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Convergence of Iterative Methods Applied to Burgers-Huxley Equation

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Abstract

In this paper, a Burgers-Huxley equation is solved by using variety of methods: the Adomian's decomposition method , modified Adomian's decomposition method , variational iteration method , modified variational iteration method, homotopy perturbation method, modified homotopy perturbation method and homotopy analysis method. The approximate solution of this equation is calculated in the form of series whose components are computed by applying a recursive relation. Consequently, the existence and uniqueness of the solution and the convergence of the proposed methods are proved. Furthermore, a numerical example is studied to demonstrate the accuracy of the presented methods.

Keywords : Burgers-Huxley equation, Adomian decomposition method (ADM) , Modified Adomian decomposition method (MADM), Variational iteration method (VIM) , Modified variational iteration method (MVIM), Homotopy perturbation method (HPM), Modified homotopy perturbation method (MHPM), Homotopy analysis method (HAM).

1 Introduction

Burgers-Huxley equation playes an important role in mathematical physics. In recent years some works have been done in order to find the numerical solution to this equation, for example [4, 9, 10, 17, 18, 23, 24, 27]. In this work, we develope the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM to solve the Burgers-Huxley equation as follows:

$$u_t + \alpha u^{\delta} u_x - u_{xx} = \beta u(1 - u^{\delta})(u^{\delta} - \gamma), \tag{1.1}$$

where α, β , δ and γ are some arbitrary constants. With the initial conditions:

$$u(x, 0) = [\tfrac{\gamma}{2} + \tfrac{\gamma}{2} tanh(\sigma\gamma x)]^{\frac{1}{\delta}} = f(x), \tag{1.2}$$

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where,

$$\sigma = \frac{\delta(\rho-\alpha)}{4(1+\delta)},$$

$$\rho = \sqrt{\alpha^2 + 4\beta(1+\delta)}$$

The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq. (1.1). In section 3 we prove the existence , uniqueness of the solution and convergence of the proposed methods. Finally, the numerical example and computational complexity of the proposed methods are shown in section 4.

In order to obtain an approximate solution of Eq. (1.1), let us integrate one time Eq. (1.1) with respect to t using the initial conditions we obtain,

$$u(x,t) = f(x) - \alpha \int_0^t F_1(u(x,t))\ dt + \int_0^t D^2(u(x,t))\ dt + \beta \int_0^t F_2(u(x,t))\ dt, \tag{1.3}$$

where,

$$D^2(u(x,t)) = \frac{\partial^2 u(x,t)}{\partial x^2},$$

$$F_1(u(x,t)) = u^\delta(x,t) u_x(x,t)\ dt,$$

$$F_2(u(x,t)) = u(x,t)(1 - u^\delta(x,t))(u^\delta(x,t) - \gamma).$$

In Eq. (1.3), we assume $f(x)$ is bounded for all x in $J = [a,T](a,T \in \mathbb{R})$. The terms $D^2(u(x,t))$, $F_1(u(x,t))$ and $F_2(u(x,t))$ are Lipschitz continuous with

$$|D^2(u) - D^2(u^*)| \leq L_1 |u - u^*|,$$

$$|F_1(u) - F_1(u^*)| \leq L_2 |u - u^*|,$$

$$|F_2(u) - F_2(u^*)| \leq L_3 |u - u^*|.$$

2 The iterative methods

2.1 Description of the MADM and ADM

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu + Ru + Nu = g_1, \tag{2.4}$$

where $u(x,t)$ is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu which represents the nonlinear terms, and g_1 is the source term. Applying the inverse operator L^{-1} to both sides of Eq. (2.4), and using the given conditions we obtain

$$u(x,t) = f_1(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{2.5}$$

where the function $f_1(x)$ represents the terms arising from integrating the source term g_1. The nonlinear operator $Nu = G_1(u)$ is decomposed as

$$G_1(u) = \sum_{n=0}^{\infty} A_n, \tag{2.6}$$

where $A_n,\ n \geq 0$ are the Adomian polynomials determined formally as follows :

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i u_i)]]_{\lambda=0}. \tag{2.7}$$

The first Adomian polynomials (introduced in [5, 12, 28]) are:

$$\begin{aligned} A_0 &= G_1(u_0), \\ A_1 &= u_1 G_1'(u_0), \\ A_2 &= u_2 G_1'(u_0) + \frac{1}{2!}u_1^2 G_1''(u_0), \\ A_3 &= u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!}u_1^3 G_1'''(u_0), ... \end{aligned} \tag{2.8}$$

2.1.1 Adomian decomposition method

The standard decomposition technique represents the solution of $u(x,t)$ in (2.4) as the following series,

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{2.9}$$

where, the components $u_0, u_1, \ldots$ can be determined recursively

$$\begin{aligned} u_0 &= f(x), \\ u_1 &= -\alpha \int_0^t A_0(x,t)\ dt + \int_0^t B_0(x,t)\ dt + \beta \int_0^t L_0(x,t)\ dt, \\ &\vdots \\ u_{n+1} &= -\alpha \int_0^t A_n(x,t)\ dt + \int_0^t B_n(x,t)\ dt + \beta \int_0^t L_n(x,t)\ dt, \quad n \geq 0. \end{aligned} \tag{2.10}$$

Substituting (2.8) into (2.10) leads to the determination of the components of u.

2.1.2 The modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz [29]. The modified form was established on the assumption that the function $f(x)$ can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x,t) = f_1(x) + f_2(x). \tag{2.11}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. It was suggested that only the part f_1 is assigned to the zeroth component u_0, whereas the remaining part f_2 is combined with the other terms given in (2.11) to define u_1. Consequently, the modified recursive relation

$$\begin{aligned} u_0 &= f_1(x), \\ u_1 &= f_2(x) - L^{-1}(Ru_0) - L^{-1}(A_0), \\ &\vdots \\ u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1, \end{aligned} \tag{2.12}$$

was developed.

To obtain the approximation solution of Eq. (1.1), according to the MADM, we can write the iterative formula (2.12) as follows:

$$\begin{aligned} &u_0 = f_1(x), \\ &u_1 = f_2(x) - \alpha \int_0^t A_0(x,t)\ dt + \int_0^t B_0(x,t)\ dt + \beta \int_0^t L_0(x,t)\ dt \\ &\vdots \\ &u_{n+1} = -\alpha \int_0^t A_n(x,t)\ dt + \int_0^t B_n(x,t)\ dt + \beta \int_0^t L_n(x,t)\ dt, \quad n \geq 1. \end{aligned} \tag{2.13}$$

The operators $D^2(u)$, $F_1(u)$ and $F_2(u)$ are usually represented by the infinite series of the Adomian polynomials as follows:

$$D^2(u) = \sum_{i=0}^{\infty} B_i,$$

$$F_1(u) = \sum_{i=0}^{\infty} A_i,$$

$$F_2(u) = \sum_{i=0}^{\infty} L_i,$$

where A_i , B_i and L_i are the Adomian polynomials. Also, we can use the following formula for the Adomian polynomials [11]:

$$\begin{aligned} A_n &= F_1(s_n) - \textstyle\sum_{i=0}^{n-1} A_i, \\ B_n &= D^2(s_n) - \textstyle\sum_{i=0}^{n-1} B_i, \\ L_n &= F_2(s_n) - \textstyle\sum_{i=0}^{n-1} L_i. \end{aligned} \tag{2.14}$$

where $s_n = \sum_{i=0}^{n} u_i(x,t)$ is the partial sum.

2.2 Description of the VIM and MVIM

In the VIM [14, 19, 20, 21, 22], the following nonlinear differential equation is considered:

$$Lu + Nu = g_1, \tag{2.15}$$

where L is a linear operator, N is a nonlinear operator and g_1 is a known analytical function. In this case, the functions u_n may be determined recursively by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,\tau)\{L(u_n(x,\tau)) + N(u_n(x,\tau)) - g_1(x,\tau)\}d\tau, \quad n \geq 0, \tag{2.16}$$

where λ is a general Lagrange multiplier which can be computed using the variational theory. Here the function $u_n(x,\tau)$ is a restricted variation which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that is identified optimally via integration by parts. The successive approximation $u_n(x,t)$, $n \geq 0$ of the solution $u(x,t)$ is readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 selects any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximations $u_n(x,t)$, $n \geq 0$ follow immediately. Consequently, the exact solution is obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{2.17}$$

The VIM is shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions.

To obtain the approximation solution of Eq. (1.1), according to the VIM, we can write iteration formula (2.16) as follows:

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + L_t^{-1}(\lambda[u_n(x,t) - f(x) + \alpha \textstyle\int_0^t (F_1(u_n(x,t))\ dt \\ &- \textstyle\int_0^t D^2(u_n(x,t))\ dt - \beta \int_0^t F_2(u_n(x,t))\ dt]),\ n \geq 0. \end{aligned} \tag{2.18}$$

where,

$$L_t^{-1}(.) = \int_0^t (.)\ d\tau.$$

To find the optimal λ, we proceed as

$$\begin{aligned} \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta L_t^{-1}(\lambda[u_n(x,t) - (x) + \alpha \textstyle\int_0^t F_1(u_n(x,t))\ dt \\ &- \textstyle\int_0^t D^2(u_n(x,t))\ dt - \beta \int_0^t F_2(u_n(x,t))\ dt]). \end{aligned} \tag{2.19}$$

From Eq. (2.19), the stationary conditions are obtained as follows:

$$\lambda' = 0 \ \ and \ \ 1 + \lambda = 0$$

Therefore, the Lagrange multipliers are identified as $\lambda = -1$ and by substituting in (2.18), the following iteration formula is obtained.

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_{n+1}(x,t) &= u_n(x,t) - L_t^{-1}(u_n(x,t) - f(x) + \alpha \textstyle\int_0^t F_1(u_n(x,t))\ dt \\ &- \textstyle\int_0^t D^2(u_n(x,t))\ dt - \beta \int_0^t F_2(u_n(x,t))\ dt), n \geq 0. \end{aligned} \tag{2.20}$$

To obtain the approximation solution of Eq. (1.1), based on the MVIM [1, 2], we write the following iteration formula:

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_{n+1}(x,t) &= u_n(x,t) - L_t^{-1}(\alpha \textstyle\int_0^t F_1(u_n(x,t) - u_{n-1}(x,t))\ dt \\ &- \textstyle\int_0^t D^2(u_n(x,t) - u_{n-1}(x,t))\ dt - \beta \int_0^t F_2(u_n(x,t) - u_{n-1}(x,t))\ dt), n \geq 0. \end{aligned} \tag{2.21}$$

Relations (2.20) and (2.21) enable us to determine the components $u_n(x,t)$ recursively for $n \geq 0$.

2.3 Description of the HAM

Consider

$$N[u] = 0,$$

where N is a nonlinear operator, $u(x,t)$ is an unknown function and x is an independent variable. let $u_0(x,t)$ denote an initial guess of the exact solution $u(x,t)$, $h \neq 0$ an auxiliary parameter, $H_1(x,t) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x,t)] = 0$ when $s(x,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] - qhH_1(x,t)N[\phi(x,t;q)] = \hat{H}[\phi(x,t;q); u_0(x,t), H_1(x,t), h, q]. \tag{2.22}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H_1(x,t)$.

Enforcing the homotopy (2.22) to be zero, i.e.,

$$\hat{H}_1[\phi(x,t;q); u_0(x,t), H_1(x,t), h, q] = 0, \tag{2.23}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH_1(x,t)N[\phi(x,t;q)]. \tag{2.24}$$

When $q = 0$, the zero-order deformation Eq. (2.4) becomes

$$\phi(x;0) = u_0(x,t), \tag{2.25}$$

and when $q = 1$, since $h \neq 0$ and $H_1(x,t) \neq 0$, the zero-order deformation Eq. (2.24) is equivalent to

$$\phi(x,t;1) = u(x,t). \tag{2.26}$$

Thus, according to (2.25) and (2.26), as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in homotopy [8, 13, 25, 26].

Due to Taylor's theorem, $\phi(x,t;q)$ is expanded in a power series of q as follows

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{2.27}$$

where,

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}|_{q=0}.$$

Let the initial guess $u_0(x,t)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H_1(x,t)$ be properly chosen so that the power series (2.27) of $\phi(x,t;q)$ converges at $q = 1$, then, on these assumptions, we have the solution series

$$u(x,t) = \phi(x,t;1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{2.28}$$

From Eq. (2.28), we write Eq. (2.25) as follows:

$$\begin{aligned}(1-q)L[\phi(x,t,q)-u_0(x,t)] &= (1-q)L[\textstyle\sum_{m=1}^{\infty} u_m(x,t)\ q^m] \\ &= q\ h\ H_1(x,t)N[\phi(x,t,q)]\end{aligned}$$

then, we have

$$L[\sum_{m=1}^{\infty} u_m(x,t)\ q^m] - q\ L[\sum_{m=1}^{\infty} u_m(x,t)q^m] = q\ h\ H_1(x,t)N[\phi(x,t,q)] \tag{2.29}$$

By differentiating (2.29) m times with respect to q, we obtain

$$\begin{aligned}\{L[\textstyle\sum_{m=1}^{\infty} u_m(x,t)\ q^m] - q\ L[\textstyle\sum_{m=1}^{\infty} u_m(x,t)q^m]\}^{(m)} &= \{q\ h\ H_1(x,t)N[\phi(x,t,q)]\}^{(m)} \\ &= m!\ L[u_m(x,t)-u_{m-1}(x,t)] \\ &= h\ H_1(x,t)\ m\ \tfrac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}\ .\end{aligned}$$

Therefore,

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = hH_1(x,t)\Re_m(u_{m-1}(x,t)), \tag{2.30}$$

where,

$$\Re_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}, \tag{2.31}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation Eq. (2.30) is governing the linear operator L, and the term $\Re_m(u_{m-1}(x,t))$ can be expressed simply by (2.31) for any nonlinear operator N.

To obtain the approximation solution of Eq. (1.1), according to HAM, let

$$N[u(x,t)] = u(x,t) - f(x) + \alpha \textstyle\int_0^t F_1(u(x,t))\ dt - \int_0^t D^2(u(x,t))\ dt - \beta \int_0^t F_2(u(x,t))\ dt,$$

so,

$$\Re_m(u_{m-1}(x,t)) = u_{m-1}(x,t) - f(x) + \alpha \textstyle\int_0^t F_1(u_{m-1}(x,t))\ dt - \int_0^t D^2(u_{m-1}(x,t))\ dt - \beta \int_0^t F_2(u_{m-1}(x,t))\ dt. \tag{2.32}$$

Substituting (2.32) into (2.30)

$$\begin{aligned}L[u_m(x,t)-\chi_m u_{m-1}(x,t)] &= hH_1(x,t)[u_{m-1}(x,t) + \alpha \textstyle\int_0^t F_1(u_{m-1}(x,t))\ dt \\ &\quad - \textstyle\int_0^t D^2(u_{m-1}(x,t))\ dt - \beta \int_0^t F_2(u_{m-1}(x,t))\ dt \\ &\quad +(1-\chi_m)z(x,t)(x)].\end{aligned} \tag{2.33}$$

We take an initial guess $u_0(x,t) = f(x)$, an auxiliary linear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H_1(x,t) = 1$. This is substituted into

(2.33) to give the recurrence relation

$$u_0(x,t) = f(x),$$

$$u_{n+1}(x,t) = -\alpha \int_0^t F_1(u_n(x,t))\, dt + \int_0^t D^2(u_n(x,t))\, dt + \beta \int_0^t F_2(u_n(x,t))\, dt, \quad n \geq 0. \tag{2.34}$$

Therefore, the solution $u(x,t)$ becomes

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\ &= f(x) + \sum_{n=1}^{\infty} \Big(-\alpha \int_0^t F_1(u_n(x,t))\, dt + \int_0^t D^2(u_n(x,t))\, dt + \beta \int_0^t F_2(u_n(x,t))\, dt \Big). \end{aligned} \tag{2.35}$$

Which is the method of successive approximations. If

$$\mid u_n(x,t) \mid < 1,$$

then the series solution (2.35) convergence uniformly.

2.4 Description of the HPM and MHPM

To explain HPM [6, 7, 15], we consider the following general nonlinear differential equation:

$$Lu + Nu = f(u), \tag{2.36}$$

with initial conditions

$$u(x,0) = f(x).$$

According to HPM, we construct a homotopy which satisfies the following relation

$$H(u,p) = Lu - Lv_0 + p\, Lv_0 + p\, [Nu - f(u)] = 0, \tag{2.37}$$

where $p \in [0,1]$ is an embedding parameter and v_0 is an arbitrary initial approximation satisfying the given initial conditions.

In HPM, the solution of Eq. (2.37) is expressed as

$$u(x,t) = u_0(x,t) + p\, u_1(x,t) + p^2\, u_2(x,t) + ... \tag{2.38}$$

Hence the approximate solution of Eq. (2.36) is expressed as a series of the power of p, i.e.

$$u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + ...$$

where,

$$\begin{aligned} u_0(x,t) &= f(x), \\ &\vdots \\ u_m(x,t) &= \sum_{k=0}^{m-1} -\alpha \int_0^t F_1(u_{m-k-1}(x,t))\, dt + \int_0^t D^2(u_{m-k-1}(x,t))\, dt \\ &+\beta \int_0^t F_2(u_{m-k-1}(x,t))\, dt, \quad m \geq 1. \end{aligned} \tag{2.39}$$

To explain MHPM [3,16], we consider Eq. (1.1) as

$$L(u) = u(x,t) - f(x) + \alpha \int_0^t F_1(u(x,t))\ dt - \int_0^t D^2(u(x,t))\ dt - \beta \int_0^t F_2(u(x,t))\ dt.$$

where $F_1(u(x,t)) = g_1(x)h_1(t)$, $D^2(u(x,t)) = g_2(x)h_2(t)$ and $F_2(u(x,t)) = g_3(x)h_3(t)$. We define homotopy $H(u,p,m)$ by

$$H(u,0,m) = f(u), \quad H(u,1,m) = L(u),$$

where, m is an unknown real number and

$$f(u(x,t)) = u(x,t) - f(x).$$

Typically we choose a convex homotopy by

$$H(u,p,m) = (1-p)f(u) + p\ L(u) + p\ (1-p)[m(g_1(x) + g_2(x) + g_3(x))] = 0, \quad 0 \le p \le 1. \tag{2.40}$$

where m is called the accelerating parameters, and for $m = 0$ we define $H(u,p,0) = H(u,p)$, which is the standard HPM.

The convex homotopy (2.40) continuously trace an implicity defined curve from a starting point $H(u(x,t) - f(u), 0, m)$ to a solution function $H(u(x,t), 1, m)$. The embedding parameter p monotonically increases from 0 to 1 as the trivial problem $f(u) = 0$ is continuously deformed to the original problem $L(u) = 0$.

The MHPM uses the homotopy parameter p as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n, \tag{2.41}$$

when $p \to 1$, Eq. (2.37) corresponds to the original one and Eq. (2.41) becomes the approximate solution of Eq. (1.1), i.e.,

$$u = \lim_{p \to 1} v = \sum_{m=0}^{\infty} u_m.$$

Where,

$$\begin{aligned}
u_0(x,t) &= f(x),\\
u_1(x,t) &= -\alpha \textstyle\int_0^t F_1(u_0(x,t))\ dt + \int_0^t D^2(u_0(x,t))\ dt + \beta \int_0^t F_2(u_0(x,t))\ dt\\
&\quad -m(g_1(x) + g_2(x) + g_3(x)),\\
u_2(x,t) &= -\alpha \textstyle\int_0^t F_1(u_1(x,t))\ dt + \int_0^t D^2(u_1(x,t))\ dt + \beta \int_0^t F_2(u_1(x,t))\ dt\\
&\quad +m(g_1(x) + g_2(x) + g_3(x)),\\
&\ \ \vdots\\
u_m(x,t) &= \textstyle\sum_{k=0}^{m-1} -\alpha \int_0^t F_1(u_{m-k-1}(x,t))\ dt + \int_0^t D^2(u_{m-k-1}(x,t))\ dt\\
&\quad +\beta \textstyle\int_0^t F_2(u_{m-k-1}(x,t))\ dt,\ \ m \ge 3.
\end{aligned} \tag{2.42}$$

3 Existence and convergence of iterative methods

We set,

$$\alpha_1 := T(|\alpha| L_1 + L_2 + |\beta| L_3),$$
$$\beta_1 := 1 - T(1-\alpha_1),$$
$$\gamma_1 := 1 - T\alpha_1.$$

Theorem 3.1. *Let $0 < \alpha_1 < 1$, then Burgers-Huxley equation (1.1), has a unique solution.*

Proof: Let u and u^* be two different solutions of (1.3) then

$$\begin{aligned} |u-u^*| &= \left| -\alpha \int_0^t [F_1(u(x,t)) - F_1(u^*(x,t))]\, dt + \int_0^t [D^2(u(x,t)) - D^2(u^*(x,t))] dt \right. \\ &\left. +\beta \int_0^t [F_2(u(x,t)) - F_2(u^*(x,t))] dt \right| \\ &\le |\alpha| \int_0^t |F_1(u(x,t)) - F_1(u^*(x,t))|\, dt + \int_0^t |D^2(u(x,t)) - D^2(u^*(x,t))|\, dt \\ &+ |\beta| \int_0^t |F_2(u(x,t)) - F_2(u^*(x,t))|\, dt \\ &\le T(|\alpha| L_1 + L_2 + |\beta| L_3)\ |u-u^*| \\ &= \alpha_1 |u-u^*|\,. \end{aligned}$$

From which we get $(1-\alpha_1) | u-u^* | \le 0$. Since $0 < \alpha_1 < 1$, then $| u-u^* | = 0$. Implies $u = u^*$ and the proof is completed.

Theorem 3.2. *The series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of problem (1.1) using MADM converges when $0 < \alpha_1 < 1$, $| u_1(x,t) | < \infty$.*

Proof: Denote as $(C[J], \| . \|)$ the Banach space of all continuous functions on J with the norm $\| f(t) \| = max | f(t) |$, for all t in J. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \ge m$. We prove that s_n is a Cauchy sequence in this Banach space:

$$\begin{aligned} \| s_n - s_m \| &= max_{\forall t \in J} | s_n - s_m | \\ &= max_{\forall t \in J} \left| \sum_{i=m+1}^{n} u_i(x,t) \right| \\ &= max_{\forall t \in J} \left| -\alpha \int_0^t (\sum_{i=m}^{n-1} A_i)\, dt + \int_0^t (\sum_{i=m}^{n-1} B_i)\, dt + \beta \int_0^t (\sum_{i=m}^{n-1} L_i)\, dt \right| . \end{aligned}$$

From [14], we have

$$\sum_{i=m}^{n-1} A_i = F_1(s_{n-1}) - F_1(s_{m-1}),$$
$$\sum_{i=m}^{n-1} B_i = D^2(s_{n-1}) - D^2(s_{m-1}),$$
$$\sum_{i=m}^{n-1} L_i = F_2(s_{n-1}) - F_2(s_{m-1}).$$

So,

$$\begin{aligned}\| s_n - s_m \| &= max_{\forall t \in J} \mid -\alpha \int_0^t [F_1(s_{n-1}) - F_1(s_{m-1})]\ dt + \int_0^t [D^2(s_{n-1}) - D^2(s_{m-1})]\ dt \\ &\qquad +\beta \int_0^t [F_2(s_{n-1}) - F_2(s_{m-1})]\ dt \mid \\ &\leq \mid \alpha \mid \int_0^t \mid F_1(s_{n-1}) - F_1(s_{m-1}) \mid\ dt + \int_0^t \mid D^2(s_{n-1}) - D^2(s_{m-1}) \mid\ dt \\ &+ \mid \beta \mid \int_0^t \mid F_2(s_{n-1}) - F_2(s_{m-1}) \mid\ dt \\ &\leq \alpha_1 \| s_n - s_m \| .\end{aligned}$$

Let $n = m + 1$, then

$$\begin{aligned}\| s_n - s_m \| &\leq \alpha_1 \| s_m - s_{m-1} \| \\ &\leq \alpha_1^2 \| s_{m-1} - s_{m-2} \| \\ &\vdots \\ &\leq \alpha_1^m \| s_1 - s_0 \| .\end{aligned}$$

From the triangle inquality we have

$$\begin{aligned}\| s_n - s_m \| &\leq \| s_{m+1} - s_m \| + \| s_{m+2} - s_{m+1} \| + ... + \| s_n - s_{n-1} \| \\ &\leq [\alpha_1^m + \alpha_1^{m+1} + ... + \alpha_1^{n-m-1}] \| s_1 - s_0 \| \\ &\leq \alpha_1^m [1 + \alpha_1 + \alpha_1^2 + ... + \alpha_1^{n-m-1}] \| s_1 - s_0 \| \\ &\leq \alpha_1^m [\tfrac{1-\alpha_1^{n-m}}{1-\alpha_1}] \| u_1(x,t) \| .\end{aligned}$$

Since $0 < \alpha_1 < 1$, we have $(1 - \alpha_1^{n-m}) < 1$, then

$$\| s_n - s_m \| \leq \frac{\alpha_1^m}{1-\alpha_1} max_{\forall t \in J} \mid u_1(x,t) \mid . \tag{3.43}$$

But $\mid u_1(x,t) \mid < \infty$, so, as $m \to \infty$, then $\| s_n - s_m \| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore the series is converged and the proof is completed.

Theorem 3.3. *The solution $u_n(x,t)$ obtained from the relation (2.20) using VIM, converges to the exact solution of the problem (1.1) when $0 < \alpha_1 < 1$ and $0 < \beta_1 < 1$.*

Proof:

$$\begin{aligned}u_{n+1}(x,t) &= u_n(x,t) - L_t^{-1} \Big(\Big[u_n(x,t) - f(x) + \alpha \int_0^t F_1(u_n(x,t))\ dt \\ &\qquad - \int_0^t D^2(u_n(x,t)))\ dt - \beta \int_0^t F_2(u_n(x,t)))\ dt \Big] \Big)\end{aligned} \tag{3.44}$$

$$\begin{aligned}u(x,t) &= u(x,t) - L_t^{-1} \Big(\Big[u(x,t) - f(x) + \alpha \int_0^t F_1(u(x,t))\ dt \\ &\qquad - \int_0^t D^2(u(x,t)))\ dt - \beta \int_0^t F_2(u(x,t)))\ dt \Big] \Big)\end{aligned} \tag{3.45}$$

By subtracting relation (3.44) from (3.45),

$$\begin{aligned} u_{n+1}(x,t)-u(x,t) &= u_n(x,t)-u(x,t)-L_t^{-1}(u_n(x,t)-u(x,t) \\ &+\alpha \int_0^t [F_1(u_n(x,t))-F_1(u(x,t))]\ dt \\ &-\int_0^t [D^2(u_n(x,t))-D^2(u(x,t))]\ dt \\ &-\beta \int_0^t [F_2(u_n(x,t))-F_2(u(x,t))]\ dt), \end{aligned}$$

if we set, $e_{n+1}(x,t) = u_{n+1}(x,t) - u_n(x,t)$, $e_n(x,t) = u_n(x,t) - u(x,t)$,$\mid e_n(x,t^*) \mid = max_t \mid e_n(x,t) \mid$ then since e_n is a decreasing function with respect to t from the mean value theorem we write,

$$\begin{aligned} e_{n+1}(x,t) &= e_n(x,t)+L_t^{-1}(-e_n(x,t)-\alpha \int_0^t [F_1(u_n(x,t))-F_1(u(x,t))]\ dt \\ &-\int_0^t [D^2(u_n(x,t))-D^2(u(x,t))]\ dt - \beta \int_0^t [F_2(u_n(x,t))-F_2(u(x,t))]\ dt) \\ &\le e_n(x,t)+L_t^{-1}[-e_n(x,t)+L_t^{-1} \mid e_n(x,t) \mid (T(\mid \alpha \mid L_1+L_2+\mid \beta \mid L_3)] \\ &\le e_n(x,t)-Te_n(x,\eta)+T(\mid \alpha \mid L_1+L_2+\mid \beta \mid L_3)L_t^{-1}L_t^{-1} \mid e_n(x,t) \mid \\ &\le 1-T(1-\alpha_1) \mid e_n(x,t^*) \mid, \end{aligned}$$

where $0 \le \eta \le t$. Hence, $e_{n+1}(x,t) \le \beta_1 \mid e_n(x,t^*) \mid$. Therefore,

$$\begin{aligned} \|e_{n+1}\| &= max_{\forall t \in J} \mid e_{n+1} \mid \\ &\le \beta_1 \max_{\forall t \in J} \mid e_n \mid \\ &\le \beta_1 \|e_n\|. \end{aligned}$$

Since $0 < \beta_1 < 1$, then $\|e_n\| \to 0$. So, the series converges and the proof is complete.

Theorem 3.4. *The solution $u_n(x,t)$ obtained from the relation (2.21) using MVIM for the problem (1.1) converges when* $0 < \alpha_1 < 1$, $0 < \gamma_1 < 1$.

Proof: The Proof is similar to the previous theorem.

Theorem 3.5. *If the series solution (2.34) of problem (1.1) uses HAM then it converges to the exact solution of the problem (1.1).*

Proof: We assume:

$$\begin{aligned} &u(x,t) = \textstyle\sum_{m=0}^{\infty} u_m(x,t), \\ &\widehat{F}_1(u(x,t)) = \textstyle\sum_{m=0}^{\infty} F_1(u_m(x,t)), \\ &\widehat{D}^2(u(x,t)) = \textstyle\sum_{m=0}^{\infty} D^2(u_m(x,t)), \\ &\widehat{F}_2(u(x,t)) = \textstyle\sum_{m=0}^{\infty} F_2(u_m(x,t)). \end{aligned}$$

where,

$$\lim_{m \to \infty} u_m(x,t) = 0.$$

We write,

$$\sum_{m=1}^{n}[u_m(x,t)-\chi_m u_{m-1}(x,t)] = u_1+(u_2-u_1)+...+(u_n-u_{n-1}) = u_n(x,t). \quad (3.46)$$

Hence, from (3.46),

$$\lim_{n\to\infty} u_n(x,t) = 0. \quad (3.47)$$

So, using (3.47) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = L[\sum_{m=1}^{\infty}[u_m(x,t)-\chi_m u_{m-1}(x,t)]] = 0.$$

therefore from (2.30), we obtain,

$$\sum_{m=1}^{\infty} L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = hH_1(x,t)\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) = 0.$$

Since $h \neq 0$ and $H_1(x,t) \neq 0$, we have

$$\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) = 0. \quad (3.48)$$

By substituting $\Re_{m-1}(u_{m-1}(x,t))$ into the relation (3.48) and simplifying it , we have

$$\begin{aligned}\textstyle\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) &= \textstyle\sum_{m=1}^{\infty}[u_{m-1}(x,t)+\alpha\int_0^t F-1(u_{m-1}(x,t))\,dt \\ &-\textstyle\int_0^t D^2(u_{m-1}(x,t))\,dt-\beta\int_0^t F_2(u_{m-1}(x,t))\,dt+(1-\chi_m)f(x)] \\ &= u(x,t)-f(x)+\alpha\textstyle\int_0^t \widehat{F}_1(u(x,t))\,dt-\int_0^t \widehat{D}^2(u(x,t))\,dt \\ &-\beta\textstyle\int_0^t \widehat{F}_2(u(x,t))\,dt.\end{aligned} \quad (3.49)$$

From (3.48) and (3.49), we have

$$u(x,t) = f(x)-\alpha\int_0^t \widehat{F}_1(u(x,t))\,dt+\int_0^t(\widehat{D}^2(u(x,t))\,dt+\beta\int_0^t \widehat{F}_2(u(x,t))\,dt.$$

Therefore, $u(x,t)$ must be the exact solution.

Theorem 3.6. *If* $|\ u_m(x,t)\ |\leq 1$, *then the series solution* $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ *of problem (1.1) converges to the exact solution by using HPM.*

Proof: We set,

$$\phi_n(x,t) = \sum_{i=1}^{n} u_i(x,t),$$

$$\phi_{n+1}(x,t) = \sum_{i=1}^{n+1} u_i(x,t).$$

so,

$$\begin{aligned} \mid \phi_{n+1}(x,t)-\phi_n(x,t)\mid &= D(\phi_{n+1}(x,t),\phi_n(x,t)) \\ &= D(\phi_n+u_n,\phi_n) \\ &= D(u_n,0) \\ &\leq \textstyle\sum_{k=0}^{m-1} \mid \alpha \mid \int_0^t \mid F_1(u_{m-k-1}(x,t)) \mid \ dt \\ &+ \textstyle\int_0^t \mid D^2(u_{m-k-1}(x,t)) \mid \ dt \\ &+ \mid \beta \mid \textstyle\int_0^t \mid F_2(u_{m-k-1}(x,t)) \mid \ dt. \end{aligned}$$

thus,

$$\sum_{n=0}^{\infty} \parallel \phi_{n+1}(x,t)-\phi_n(x,t) \parallel \leq m\alpha_1 \mid f(x) \mid \sum_{n=0}^{\infty}(m\alpha_1)^n.$$

Therefore,

$$\lim_{n\to\infty} u_n(x,t)=u(x,t).$$

Theorem 3.7. *If* $\mid u_m(x,t) \mid \leq 1$, *then the series solution* $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ *of problem (1.1) converges to the exact solution by using MHPM.*

Proof: The Proof is similar to the previous theorem.

Lemma 3.1. *The computational complexity of the ADM and MADM is* $O(n^3)$,*that of HAM, VIM and MVIM is* $O(n)$,*that of HPM and MHPM is* $O(n^2)$.

Proof: The number of computations including division, production, sum and subtraction.

ADM:
In step 2,
$A_n, B_n, L_n : \frac{n^2}{2} + \frac{9}{2}n + 2.$
In step 3,
$u_0 : 6.$
$u_1 : 11.$
$u_2 : 26.$
.
.
$u_{n+1} : \frac{3}{2}n^2 + \frac{27}{2}n + 11, n \geq 0.$
In step 5, the total number of the computations is equal to
$\sum_{i=0}^{n} u_i(x,t) = O(n^3).$

MADM:
In step 2,
$A_n, B_n, L_n : \frac{n^2}{2} + \frac{9}{2}n + 2.$
In step 3,
$u_0 : 6.$
$u_1 : 17.$
$u_2 : 26.$

.

.

$u_{n+1} : \frac{3}{2}n^2 + \frac{27}{2}n + 16, n \geq 1.$

In step 5, the total number of the computations is equal to

$\sum_{i=0}^{n} u_i(x,t) = O(n^3).$

VIM:

In step 2,

$u_0 : 6.$

$u_1 : 17.$

.

.

$u_{n+1} : 17, \quad n \geq 0.$

In step 4, the total number of the computations is equal to

$\sum_{i=0}^{n} u_i(x,t) = O(17n).$

MVIM:

In step 2,

$u_0 : 6.$

$u_1 : 13.$

.

.

$u_{n+1} : 13, \quad n \geq 0.$

In step 4, the total number of the computations is equal to

$\sum_{i=0}^{n} u_i(x,t) = O(13n).$

HAM:

In step 2,

$u_0 : 6.$

$u_1 : 10.$

.

.

$u_{n+1} : 10, \quad n \geq 0.$

In step 4, the total number of the computations is equal to

$\sum_{i=0}^{n} u_i(x,t) = 10n + 16 = O(10n).$

HPM:

In step 2,

$u_0 : 6.$

$u_1 : 10.$

$u_2 : 10.$

.

.

$u_{n+1} : 10n + 16, \quad n \geq 0.$

In step 4, the total number of the computations is equal to

$\sum_{i=0}^{n} u_i(x,t) = O(n^2).$

MHPM:
In step 2,

$u_0 : 6.$
$u_1 : 13.$
$u_2 : 13.$
.
.
$u_{n+1} : 10n + 10, \quad n \geq 2.$

In step 4, the total number of the computations is equal to

$u_0 + u_1 + u_2 + \sum_{i=3}^{n} u_i(x,t) = O(n^2).$

4 Numerical example

In this section, we compute a numerical example which is solved by the ADM, MADM, VIM, MVIMm HPM, MHPM and HAM. The program is provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm 1:
Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relations (2.10) for ADM , (2.13) for MADM, (2.34) for HAM, (2.39) for HPM and (2.42) for MHPM.
Step 3. If $\mid u_{n+1} - u_n \mid < \varepsilon$ then go to step 4, else $n \leftarrow n + 1$ and go to step 2.
Step 4. Print $u(x,t) = \sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.

Algorithm 2:
Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relations (2.20) for VIM and (2.21) for MVIM.
Step 3. If $\mid u_{n+1} - u_n \mid < \varepsilon$ then go to step 4, else $n \leftarrow n + 1$ and go to step 2.
Step 4. Print $u_n(x,t)$ as the approximate of the exact solution.

Example 4.1. *Consider the Burgers-Huxley equation as follows:*

$$u_t + uu_x - u_{xx} = u(1-u)(u-2),$$

subject to the initial conditions:

$$f(x) = 1 + tanh(\frac{1}{2}x).$$

Table 1, shows that, approximate solution of the Burgers-Huxley equation is convergent with 4 iterations by using the HAM . By comparing the results of Table 1 , we can observe that the HAM is of higher level of convergence than the ADM, MADM, VIM, MVIM, HPM and MHPM.

Table 1
Numerical results for Example (4.1)

(x,t)	Errors						
	ADM(n=16)	MADM(n=13)	VIM(n=9)	MVIM(n=7)	HPM(n=8)	MHPM(n=7)	HAM(n=4)
(0.1,0.15)	0.080623	0.072725	0.051647	0.042458	0.060658	0.042673	0.033365
(0.2,0.18)	0.081482	0.073668	0.054562	0.043461	0.062763	0.044658	0.023706
(0.3,0.28)	0.083772	0.074235	0.055651	0.043736	0.063495	0.045277	0.034458
(0.4,0.34)	0.084785	0.074788	0.056351	0.044386	0.063756	0.046674	0.036742
(0.5,0.4)	0.085562	0.075325	0.057744	0.044825	0.064347	0.047706	0.037173
(0.7,0.45)	0.086687	0.075864	0.058443	0.045746	0.064832	0.048248	0.038785

5 Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of non-linear problems with the approximations which rapidly converge to exact solutions. In this work, the HAM has been successfully employed to obtain the approximate solution to analytical solution of the Burgers-Huxley equation. For this purpose, we have showed that the HAM is of higher level of convergence than the ADM, MADM, VIM, MVIM, HPM and MHPM. Also, the number of computations in HAM is less than the number of computations in ADM, MADM, VIM, MVIM, HPM and MHPM.

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