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Numerical solution of two-dimensional fuzzy Fredholm integral equations using collocation fuzzy wavelet like operator

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Abstract

In this paper, first we propose a new method to approximate the solution of two-dimensional linear fuzzy Fredholm integral equations of the second kind based on the fuzzy wavelet like operator. Then, we discuss and investigate the convergence and error analysis of the proposed method. Finally, to show the accuracy of the proposed method, we present two numerical examples.

Keywords : Fuzzy linear system; Two-dimensional fuzzy Fredholm integral equation; Fuzzy wavelet like operator.

1 Introduction

The concept of fuzzy integrals was initiated by Dubois and Prade [11] and then investigated by Kaleva [21], Goetschel and Voxman [20], Nanda [25] and others. In [35], the Henstock integral of fuzzy-valued functions is defined, while the fuzzy Riemann integral and its numerical integration was investigated by Wu in [36]. In [17], the authors introduced some quadrature rules for the integral of fuzzy-number-valued mappings. Kaleva [21] proposed the existence and uniqueness of the solution of fuzzy differential equations using the Banach fixed point principle. Mordeson and Newman [24], started the study of the subject of fuzzy integral equations (FIE).
Many authors applied the Banach fixed point principle, as a powerful tool, to show the existence and uniqueness of the solution of FIE [5, 6, 16, 29, 30, 31]. In [19, 26], sufficient conditions for bounded solutions of FIE are given. Recently, the iterative techniques are applied to solve fuzzy Fredholm integral equations of the second kind (FFIE-2) by researchers [7, 15, 28]. Friedman et al. [16] presented a numerical algorithm to solve FFIE-2 based on successive approximations method. Also, Friedman et al. [17] investigated numerical procedures for solving such equations by using the embedding method. In [8], the successive approximations method is used for solving nonlinear fuzzy Fredholm integral equations. The authors of [44] presented iterative method and quadrature rules for solving nonlinear FFIE-2. In [9], Bica et al. developed an iterative numerical method to solve fuzzy Hammerstein-Voltera integral equations with constant delay. In [10], the same method has been applied to obtain the solution that take values in the set of right-sided fuzzy numbers for a fuzzy Volterra integral equation with constant delay arising in epidemiology. For numeric-analytic methods to solve FFIE-2, one can refer [1, 4, 12, 13, 14, 18, 27, 32, 33, 34, 43]. Since many real-valued problems in engineering and mechanics can be brought in the form of two-dimensional fuzzy Fredholm integral equa-

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tions (2DFFIE), it is important that we develop quadrature rules and numerical methods for solving such equations. Recently, some researchers investigated solving such equations. In [?], the authors applied modified Homotopy perturbation method to solve 2DFFIE. The authors of [38] proposed quadratur rules for numerical solution of two-dimensional fuzzy integrals. In this work, the authors also obtained numerical solution of linear 2DFFIE by using iterative technique. Also, solving nonlinear 2DFFIE by using quadrature rules is done in [42].
Recently several researchers have attempted to develop "fuzzy wavelets" based models and systems. Wavelet theory is a relatively new and an emerging area in mathematical research. Also, wavelets are the suitable and powerful tool for approximating functions based on wavelet basis functions. As we know, most of the methods to solve integral equations lead us to solve the linear systems, but the singularity of these systems may cause some problems. So, by using fuzzy wavelet like operator with collocation points to obtain numerical solution of 2FFIE, such linear systems can be as sparse linear systems and it can reduce the cost of computation. In this paper, by using fuzzy wavelet like operator, we propose a numerical method to approximate the solution of linear 2FFIE.

$$f(x,y) = g(x,y) \oplus \lambda \int_c^d \int_a^b K(x,y,s,t) \otimes f(s,t)dsdt \tag{1.1}$$

where $K(x,y,s,t)$ is an arbitrary kernel function over the square $[a,b]\times[c,d]$, and $g(x,y)$ is a fuzzy real valued function of x and y. Also, we present the error estimation for approximating the solution of equation(1.1).
The rest of the paper is organized as follows: In Section 2, we review some elementary concepts of the fuzzy set theory and modulus of continuity. In Section 5, we present the method for approximate the solution of (1.1) by using fuzzy wavelet like operator. The error estimation of this method is proved in Section 4. Finally in Section 5, we give two numerical examples for applicability of the proposed method and compare the numerical results with the exact solutions.

2 Preliminaries

In this Section, we review some necessary backgrounds and notions of fuzzy sets theory.

Definition 2.1 *[3] A fuzzy number is a function $u : R \longrightarrow [0,1]$ with the following properties:*
1. u is normal, that is $\exists x_0 \in R$ such that $u(x_0) = 1$,
2. u is fuzzy convex set
$(i.e. u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\} \ \ \forall x,y \in R, \lambda \in [0,1])$,
3. u is upper semicontinuous on R,

4. The $\overline{\{x \in R : u(x) > 0\}}$ is compact set.

The set of all fuzzy numbers denoted by R_F.

Definition 2.2 *[17] Suppose that $u \in R_F$. The r-level set of u is denoted by $[u]^r = [u_-^{(r)}, u+^{(r)}]$ and is defined by $[u]^r = \{x \in R; u(x) \geq r\}$, where $0 < r \leq 1$. Also, $[u]^0$ is called the support of u and it is given as $[u]^0 = \overline{\{x \in R : u(x) > 0\}}$. It follows that the r-level sets of u are closed and bounded intervals in R.*

It is well-known that the addition and multiplication operations of real numbers can be extended to R_F. In other words, for $u, v \in R_F$ and $\lambda \in R$, we define uniquely the sum $u \oplus v$ and the product $\lambda \otimes u$, by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \otimes u]^r = \lambda[u]^r, \quad \forall r \in [0,1]$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subset of R) and $\lambda[u]^r$ means the usual product between a scalar and a subset of R. We use the same symbol $\sum$ both for the sum of real numbers and for the sum $\oplus$ (when the terms are fuzzy numbers).

Definition 2.3 *[17] An arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions $(\underline{u}(r), \bar{u}(r)), 0 \leq r \leq 1$, which satisfy the following requirements:*
1. $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0,1]$,
2. $\bar{u}(r)$ is a bounded left continuous nonincreasing function over $[0,1]$,
3. $\underline{u}(r) \leq \bar{u}(r), \quad 0 \leq r \leq 1$.
4. The addition and scalar multiplication of fuzzy numbers in R_F are defined as follows:

$$u \oplus v = (\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r))$$

$$\lambda \otimes u = \begin{cases} (\lambda\underline{u}(r), \lambda\bar{u}(r)) & \lambda \geq 0 \\ \lambda\bar{u}(r), \lambda\underline{u}(r)) & \lambda < 0. \end{cases}$$

Definition 2.4 *[3] For arbitrary fuzzy numbers $u = (\underline{u}, \bar{u})$ and $v = (\underline{v}, \bar{v})$ the quantity*

$$D(u,v) = \sup_{r\in[0,1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\}$$

is called the distance between u and v. It is shown that (R_F, D) *is a complete metric space with the following properties [7]:*

1.$D(u \oplus w, v \oplus w) = D(u, v), \ \forall u, v, w \in R_F$,
2.$D(k \otimes u, k \otimes v) = |k|D(u, v), \ \forall u, v \in R_F, \forall k \in R$,
3.$D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e), \forall u, v, e \in R_F$.

Definition 2.5 *[11] Let* $f, g : [a, b] \times [c, d] \longrightarrow R_F$ *be fuzzy number valued functions. The uniform distance between f and g is defined by*

$$D^*(f, g) = \sup\{(D(f(x, y), g(x, y)); x \in [a, b], y \in [c, d]\}.$$

Definition 2.6 *[40] Let* $f, g : [a, b] \times [c, d] \longrightarrow R_F$ *. For each partition* $p = \{x_1, x_2, \cdots, x_n\}$ *of* $[a, b]$ *and* $q = \{y_1, y_2, \cdots, y_n\}$ *of* $[c, d]$ *and for arbitrary* $\xi_i : x_{i-1} \leq \xi_i \leq x_i, 2 \leq i \leq m$ *and for arbitrary* $\eta_j : y_{j-1} \leq \eta_j \leq y_j, 2 \leq j \leq n$, *let*

$$R_p = \sum_{i=2}^{m} \sum_{j=2}^{n} f(\xi_i, \eta_j) \otimes (x_i - x_{i-1})(y_j - y_{j-1}).$$

The function f is called two- dimensional Riemann integrable to $I \in R_F$ if for every $\epsilon > 0$,

$$D(I, R_p) < \epsilon$$

In this case, we have

$$I = (FR) \int_c^d \int_a^b f(x, y) dx dy.$$

Definition 2.7 *[42] A function* $f : [a, \ b] \times [c, \ d] \to R_F$ *is said to be continuous in* $x_0 \in [a, b], y_0 \in [c, d]$ *if for each* $\epsilon > 0$ *there exist* $\delta > 0$ *such that* $D(f(x, y), f(x_0, y_0)) < \epsilon$, *whenever* $x \in [a, b], y \in [c, d]$ *and* $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$. *We say that f is fuzzy continuous on* $[a, b] \times [c, d]$ *if f is continuous at each* $(x_0, y_0) \in [a, \ b] \times [c, \ d]$.

The space of all such functions is denoted by $C_F([a, \ b] \times [c, \ d])$.

Lemma 2.1 *[40] If* $f, g : [a, b] \times [c, d] \to R_F$ *are fuzzy continuous functions, then the function* $F : [a, b] \times [c, \ d] \to R^+$ *defined by*

$$F(x_J y) = D(f(x, y), \ g(x_1 y))$$

is continuous on $[a, b]$, and $[c, d] \longrightarrow R^+$
defined by $F(x, y) = D(f(x, y), g(x, y))$ is continuous on $[a, b]$,
and

$$D((FR) \int_c^d \int_a^b f(x, y) dx dy, (FR) \int_c^d \int_a^b g(x, y) dx dy)$$
$$\leq \int_c^d \int_a^b D(f(x, y), g(x, y)) dx dy.$$

Definition 2.8 *. Let* $f : [a, b] \times [c, d] \to R_F$. *One call f a uniformly continuous fuzzy real number valued function, if and only if for any* $\epsilon > 0$ *there exists* $\delta > 0$ *whenever* $\sqrt{(x - s)^2 + (y - t)^2} \leq \delta; x, s \in [a, b], y, t \in [c, d]$, *implies that* $D(f(x, y) f(s, t)) \leq \epsilon$. *one denotes it as* $f \in C_F^U([a, b])$.

Definition 2.9 *[39] A function* $f : [a, b] \times [c, d] \to R_F$ *is said to be bounded if there exist M such that* $\|f(x, y)\|_F \leq M$ *for any* $(x, y) \in [a, b] \times [c, d]$, *where* $||f(x, y)||_F = D(f(x, y), 0)$.

Corollary 2.1 *[39] If* $f \in C_F([a, b] \times [c, d])$ *, its definite integral exists [42], furthemore*

1. $(FR)\underline{\int_c^d \int_a^b f(x, y, r) dx dy} = \int_c^d \int_a^b \underline{f}(x, y, r) dx dy,$

2. $(FR)\overline{\int_c^d \int_a^b f(x, y, r)\, dx dy} = \int_c^d \int_a^b \overline{f}(x, y, r)\, dx dy$

Remark 2.1 Consider two-dimensional fuzzy Fredholm integral equation of the second kind

$$f(x, y, r) = g(x, y, r) + \lambda \int_c^d \int_a^b K(x, y, s, t) f(s, t, r) ds dt$$

In order to design numerical scheme for solving above equation, we first replace it by the system

$$\underline{f}(x, y, r) = \underline{g}(x, y, r) + \lambda \int_c^d \int_a^b \underline{K(x, y, s, t) f(s, t, r)} ds dt$$

$$\overline{f}(x, y, r) = \overline{g}(x, y, r) + \lambda \int_c^d \int_a^b \overline{K(x, y, s, t) f(s, t, r)}\, ds dt$$

where

$$\underline{K(x, y, s, t) f(s, t, r)} = \begin{cases} K(x, y, s, t)\underline{f}(s, t, r) & K(x, y, s, t) \geq 0 \\ K(x, y, s, t)\overline{f}(s, t, r) & K(x, y, s, t) < 0 \end{cases}.$$

$$\overline{K(x, y, s, t) f(s, t, r)} = \begin{cases} K(x, y, s, t)\overline{f}(s, t, r) & K(x, y, s, t) \geq 0 \\ K(x, y, s, t)\underline{f}(s, t, r) & K(x, y, s, t) < 0 \end{cases}.$$

Corollary 2.2 *[40] If* $f, g \in R_F$ *are Henstock integrale mappings on* $[a, b] \times [c, d]$ *and if* $D(f(x, y), g(x, y))$ *is Lebesgue integrable, then*

$$D((FH) \int_c^d \int_a^b f(x, y) dx dy, (FH) \int_c^d \int_a^b g(x, y) dx dy)$$
$$\leq (L) \int_c^d \int_a^b D(f(x, y), g(x, y)) dx dy.$$

Definition 2.10 *[3] Let* $f, g : [a, b] \times [c, d] \to R_F$ *be a bounded function, then function*
$\omega_{[a,b]\times[c,d]}(f, 0) : R^+ \cup \{0\} \to R^+$,
$\omega_{[a,b]\times[c,d]} = \sup\{D(f(x, y), f(s, t))| \ x, s \in [a, b]; y, t \in [c, d], \sqrt{(x - s)^2 + (y - t)^2} \leq \delta\}$

where R^+ is the set of positive real numbers, is called the modulus of continuity of f on $[a,b] \times [c,d]$.

Some properties of the modulus of continuity are presented below: [7]

1. $D(f(x,y), f(s,t)) \leq$
$$\omega_{[a,b]\times[c,d]}(f, \sqrt{(x-s)^2+(y-t)^2})$$
, for any $x, s \in [a,b]$ and $y, t \in [c,d]$;

2. $\omega_{[a,b]\times[c,d]}(f,\delta)$ is increasing function of δ,

3. $\omega_{[a,b]\times[c,d]}(f,0) = 0$,

4. $\omega_{[a,b]\times[c,d]}(f,\delta_1+\delta_2) \leq \omega_{[a,b]\times[c,d]}(f,\delta_1) + \omega_{[a,b]\times[c,d]}(f,\delta_2)$ for any $\delta_1, \delta_2 \geq 0$ and $f : [a,b]\times[c,d] \longrightarrow R_F$,

5. $\omega_{[a,b]\times[c,d]}(f,n\delta) \leq n\omega_{[a,b]\times[c,d]}(f,\delta)$ for any $\delta \geq 0, n \in N$,and $f : [a,b]\times[c,d] \rightarrow R_F$.

6. $\omega_{[a,b]\times[c,d]}(f,\lambda\delta) \leq ([\lambda]+1)\omega_{[a,b]\times[c,d]}(f,\delta)$ for any $\delta, \lambda \geq 0$, and any $f : [a,b]\times[c,d] \rightarrow R_F$, where $[.]$ is the ceiling of the number.

In [2], the following theorem is proved.

Corollary 2.3 *[2] Let $f\epsilon C_F([a,b]\times[c,d])$ and the scaling function $\varphi(x,y)$ a real-valued bounded function with it supp $\varphi(x,y) \subseteq [-\alpha,\alpha]\times[-\beta,\beta], 0<\alpha<+\infty, 0<\beta<+\infty, \varphi(x,y) \geq 0$ such that*
$$\sum_{j=-\infty}^{+\infty}\sum_{i=-\infty}^{+\infty}\varphi(x-i)\varphi(y-j) = 1$$
on $[a,b]\times[c,d]$. For $k \in Z^+, x, y \in [a,b]\times[c,d]$, put

$$(B_k f)(x,y) = \sum_{j=-\infty}^{+\infty}\sum_{i=-\infty}^{+\infty} f(\frac{i}{2^k},\frac{j}{2^k})\otimes \varphi(2^k x - i)\varphi(2^k y - j) \tag{2.2}$$

which is a fuzzy-wavelet-like operator. Then
$$D((B_k f)(x,y), f(x,y)) \leq \omega[a,b]\times[c,d](f, \frac{\alpha}{2^k}, \frac{\beta}{2^k})$$
$$D^*((B_k f), f) \leq \omega_{[a,b]\times[c,d]}(f, \frac{\alpha}{2^k}, \frac{\beta}{2^k})$$
for all $x, y \in R$ and $k \in Z^+$. If $f \in C_F^U([a,b]\times[c,d])$, then as $k \rightarrow +\infty$ one gets $\omega_{[a,b]\times[c,d]}(f, \frac{\alpha}{2^k}, \frac{\beta}{2^k}) \rightarrow 0$ and $\lim_{k\rightarrow+\infty} B_k f = f$, point wise and uniformly with rates.

3 Solving 2FFIE of the second kind

Here, we use fuzzy wavelet like operator defined by (2.2) due to approximate solution of equation (1.1). To do this, we approximate the solution of (1.1) by (2.2). So, by substituting (2.2) in (1.1) we conclude that
$$\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} f(\frac{i}{2^k},\frac{j}{2^k})\otimes\varphi(2^k x-i).\varphi(2^k y-j)$$
$$\cong g(x,y) + \lambda\int_c^d\int_a^b K(x,y,s,t)\otimes$$
$$\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} f(\frac{i}{2^k},\frac{j}{2^k})\otimes(\varphi(2^k s-i)\varphi(2^k t-j)dsdt \quad (*)$$

By using $2^k s - i = u$ and $2^k t - j = v$, we get
$$(*) \cong g(x,y) +$$
$$\frac{\lambda}{2^{2k}}\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} f(\frac{i}{2^k},\frac{j}{2^k})\otimes$$
$$\int_{2^k c-j}^{2^k d-j}\int_{2^k a-i}^{2^k b-i} K(x,y,\frac{u+i}{2^k},\frac{v+j}{2^k})\varphi(u)\varphi(v)dudv \tag{3.3}$$

Now, by using the following scaling function [38]
$$\varphi(x,y) = \begin{cases} 1 & -\frac{1}{2} \leq x,y \leq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{3.4}$$

in (3.3), we conclude that
$$\sum_{j=2^k c}^{2^k d-1}\sum_{i=2^k a}^{2^k b-1} f(\frac{i}{2^k},\frac{j}{2^k})\otimes\varphi(2^k x-i)\varphi(2^k y-j) \cong$$
$$g(x,y) + \frac{\lambda}{2^{2k}}$$
$$\sum_{j=2^k c}^{2^k d-1}\sum_{i=2^k a}^{2^k b-1} f(\frac{i}{2^k},\frac{j}{2^k})\otimes$$
$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\frac{1}{2}}^{\frac{1}{2}} K(x,y,\frac{u+i}{2^k},\frac{v+j}{2^k})\varphi(u)\varphi(v)dudv \tag{3.5}$$

For fixed k, suppose that

$$A_{i,j,k}(x,y) = \frac{\lambda}{2^{2k}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} K(x, y, \frac{u+i}{2^k}, \frac{v+j}{2^k})$$

$$\varphi(u)\varphi(v)dudv,$$

$$i = 2^k a, 2^k a+1, \cdots, 2^k b-1,$$

$$j = 2^k c, \; 2^k c+1, \cdots, 2^k d-1$$

Clearly, we can write (3.5) in the following form

$$\sum_{j=2^k c}^{2^k d-1} \sum_{i=2^k a}^{2^k b-1} f(\frac{i}{2^k}, \frac{j}{2^k}) \otimes \varphi(2^k x_p - i)\varphi(2^k y_q - j)$$

$$\cong g(x_p, y_q) \oplus \sum_{j=2^k c}^{2^k d-1} \sum_{i=2^k a}^{2^k b-1} f(\frac{i}{2^k}, \frac{j}{2^k}) \otimes A_{i,j,k}(x_p, y_q) \tag{3.6}$$

Now, we use collocation points, $x_p = \frac{p}{2^k}$, $y_q = \frac{q}{2^k}$, $p = 2^k a, 2^k a+1, \ldots, 2^k b-1, q = 2^k c, 2^k c + 1 \cdots 2^k d - 1$ in (3.6). We have:

$$\sum_{j=2^k c}^{2^k d-1} \sum_{i=2^k a}^{2^k b-1} f(\frac{i}{2^k}, \frac{j}{2^k}) \otimes \varphi(2^k x_p - i)\varphi(2^k y_q - j)$$

$$= g(x_p, y_q) \oplus \sum_{j=2^k c}^{2^k d-1} \sum_{i=2^k a}^{2^k b-1} f(\frac{i}{2^k}, \frac{j}{2^k}) \otimes A_{i,j,k}(x_p, y_q)$$

$$p = 2^k a, \; 2^k a+1, \ldots, \; 2^k b-1$$

$$q = 2^k c, \; 2^k c+1, \; \ldots, \; 2^k d-1$$

Thus, we obtain the following $2n \times 2n$, $n = 2^k(b-a)$, fuzzy linear system of equations:

$$C \otimes X = Y \oplus B \otimes X,$$

where

$$X = (f(a,c), f(a+\frac{1}{2^k}, c+\frac{1}{2^k}), \cdots, f(b-\frac{1}{2^k}, d-\frac{1}{2^k}))^t$$

$$Y = (g(a,c), g(a+\frac{1}{2^k}, c+\frac{1}{2^k}), \cdots, g(b-\frac{1}{2^k}, d-\frac{1}{2^k}))^t,$$

$$B = (B_{i,j})_{2n\times 2n}, B_{ij} = A_{i,j,k}(x_p, y_q),$$

$$C = (c_{i,j})_{2n\times 2n}, c_{ij} = \varphi(2^k x_p - i)\varphi(2^k y_q - j)$$

Here, we suppose that $b - a \in N, d - c \in N$, where N is set of natural numbers. Clearly, the above system is a dual fuzzy linear system. For solving this system, one can refer to [37]. Finally, by solving this system and using Eq (2.2), we can present the approximate solution of (1.1).

4 Error Estimation

Corollary 4.1 *Consider linear 2FFIE of the second kind as follows:*

$$f(x,y) = (g(x,y) \oplus \lambda \int_c^d \int_a^b K(x,y,s,t) \otimes f(s,t)dsdt,$$

where $K(x,y,s,t)$ is an arbitrary continues kernel function having same sign in the square $a \le x, s \le b, c \le y, t \le d$, and $g(x,y) \neq 0$ is a continuous fuzzy function over $a \le x \le b, c \le y \le d$. Under hypothesis of Theorem(2.3) , we have:

$$D(f(x,y), \; (B_k f)(x,y)) \le M|\lambda|(b-a)(d-c)$$

$$\omega_{[a,b]\times[c,d]}(f, \frac{\alpha}{2^k}, \frac{\beta}{2^k})$$

where $M = \max|K(x,y,s,t)|, a \le x, s \le b$ and $c \le y, t \le d$, and

$$(B_k f)(x,y) = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f(\frac{i}{2^k}, \frac{j}{2^k}) \otimes$$

$$(\varphi(2^k x - i)\varphi(2^k y - j)$$

is the approximate solution of equation (1.1).
Proof. We would like to estimate

$$D(f(x,y), \; (B_k f)(x,y)) = D(g(x,y) \oplus$$

$$\lambda \int_c^d \int_a^b K(x,y,s,t) \otimes f(s,t)dsdt,$$

$$g(x,y) \oplus \lambda \int_c^d \int_a^b K(x,y,s,t) \otimes (B_k f)(s, \; t)dsdt) =$$

$$D(\lambda \int_c^d \int_a^b K(x,y,s,t) \otimes f(s,t)dsdt,$$

$$\lambda \int_c^d \int_a^b K(x,y,s,t) \otimes (B_k f)(s,t)dsdt)$$

$$\le |\lambda| \int_c^d \int_a^b |K(x,y,s,t)|.D(f(s,t), (B_k f)(s,t))dsdt$$

$$\le |\lambda| \int_c^d \int_a^b M.D^*(f, \; B_k f)dsdt \le$$

$$M|\lambda|(b-a)(d-c)\omega_{[a,b]\times[c,d]}(f, \frac{\alpha}{2^k}, \frac{\beta}{2^k}) \blacksquare$$

It is obvious that

$$(\overline{B_k f})(x,y) = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \overline{f}(\frac{i}{2^k}, \frac{j}{2^k}) \otimes$$

$$\varphi(2^k x - i)\varphi(2^k y - j)$$

$$(\underline{B_k} f)(x,y) = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \underline{f}(\frac{i}{2^k}, \frac{j}{2^k}) \otimes$$

$$\varphi(2^k x - i)\varphi(2^k y - j) \; .$$

Now, we consider two cases as follows:

$$(a)\ K(x,y,s,t) \geq 0,$$

$$(b)\ K(x,y,s,t) < 0.$$

In case (a) , we have:

$$\overline{e} = \overline{f}(x,y) - \overline{B_k f})(x,y) = [\overline{g}(x,y)+$$

$$\lambda \int_c^d \int_a^b K(x,y,s,t) \otimes \overline{f}(s,t)dsdt]-$$

$$[\overline{g}(x,y) + \lambda \int_c^d \int_a^b K(x,y,s,t)(\overline{B_k f})(s,t)dsdt]$$

$$= \lambda \int_c^d \int_a^b K(x,y,s,t) \otimes \overline{f}(s,t)dsdt-$$

$$\lambda \int_c^d \int_a^b K(x,y,s,t)(\overline{B_k f})(s,t)dsdt$$

$$= \lambda \int_c^d \int_a^b K(x,y,s,t)[\overline{f}(s,t) - (\overline{B_k f})(s,t)dsdt$$

similary we have:

$$\underline{e} = \underline{f}(s,t) - (\underline{B_k} f)(s,t)$$

$$\underline{g}(x,y) + \lambda \int_c^d \int_a^b K(x,y,s,t)\underline{f}(s,t)dsdt]$$

$$-[\underline{g}(x,y) + \lambda \int_c^d \int_a^b K(x,y,s,t)(\underline{B_k} f)(s,t)dsdt].$$

$$= \lambda \int_c^d \int_a^b K(x,y,s,t)\underline{f}(s,t)dsdt$$

$$-\lambda \int_c^d \int_a^b K(x,y,s,t)(\underline{B_k} f)(s,t)dsdt$$

.

$$= \lambda \int_c^d \int_a^b K(x,y,s,t)[\underline{f}(s,t) - \underline{B_k} f)(s,t)]dsdt$$

Hence

$$e = \overline{e} - \underline{e} = \lambda \int_c^d \int_a^b K(x,y,s,t)[\overline{f}(s,t) - (\overline{B_k f})(s,t)dsdt$$

$$\lambda \int_c^d \int_a^b K(x,y,s,t)[\underline{f}(s,t) - (\underline{B_k f}(s,t)]dsdt$$

$$= \lambda \int_c^d \int_a^b K(x,y,s,t)([\overline{f}(s,t) - (\underline{B}_k f)(s,t)])$$

$$+[\underline{f}(s,t) - (\underline{B_k} f)(s,t)])dsdt,$$

and

$$\|e\| = \|\overline{e} + \underline{e}\| \leq |\lambda| \int_c^d \int_a^b \|K(x,y,s,t)\| [\|\overline{f}(s,t) - (\overline{B_k f})(s,t)\|]$$

$$+\|\underline{f}(s,t) - (\underline{B_k} f)(s,t)\|]dsdt.]$$

Since $K(x,y,s,t)$ is an arbitrary continues kernel function over the square $a \leq x,s \leq b, c \leq y,t \leq d$,there exist $M > 0$ such that $M = \max|K(x,y,s,t)|$. So, we can write

$$\|e\| \leq |\lambda| M \int_c^d \int_a^b [\|\overline{f}(s,t) - \overline{(B_k f)}(s,t)\|$$

$$+ \|\underline{f}(s,t) - (\underline{B_k} f)(s,t)\|]dsdt. \tag{4.7}$$

On the other hand, we have:

$$\overline{f}(s,t) - \overline{B_k} f(s,t) = \overline{f}(s,t) \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \varphi(2^k x - i)\varphi(2^k y - j) -$$

$$\sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \overline{f}(\frac{i}{2^k}, \frac{j}{2^k}) \otimes \varphi(2^k x - i)\varphi(2^k y - j)$$

$$= \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} [\overline{f}(s,t) - \overline{f}(\frac{i}{2^k}, \frac{j}{2^k})]\varphi(2^k x - i)(p(2^k y - j),$$

Therefore

$$\|\overline{f}(s,t) - (\overline{B_k f})(s,t)\| = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} [\overline{f}(s,t) - \overline{f}(\frac{i}{2^k}, \frac{j}{2^k})]$$

$$\varphi(2^k x - i)(\varphi(2^k y - j)\|$$

$$\leq \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \|[\overline{f}(s,t) - \overline{f}(\frac{i}{2^k}, \frac{j}{2^k})]\| \varphi(2^k x - i)(\varphi(2^k y - j)$$

$$\leq \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \omega[\overline{f}\|x - \frac{i}{2^k}, y - \frac{j}{2^k}\|]$$

$$\varphi(2^k x - i)(\varphi(2^k y - j), \quad (**)$$

Notice that, under hypotheses of Theorem (2.3), we conclude that

$$-\alpha \leq 2^k x - i \leq \alpha \Rightarrow -\frac{\alpha}{2^k} \leq x - \frac{i}{2^k} \leq \frac{\alpha}{2^k}$$

$$\Rightarrow |x - \frac{i}{2^k}| \leq \frac{\alpha}{2^k},$$

$$-\beta \leq 2^k x - i \leq \beta \Rightarrow -\frac{\beta}{2^k} \leq x - \frac{i}{2^k} \leq \frac{\beta}{2^k}$$

$$\Rightarrow |x - \frac{i}{2^k}| \leq \frac{\beta}{2^k}.$$

Hence

$$(**) \leq \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \varphi(2^k x - i)\varphi(2^k y - j)$$

$$\omega_{[a,b]\times[c,d]}(\overline{f}, \frac{\alpha}{2^k}, \frac{\beta}{2^k}) = \omega_{[a,b]\times[c,d]}(\overline{f}, \frac{\alpha}{2^k}, \frac{\beta}{2^k})$$

Therefore, *as* $k \to +\infty$ we get

$$\|\overline{f}(s,t) - \overline{(B_k f)}(s,t)\| \to 0$$

Similarly, we conclude

$$\|\underline{f}(s,t) - (\underline{B_k} f)(s,t)\| \to 0, \ as\ k \to +\infty.$$

So we obtain that

$$\|e\| = \|\overline{e} + \underline{e}\| \to 0,$$

as $k \to +\infty$.

In case (b) we have $K(x, y, s, t) < 0$. So

$$\overline{e} = [\overline{g}(x, y) + \lambda \int_c^d \int_a^b K(x, y, s, t)\underline{f}(s, t)dsdt$$

$$-[\overline{g}(x, y) + \lambda \int_c^d \int_a^b K(x, y, s, t)(\underline{B_k} f)(s, t)dsdt]$$

$$= +\lambda \int_c^d \int_a^b K(x, y, s, t)(\underline{f}(s, t) - \underline{B_k} f)(s, t)dsdt$$

and similarly way, we have:

$$\underline{e} = \lambda \int_c^d \int_a^b K(x, y, s, t)[\overline{f}(s, t) - (\overline{B_k f})(s, t)]dsdt$$

Hence

$$\|e\| = \|\overline{e} + \underline{e}\|$$

$$\leq |\lambda| M \int_c^d \int_a^b \|K(x, y, s, t)\|[\|\underline{f}(s, t) - (\underline{B_k} f)(s, t)\| +$$

$$\|\overline{f}(s, t) - (\overline{B_k f})(s, t)\|]dsdt$$

$$\leq |\lambda| M(b - a)(d - c)[\omega_{[a,b]\times[c,d]}(\underline{f}, \frac{\alpha}{2^k}, \frac{\beta}{b^k})$$

$$+\omega_{[a,b]\times[c,d]}(\overline{f}, \frac{\alpha}{2^k}, \frac{\beta}{2^k})$$

Clearly, we have:

$\|e\| = \|\overline{e} + \underline{e}\| \to 0$, *as* $k \to +\infty$.

5 Numerical examples

To illustrate the efficiency of the presented method in Section , we give two examples. Also, we compare the numerical solutions obtained by using the proposed method with the exact solutions. Trough this section, we suppose that

$$\varphi(x, y) = \begin{cases} 1 & -\frac{1}{2} \leq x, y \leq \frac{1}{2} \\ 0 & otherwise \end{cases}$$

Example 5.1 *Consider the following linear 2FFIE of the second kind*

$$\underline{g}(x, y, r) = (x.sin\frac{y}{2})(r^2 + r)$$

$$\overline{g}(x, y) = (x.sin\frac{y}{2})(4 - r^3 - r)$$

$$K(x, y, s, t) = x^2ys; 0 \leq x, y, s, t \leq 1$$

Also, let $a = o, b = 1$. The exact solution of this example is given by

$$\underline{f}(x, y, r) = [(x.sin\frac{y}{2}) - \frac{16}{21}(cos\frac{1}{2} - 1).x^2y](r^2 + r)$$

$$\overline{f}(x, y, r) = [(x.sin\frac{y}{2}) - \frac{16}{21}(cos\frac{1}{2} - 1).x^2y](4 - r^3 - r)$$

By using the proposed method in Section 5, we can present the approximate solution for this example. To compare the numerical results with the exact solution for different values of x, y and k, see Tables 1-3.

Example 5.2 *Consider the following linear 2FFIE ofthe second kind*

$$\underline{g}(x, y, r) = r(\frac{1}{3}r + \frac{8}{3})(1 + x + y - \frac{7}{12}xy)$$

$$\overline{g}(x, y) = (2r^2 - 4r + 5)(1 + x + y - \frac{7}{12}xy)$$

$$K(x, y, s, t) = xyst; \quad 0 \leq x, y, s, t \leq 1$$

Also, let $a = 0, b = 1$. The exact solution of this example is given by

$$\underline{f}(x, y, r) = r(\frac{1}{3}r + \frac{8}{3})(x + y + 1)$$

$$\overline{f}(x, y, r) = (2r^2 - 4r + 5)(x + y + 1)$$

By using the proposed method in Section 5, we can present the approximate solution for this example. To compare the numerical results with the exact solution for different values of x, y and k, see Tables 4-5.

6 Conclusion

To approximate the solution of 2FFIE of the second kind, a new approach based on fuzzy wavelet like operator via a real-valued scaling function and collocation method is proposed. Convergence analysis of the proposed method is investigated by using the modulus of continuity in one theorem. To illustrate the efficiency of the presented method, two examples are given. Comparing the numerical solutions with the exact solution show that the proposed method can be a suitable method for solving 2FFIE ofthe second numerically .

Table 1: Numerical results for Example 5.1 for $k = 2, x = \frac{1}{4}, y = \frac{1}{2}$

r	$\lvert \underline{f}(x,y) - (\underline{B_k} f)(x,y)\rvert$	$\lvert \overline{f}(x,y) - \overline{B_k f}(x,y)\rvert$
0.1	0.000869398	0.0308162
0.2	0.00189687	0.0299705
0.3	0.00189687	0.02903
0.4	0.00442603	0.0279472
0.5	0.00592771	0.0266747
0.6	0.00758747	0.0251651
0.7	0.00940531	0.023371
0.8	0.0113812	0.0212449
0.9	0.0135152	0.0187395
1.0	0.0158072	0.0158072

Table 2: Numerical result for Example 5.1 *with* $k = 3, x = \frac{1}{8}, y = \frac{2}{8}$

r	$\lvert \underline{f}(x,y) - (\underline{B_k} f)(x,y)\rvert$	$\lvert \overline{f}(x,y) - \overline{B_k f}(x,y)\rvert$
0.1	0.0000430158	0.00152471
0.2	0.0000938526	0.00148287
0.3	0.000152511	0.00143634
0.4	0.00021899	0.00138276
0.5	0.00029329	0.0013198
0.6	0.000375411	0.00124511
0.7	0.000465353	0.00115634
0.8	0.000563116	0.00103115
0.9	0.0006687	0.000927186
1.0	0.000782105	0.000782105

Table 3: Numerical results for Example 5.1 for $k = 4, x = \frac{1}{16}, y = \frac{1}{8}$

r	$\lvert \underline{f}(x,y) - (\underline{B_k} f)(x,y)\rvert$	$\lvert \overline{f}(x,y) - \overline{B_k f}(x,y)\rvert$
0.1	5.0409210^{-6}	0.00152471
0.2	0.0000109984	0.00148287
0.3	0.0000178724	0.00143634
0.4	0.0000256629	0.00138276
0.5	0.0000343699	0.0013198
0.6	0.0000439935	0.00124511
0.7	0.0000545336	0.00115634
0.8	0.0000783634	0.00103115
0.9	0.0006687	0.000927186
1.0	0.0000916531	0.000782105

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Table 4: Numerical results for Example 5.2 for $k = 2, x = \frac{1}{4}, y = \frac{1}{4}$

r	$\lvert \underline{f}(x,y) - (\underline{B_k f})(x,y) \rvert$	$\lvert \overline{f}(x,y) - \overline{B_k f}(x,y) \rvert$
0.1	0.0235411	0.402814
0.2	0.0476634	0.37317
0.3	0.072367	0.347013
0.4	0.0976519	0.324344
0.5	0.123518	0.305162
0.6	0.149965	0.289468
0.7	0. 17 6994	0.277262
0.8	0.204604	0.268543
0.9	0.232795	0.263311
1.0	0.261568	0.261568

Table 5: Numerical results for Example 5.2 for $k = 3, x = \frac{1}{8}$, y $= \frac{1}{8}$

r	$\lvert \underline{f}(x,y) - (\underline{B_k f})(x,y) \rvert$	$\lvert \overline{f}(x,y) - \overline{B_k f}(x,y) \rvert$
0.1	0.00283428	0.0484977
0.2	0.00573855	0.0449286
0.3	0.00871279	0.04177 94
0.4	0.117557	0.0390501
0.5	0.0148712	0.0367407
0.6	0.0180554	0.0348512
0.7	0.0213096	0.0333815
0.80.0246338	0.0323318	
0.9	0.0280279	0.031702
1.0	0.031492	0.031492

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