity theory on the nonlinear free vibration of FG nano-beams, variations of the frequency ratios versus gradient index, amplitude ratio, length of the FG nano-beam, linear and nonlinear foundation stiffness and temperature rise are presented in this section.

$$\begin{aligned}&\text{Frequency Ratio (NL/NL)}\\&=\frac{\text{Nonlocal or SIG nonlinear natural frequency}}{\text{classical nonlinear natural frequency}}\\&\text{Frequency Ratio (L/L)}\\&=\frac{\text{Nonlocal or SIG linear natural frequency}}{\text{classical linear natural frequency}}\\&\text{Frequency Ratio (NL/L)}\\&=\frac{\text{SIG nonlinear natural frequency}}{\text{SIG linear natural frequency}}\end{aligned} \qquad (62)$$

A FG nano-beam with squared cross-section $(b = h = 0.05L)$ is considered as a case study to illustrate general behavior of functionally graded nano-beams. The nano-beams are assumed to be made of Si_3N_4-SUS304 whose material properties are listed in “Table 3”.The half-wave number $n = 1$ and the relation between static and dynamic length scale parameters for strain-inertia gradient theory is assumed to be $l_d = 3l_s$ [46].As mentioned earlier, in non-local elasticity theory the parameter μ (nm) is the small-scale parameter revealing the small-scale effect on the responses of nano-size structures. In the present study, a conservative estimate of the small-scale parameter (μ) is considered to be in the range of 0-2 nm [6].

Table 3 Material properties of FG nano-beam [47]

Material	Young modulus,	Poisson's ratio	Mass density,	thermal expansion coefficient,
	GPa		kg/m^3	$10^{-6}/K^{\circ}$
SUS304	207.8	0.3178	8166	12.33
Si_3N_4	322.3	0.24	2370	5.87

It is noted that, increasing in temperature reduces the linear and nonlinear natural frequencies. "Fig. 2" shows that when temperature rises to $\Delta T > 186^{℃}$ for nonlocal and $\Delta T > 273^{℃}$ for SIG theory the buckling occurs. So, in all of analyzes the temperature increase is less than this suffering.

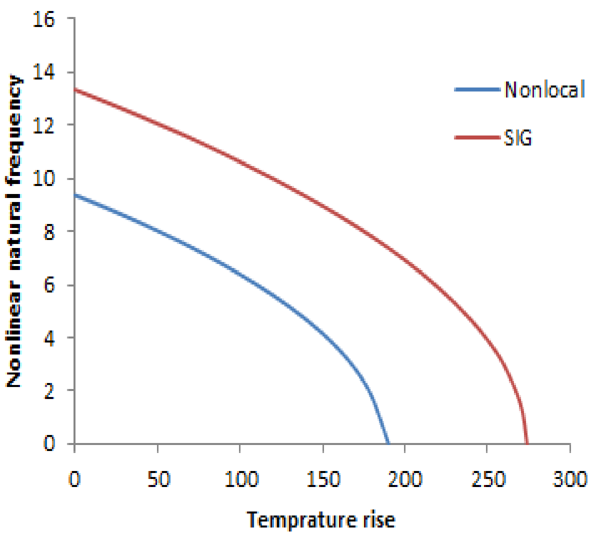


Fig. 2 Variations of the nonlinear natural frequency versus temperature rise. Buckling of FG nano-beam calculated based on Non-local theory and SIG theory. $\mu = 2 \times 10^{-9}, P = 2, a_0/r = 1$.

Some selected numerical results are also presented in "Tables 4 to 5" as benchmark in the future works.

5.2.1. Influence of the FG nano-beam length on the frequency ratio

Firstly, we turn our attention to the variations of the fundamental linear and nonlinear frequency ratios versus the length of FG nano-beam with elastic foundation and thermal loading based on non-local and SIG elasticity theory. The variations are plotted in "Fig. 3" for various amplitude ratios ($a_0 = 0,1,2$) when the gradient index is set to be $P = 2$.

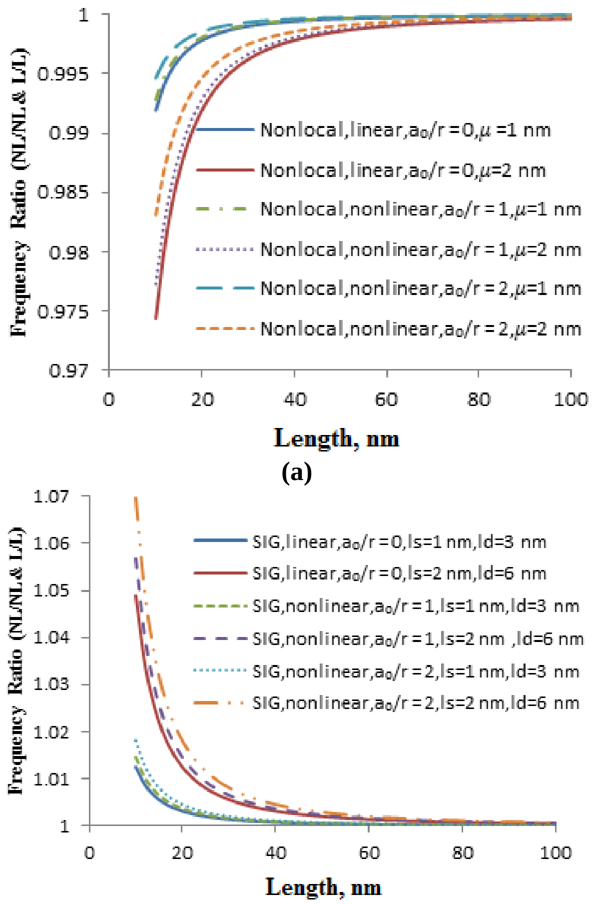
Table 4 Nonlinear frequencies and nonlinear frequency ratios of FG nano-beam upon non-local theory for various amplitude ratios, linear and nonlinear elastic foundation parameter and temperature rise

Nonlinear Frequencies				T	K_L	$\frac{a_0}{r}$
K_{NL}						
0		50				
K_S		K_S				
0	25	0	25			
9.3738	18.2923	10.77	19.0456	0	0	1
9.427	18.3196	10.8163	19.0718	0	1	1
11.7417	19.6114	12.8838	20.3158	0	50	1
6.3871	16.9569	8.3018	17.7668	100	0	1
9.5286	18.3721	10.905	19.1222	100	50	1
11.9147	19.7155	15.9518	22.3875	0	0	2
13.8549	20.9451	17.4488	23.4776	0	50	2
9.7409	18.4831	14.4009	21.3102	100	0	2
12.0368	19.7895	16.0432	22.4527	100	50	2
15.2352	21.8827	22.0281	27.0550	0	0	3
$\mu = 2\,nm$, $L = 10\,nm, P = 2$, $l/h = 20$						

Table 5 Nonlinear frequencies and nonlinear frequency ratios of FG nano-beam upon SIG theory for various amplitude ratios, linear and nonlinear elastic foundation parameter and temperature rise

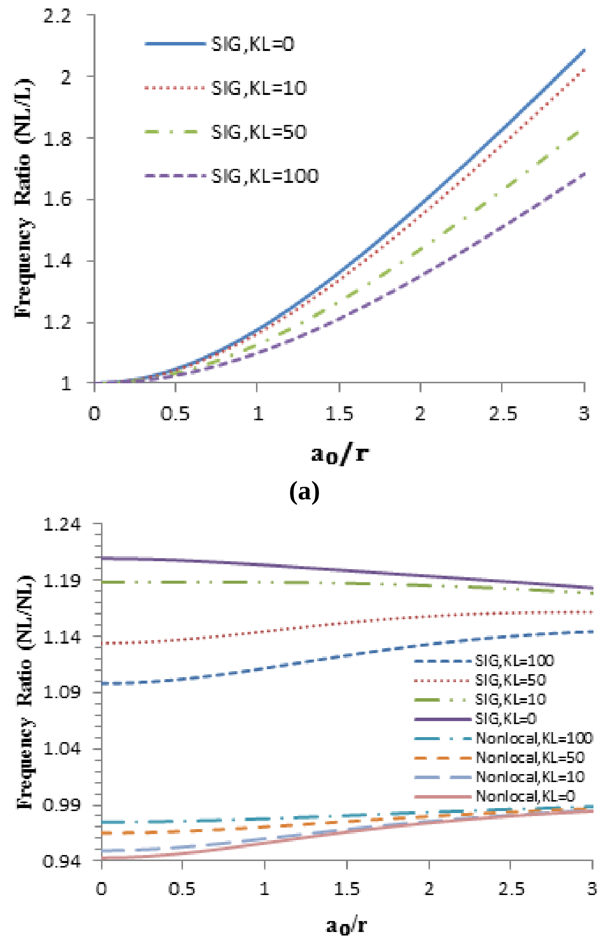
Nonlinear Frequencies				T	K_L	$\frac{a_0}{r}$
K_{NL}						
0		50				
K_S		K_S				
0	25	0	25			
13.3365	20.5489	14.3429	21.2159	0	0	1
13.3736	20.5730	14.3774	21.2393	0	1	1
15.0794	21.7206	15.9764	22.3526	0	50	1
10.6221	18.9004	11.8611	19.6235	100	0	1
12.7418	20.1680	13.7917	20.8472	100	50	1
17.2657	23.2492	20.2147	25.5158	0	0	2
18.6341	24.2827	21.3953	26.4609	0	50	2
15.2838	21.8178	18.5506	24.2188	100	0	2
16.8143	22.9160	19.8306	25.2126	100	50	2
22.2317	27.0823	27.1963	31.2865	0	0	3
$l_s = 2\,nm$, $l_d = 6\,nm$, $L = 10\,nm, P = 2$, $l/h = 20$						

From "Fig. 3(a) and Fig. 3(b)", it is observed that by increasing the length of FG nano-beam, the linear and nonlinear frequency ratios obtained from SIG theory descending approaches the local limit while in the non-local theory, the linear and nonlinear frequency ratios ascending approaches the local limit.



(b)

Fig. 3 Variations of the frequency ratios versus the FG nano-beam length for various amplitude ratios ($a_0/r = 0,1,2$) and length scale parameters ($\mu = 1,2\ nm\ and\ l_d = 3\ \&\ 6\ nm$): (a): Non-local theory and (b): SIG theory. ($K_L = 10, K_{NL} = 10, K_S = 5, \Delta T = 100, P = 2$)



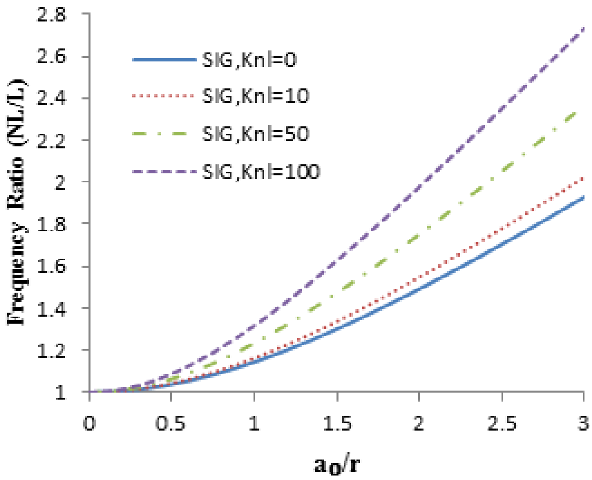
(b)

Fig. 4 Variations of the frequency ratios versus the amplitude ratio for various linear foundation parameter values for the FG nano-beam: (a): SIG theory and (b): comparison between non-local and SIG elasticity theory. ($K_{NL} = 10, K_S = 5, \Delta T = 100, P = 2, L = 10\ nm, \mu = 2\ nm$)

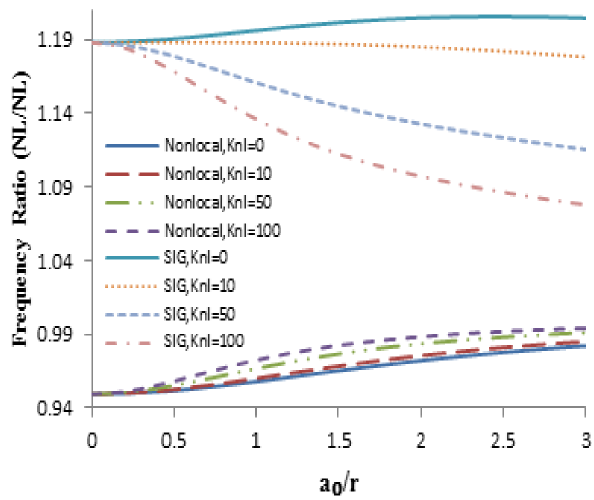
5.2.2. Influence of elastic foundation parameters on the frequency ratio

Next, we investigate the effects of elastic foundation parameters on the nonlinear free vibrations behavior of FG nano-beams based on non-local and strain-inertia gradient elasticity theory. "Fig. 4 and Fig. 5" demonstrate the effects of the linear and nonlinear elastic foundation parameters on frequency ratios of simply supported FG nano-beam when the gradient index is set to be $P = 2$ and static and dynamic length scale parameters are $l_s = 2\ nm$, $l_d = 6\ nm$ and $L = 10\ nm$, $l/h = 20$. The following dimensionless foundation parameters $k_L L^4/(bD^*)$, $k_S L^2/(bD^*)$ and $k_{NL} r^2 L^4/(bD^*)$ are used in the plot of "Fig. 4 and Fig. 5".

It can be seen from "Fig. 4(a)" that increasing values of the linear elastic foundation parameter leads to decreasing the frequency ratios at a constant amplitude ratio. It should be noted that the linear elastic foundation parameter increases nano-beam stiffness and also, linear and nonlinear frequencies but the increase of linear frequencies is more than nonlinear frequencies. At "Fig. 4(b)", the trend of non-local elasticity theory is compared with SIG theory. It can be observed from "Fig. 4(b)" that increasing values of the linear elastic foundation parameter according to SIG theory leads to decreasing the nonlinear frequency ratios at a constant amplitude ratio but the non-local theory will cause increasing the frequency ratios.



(a)



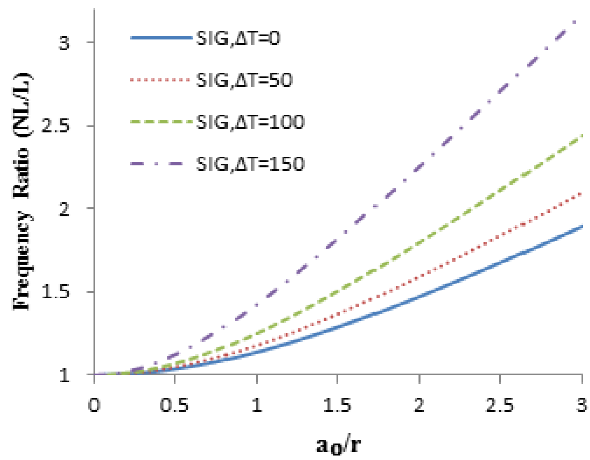
(b)

Fig. 5 Variations of the frequency ratios versus the amplitude ratio for various nonlinear foundation parameter values for the FG nano-beam: (a): SIG theory and (b): comparison between non-local and SIG elasticity theory. $(K_L = 10, K_S = 5, \Delta T = 100, P = 2, L = 10\ nm\ , \mu = 2\ nm)$

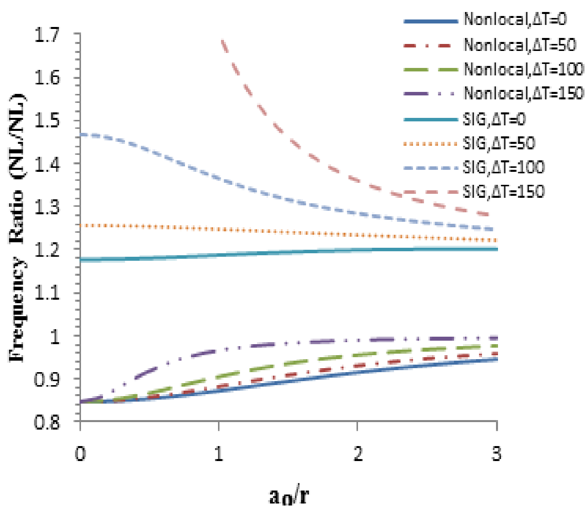
An increase in the value of the nonlinear elastic foundation parameter leads to increasing at frequency ratios at a constant amplitude ratio. This interesting behavior is shown in “Fig. 5(a)”. According to “Fig. 5(b)”, if in SIG theory$(k_{NL} < 10\ )$ then the nano-beam displays hardening type behavior and if $(k_{NL} > 10)$ then the nano-beam displays softening type behavior. But the non-local theory shows a constantly ascending behavior by extending amplitude ratio.

5.2.3. Influence of temperature rise on the frequency ratio

Then verification is carried out for the nonlinear vibration problem of a uniformly heated simply supported FG nano-beam based on non-local and SIG elasticity theory. The dependences of frequency ratios on the temperature rise under some specific values of static and dynamic length scale parameters $(l_s = 2\ nm\ \&l_d = 6\ nm)$ and $= 2$, $L = 10\ nm$ and $l/h = 20$ are plotted in “Fig. 6”, in which, $\Delta T = 0$ implies an unheated FG nano-beam.



(a)



(b)

Fig. 6 Variations of the frequency ratios versus the amplitude ratio for various temperature rise values for the FG nano-beam: (a): SIG theory and (b): comparison between non-local and SIG elasticity theory. $(K_L = 10, K_{NL} = 10, K_S = 5, P = 2, L = 10\ nm\ , \mu = 2\ nm)$

It is found from “Fig. 6(a)” that the increment in temperature rise leads to an increment of the frequency ratio. It can be seen from “Fig.6 (b)” that increase in temperature rise leads to increasing the nonlinear frequency ratios in non-local and SIG elasticity theory. In addition, upon SIG elasticity theory, the temperature rise effects are more obvious for lower amplitude ratios.

6.2.4. Influence of length scale parameters on the frequency ratio

In order to investigate the effect of the length scale parameters on the natural frequency of FG nano-beam, a curve has been plotted as a function of the non-local parameter for three elasticity theories when $(l_s = \mu \,\&\, l_d = 3l_s)$.

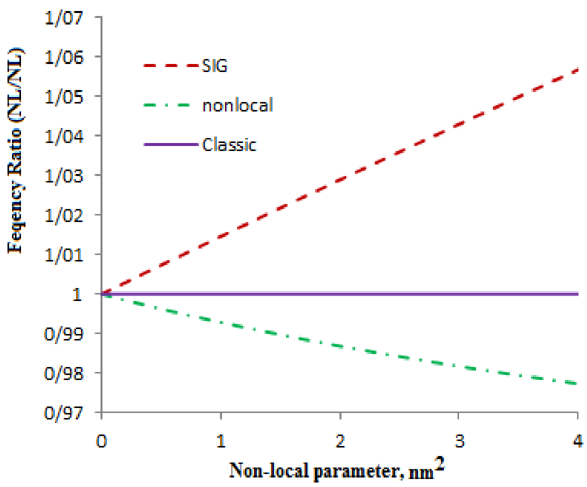


Fig. 7 Variations of the frequency ratios versus the non-local parameter for the FG nano-beam for comparison between SIG, Non-local and Classic elasticity theory. $(a_0/r = 1, K_L = 10, K_{NL} = 10, K_S = 5, \Delta T = 100, P = 2, L = 10\,nm\,)$

It is observed from “Fig. 7” that by increasing the non-local parameter, the nonlinear frequency ratio of SIG theory has an increasing trend while the nonlinear frequency ratio of non-local elasticity theory has a decreasing trend. This implies that the overall stiffness of FG nano-beam is significantly reduced due to the effects of the non-local parameter. The nonlinear frequency ratios that are obtained from SIG theory are greater than non-local and classical theory and smaller than strain gradient theory.

5.2.5. Influence of the gradient index on the frequency ratio

Finally, variations of the linear and nonlinear frequency ratios of simply supported FG non-beam with elastic foundation and thermal loading versus the gradient index based on non-local and SIG elasticity theory are presented in “Fig. 8” for various amplitude ratios $(a_0/r = 0,1,2)$ and static and dynamic length scale parameters $(l_s = 1{,}2\,nm \,\&l_d = 3{,}6\,nm)$ when the length of FG non-beam is L = 10 nm. It is clearly understood from “Fig. 8” that by increasing gradient index, its effect on the variation of frequency ratios in both theories would be reduced.

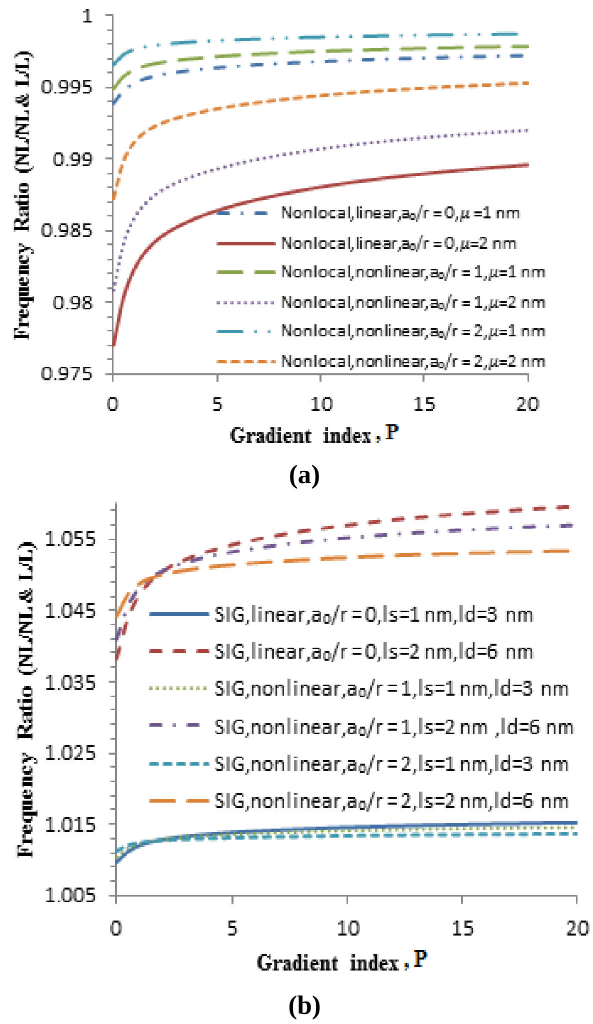


Fig. 8 Variations of the frequency ratios versus the gradient index for the FG nano-beam for various amplitude ratio and length scale parameters: (a): Non-local theory and (b): SIG theory. $(K_L = 10, K_{NL} = 10, K_S = 5, \Delta T = 100, L = 10\,nm\,)$

5 CONCLUSION

In this work, the nonlinear free vibration analysis of FG nano-beam in thermal environment and resting on nonlinear elastic foundation was investigated based upon non-local and SIG elasticity theories. The Euler–Bernoulli beam theory including von Karman geometric nonlinearity is employed to model the FG nano-beam. Explicit formulas are proposed for Euler–Bernoulli model relevant to each type of gradient theory to evaluate the natural frequencies of FG nano-beam. The analytical solution for natural frequencies is established using homotopy analysis method.

From the numerical results, it can be concluded that the frequency ratios predicted by HA method and exact solution are in good agreement. It can be seen that by increasing the length of FG nano-beam, the linear and nonlinear frequency ratios obtained from SIG theory descending approaches the local limit but in the non-local theory the linear and nonlinear frequency ratios ascending approaches the local limit. In non-local and SIG theory, the nonlinear frequency ratios increase as the linear and nonlinear elastic foundation parameter is decreased. The effects of temperature rise on the nonlinear frequency ratios in non-local and SIG theory are the same. It is founded that by increasing the length scale parameters, the nonlinear frequency ratio for SIG theory has an increasing trend and in non-local elasticity theory has a decreasing trend. The frequency ratios and linear and nonlinear natural frequencies that are obtained from strain-inertia gradient theory are greater than non-local and classical theory and smaller than strain gradient theory.

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