



Two new three and four parametric with memory methods for solving nonlinear equations

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Abstract

In this study, based on the optimal free derivative without memory methods proposed by Cordero et al. [A. Cordero, J.L. Hueso, E. Martinez, J.R. Torregrosa, Generating optimal derivative free iterative methods for nonlinear equations by using polynomial interpolation, *Mathematical and Computer Modeling*. 57 (2013) 1950-1956], we develop two new iterative with memory methods for solving a nonlinear equation. The first has two steps with three self-accelerating parameters, and the second has three steps with four self-accelerating parameters. These parameters are calculated using information from the current and previous iteration so that the presented methods may be regarded as the with memory methods. The self-accelerating parameters are computed applying Newton's interpolatory polynomials. Moreover, they use three and four functional evaluations per iteration and corresponding R-orders of convergence are increased from 4 ad 8 to 7.53 and 15.51, respectively. It means that, without any new function calculations, we can improve convergence order by 93% and 96%. We provide rigorous theories along with some numerical test problems to confirm theoretical results and high computational efficiency.

Keywords : Nonlinear equation; With memory method; R-order of convergence; Self accelerating parameter; Efficiency index.

1 Introduction

Multi-point iterative with and without memory methods for solving a nonlinear equation are great importance among the researchers in this field. Without doubt, Traub [14] and Ostrowski [12] made major contributions. Kung and Traub conjectured any optimal multi-point without memory method has convergence order 2^n using exactly $n + 1$ functional evaluations per full cycle [4].

In addition, Ostrowski introduced a criteria for

comparing different methods, say, efficiency index which is defined by $EI = p^{\frac{1}{n}}$, where p and n represent convergence order and functional evaluations, respectively.

Based on Kung and Traub's conjecture, during the last two decades many researchers have designed many optimal without memory methods [1,5,17] and over recent years several with memory methods have presented [2,3] and [6-11], too. In this study, we consider two kind of Cordero et al.'s methods [1] and try to develop two new with memory methods. Our improvements have not been studied before. As the main contribution of this work, convergence orders have been increased from 4 and 8 to 7.53 and 15.51, respectively, without any new functional evaluations.

First, we modify the first two and three steps of

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Cordero et al.'s method [1] in such a way that they are still optimal. Then, introducing the best approximations for considered accelerators, we attempt to derive our new with memory methods.

The rest of this paper is organized as follows: Section 2 is devoted to modifications of the two and three steps of Cordero et al.'s methods [1]. Section 3 concerns with developing new with memory methods. Section 4 includes some numerical performances. And, Section 5 comprises conclusion remarks.

2 Modified three- and Four-parametric methods

2.1 three-parametric two-point method

In this section, our goal is to modify two optimal without memory methods by Cordero et al. [1]. Let start by the following optimal two-point without memory method

$$\begin{cases} z_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)}{p'_2(z_n)}, \end{cases} \quad (2.1)$$

with this error equation

$$e_{n+1} = (1 + f'(\alpha))^2 c_2 (c_2^2 - c_3) e_n^4 + O(e_n^5). \quad (2.2)$$

Where $w_n = x_n + f(x_n)$, and $p_2(z_n)$ is the interpolating polynomial of the points $(x_n, f(x_n)), (w_n, f(w_n)), (z_n, f(z_n))$. This polynomial can be written as

$$p_2(x) = \frac{x_n - z_n}{x_n - w_n} f[w_n, z_n] + \frac{w_n - z_n}{w_n - x_n} f[x_n, z_n], \quad (2.3)$$

so, we have

$$p'_2(z_n) = f[x_n, z_n] - f[x_n, w_n] + f[w_n, z_n]. \quad (2.4)$$

Now, we consider the following modification of (2.1) by adding three free parameters γ, p , and λ

$$\begin{cases} w_n = x_n + \gamma f(x_n), \\ z_n = x_n - \frac{f(x_n)}{f[x_n, w_n] + p f(w_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{p'_2(z_n) + \lambda(z_n - x_n)(z_n - w_n)}. \end{cases} \quad (2.5)$$

This method is of four convergence order and we state it formally in the following theorem

Theorem 2.1 Let $f : D \rightarrow \mathbb{R}$ be sufficiently differentiable function with a simple root $\alpha \in D$, $D \subset \mathbb{R}$ be an open set, x_0 be close enough to α , then the method (2.5) is at least of fourth-order, and satisfies α the error equation

$$e_{n+1} = \frac{(1+\gamma f'(\alpha))^2 (p+c_2)(\lambda+c_2 f'(\alpha)(p+c_2)-c_3 f'(\alpha)) e_n^4}{f'(\alpha)} + O(e_n^5), \quad (2.6)$$

where $e_n = x_n - \alpha$ and $c_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$.

Proof. We use the Mathematica for finding the error equation.

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f[e_] = f1a * (e + Sum[k=2^4 c_k * e^k]);
ew = e + 'f[e] (*ew = w - alpha*);
f[x_, y_] := (f[x]-f[y])/(x-y);
ez = e - Series[f[e]/(f[e,ew]+pf[ew]), {e, 0, 4}]
(*ez = z - alpha*)
Out[a] : (1 + f1a')(p + c2)e^2 + 0[e]^3.
p2[t_] = a0 + a1(t - ez) + a2(t - ez)^2;
dp2[t_] := a1 + 2a2(t - ez);
a1 := f[e, ez] - f[e, ew] + f[ew, ez];
en+1 = ez - (f[ez]/(dp2[ez]+(ez-e)(ez-ew)))//Simplify;
Out[b] : ((1+f1a')^2(p+c2)(lambda+c2 f1a(p+c2)-c3 f1a) e^4) / f1a + 0[e]^5.
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Therefore, we have

$$e_{n+1} = \frac{(1+\gamma f'(\alpha))^2 (p+c_2)(\lambda+c_2 f'(\alpha)(p+c_2)-c_3 f'(\alpha)) e_n^4}{f'(\alpha)} + O(e_n^5).$$

2.2 Four-parametric three-point method

Now, we consider the following optimal three-point without memory method that has proposed by Cordero et al. [1]

$$\begin{cases} z_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ u_n = z_n - \frac{f(z_n)}{p'_2(z_n)}, \\ x_{n+1} = u_n - \frac{f(u_n)}{p'_3(u_n)}, \end{cases} \quad n = 0, 1, \dots, \quad (2.7)$$

with this error equation

$$e_{n+1} = (1 + f'(\alpha))^4 c_2^2 (c_2^2 - c_3) (c_2^3 - c_2 c_3 + c_4) e_n^8 + O(e_n^9). \quad (2.8)$$

Where $p_3(u_n)$ is the interpolating polynomial of the points $(x_n, f(x_n)), (w_n, f(w_n)), (z_n, f(z_n))$,

$(u_n, f(u_n))$. This polynomial can be written as

$$p_3(x) = \frac{(x_n - u_n)(z_n - u_n)}{(x_n - w_n)(z_n - w_n)} f[w_n, u_n] + \frac{(w_n - u_n)(z_n - u_n)}{(w_n - x_n)(z_n - x_n)} f[x_n, u_n] + \frac{(w_n - u_n)(x_n - u_n)}{(w_n - z_n)(x_n - z_n)} f[z_n, u_n], \tag{2.9}$$

so, we have

$$p'_3(u_n) = - \left(f[z_n, u_n](w_n - u_n)(x_n - u_n)(x_n - w_n) + f[x_n, u_n](w_n - u_n)(z_n - u_n)(w_n - z_n) + f[w_n, u_n](x_n - u_n)(z_n - u_n)(z_n - x_n) \right) / \left((x_n - u_n)((w_n - u_n)(1 - x_n - z_n + 2u_n) + (x_n - u_n)(z_n - u_n)) \right). \tag{2.10}$$

Then, we modify (2.7) as follows similar to (2.5)

$$\begin{cases} w_n = x_n + \gamma f(x_n), \\ z_n = x_n - \frac{f(x_n)}{f[x_n, w_n] + p f(w_n)}, \\ u_n = z_n - \frac{f(z_n)}{p'_2(z_n) + \lambda(z_n - x_n)(z_n - w_n)}, \\ x_{n+1} = u_n - \frac{f(u_n)}{p'_3(u_n) + \beta(u_n - w_n)(u_n - x_n)(u_n - z_n)}. \end{cases} \tag{2.11}$$

This method is of eight convergence order and we demonstrate it officially in the following theorem

Theorem 2.2 *Let $f : D \rightarrow \mathbb{R}$ be sufficiently differentiable function with a simple root $\alpha \in D$, $D \subset \mathbb{R}$ be an open set, x_0 be close enough to α , then the method (2.11) is at least of eighth-order, and satisfies α the error equation*

$$e_{n+1} = \left((1 + \gamma f'(\alpha))^4 (p + c_2)^2 (\lambda + c_2 f'(\alpha) (p + c_2) - c_3 f(\alpha)) (-\beta + c_2 (\lambda + c_2 f'(\alpha) (p + c_2) - c_3 f(\alpha)) + c_4 f'(\alpha)) e_n^8 \right) f'(\alpha)^2 + O(e_n^9), \tag{2.12}$$

where $e_n = x_n - \alpha$ and $c_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$.

Proof. We use the Mathematica for finding the error equation.

$$\begin{aligned} f[e.] &= f1a * (e + \sum_{k=2}^5 c_k * e^k); \\ ew &= e + 'f[e] (*ew = w - \alpha*); \\ f[x-, y-] &:= \frac{f[x]-f[y]}{x-y}; \\ ez &= e - Series[\frac{f[e]}{f[e,ew]+pf[ew]}, \{e, 0, 8\}] \end{aligned}$$

$$\begin{aligned} (*ez = z - \alpha*) \\ Out[a] &: (1 + f1a')(p + c_2)e^2 + O[e]^3. \\ p_2[t.] &= a_0 + a_1(t - ez) + a_2(t - ez)^2; \\ dp_2[t.] &:= a_1 + 2a_2(t - ez); \\ a_1 &:= f[e, ez] - f[e, ew] + f[ew, ez]; \\ eu &= ez - \frac{f[ez]}{dp_2[ez] + f[ez] / (ez - e)} // Simplify \\ (*eu = u - \alpha*) \\ Out[b] &: \frac{(1 + f1a')^2 (p + c_2) (-c_2 f1a (p + c_2) - c_3 f1a) e^4}{f1a} + O[e]^5. \\ p_3[t.] &= b_0 + b_1(t - eu) + b_2(t - eu)^2 + b_3(t - eu)^3; \\ dp_3[t.] &:= b_1 + 2b_2(t - eu) + 3b_3(t - eu)^2; \\ b_1 &:= - \left(f[ez, eu](ew - eu)(e - eu)(e - ew) + f[e, eu](ew - eu)(ew - eu)(ew - ez) + f[ew, eu](e - eu)(ez - eu)(ez - e) \right) / \left((e - eu)((ew - eu)(1 - e - ez + 2eu) + (e - eu)(ez - eu)) \right); \end{aligned}$$

$$\begin{aligned} e_{n+1} &= eu - \frac{f[eu]}{dp_3[eu] + f[eu] / ((e - ew)(e - e)(e - ez))} \\ Out[c] &: \left((1 + f1a')^4 (p + c_2)^2 (-c_2 f1a (p + c_2) - c_3 f1a) (-\beta + c_2 (\lambda + c_2 f1a (p + c_2) - c_3 f1a) + c_4 f1a) e^8 \right) / f1a^2 + O[e]^9. \end{aligned}$$

Therefore, we gain

$$e_{n+1} = \left((1 + \gamma f'(\alpha))^4 (p + c_2)^2 (\lambda + c_2 f'(\alpha) (p + c_2) - c_3 f'(\alpha)) (-\beta + c_2 (\lambda + c_2 f'(\alpha) (p + c_2) - c_3 f'(\alpha)) + c_4 f'(\alpha)) e_n^8 \right) / f'(\alpha)^2 + O(e_n^9).$$

3 The developments of new with memory methods

3.1 A new family of two-step with memory methods

In structure of iterative with memory methods by using free parameters and utilization suitable approximations for them, without any new function evaluations, one is able to increase the convergence order of an optimal method by applying the previous and current information of iterations. To this end, we modify $\gamma \rightarrow \gamma_n$, $p \rightarrow p_n$, and $\lambda \rightarrow \lambda_n$. According to error equation (2.6),

to increase convergence order, we consider

$$\begin{cases} 1 + \gamma_n f'(\alpha) = 0, \\ p_n + c_2 = 0, \\ \lambda_n + f'(\alpha)c_2(p + c_2) - f'(\alpha)c_3 = 0, \end{cases} \quad (3.13)$$

So, we have

$$\gamma_n = -\frac{1}{f'(\alpha)}, \quad p_n = -\frac{f''(\alpha)}{2!f'(\alpha)}, \quad \lambda_n = \frac{f'''(\alpha)}{3!}. \quad (3.14)$$

Because α is unknown, therefore, we can not compute $f^{(j)}(\alpha), j = 1, 2, 3$. Then, we use interpolation to approximate them as follows

$$\gamma_n = -\frac{1}{N'_3(x_n)}, \quad p_n = -\frac{N''_4(w_n)}{2!N'_4(w_n)}, \quad \lambda_n = \frac{f'''_5(z_n)}{3!},$$

where $N_3(t)$ and $N_4(t)$ are Newton's interpolatory polynomials of third and fourth degrees. Hence, the new with memory method is given by

$$\begin{cases} x_0, \gamma_0, p_0 \text{ are given, then } w_0 = x_0 + \gamma_0 f(x_0), \\ \gamma_n = -\frac{1}{N'_3(x_n)}, \quad p_n = -\frac{N''_4(w_n)}{2!N'_4(w_n)}, \quad \lambda_n = \frac{f'''_5(z_n)}{3!}, \\ w_n = x_n + \gamma_n f(x_n), \\ z_n = x_n - \frac{f(x_n)}{f[x_n, w_n] + p_n f(w_n)}, \quad n = 1, 2, \dots, \\ x_{n+1} = z_n - \frac{f(z_n)}{p'_2(z_n) + \lambda_n(z_n - x_n)(z_n - w_n)}. \end{cases} \quad (3.15)$$

To prove its convergence order, we need to following lemma

Lemma 3.1 *If $\gamma_n = -1/N'_3(x_n)$ and $p_n = -N''_4(w_n)/(2N'_4(w_n)), n = 1, 2, \dots$, then the estimates*

$$1 + \gamma_n f'(\alpha) \sim e_{n-1,z} e_{n-1,w} e_{n-1}, \quad (3.16)$$

and

$$c_2 + p_n \sim e_{n-1,z} e_{n-1,w} e_{n-1}, \quad (3.17)$$

and

$$\begin{aligned} \lambda_n + f'(\alpha)c_2(p_n + c_2) - f'(\alpha)c_3 \\ \sim e_{n-1,z} e_{n-1,w} e_{n-1}, \end{aligned} \quad (3.18)$$

hold.

Proof. Similar to Lemma 1 in [15] and Lemma 4 and 6 in [9].

The following theorem determines the convergence order of the two-point iterative with memory method (3.15).

Theorem 3.1 *If an initial estimation x_0 is close enough to a simple root α of $f(x) = 0$, being f a real sufficiently differentiable function, then the R-order of convergence of the two-point method with memory (3.15) is at least 7.5311.*

Proof. Let $\{x_n\}$ have converged order R . Then, we can write

$$e_{n+1} \sim e_n^R, \quad e_n = x_n - \alpha, \quad (3.19)$$

Hence

$$e_{n+1} \sim e_n^R = (e_{n-1}^R)^R = e_{n-1}^{R^2}. \quad (3.20)$$

Suppose sequences $\{w_n\}$ and $\{y_n\}$ have converged p and q , respectively,

$$e_{n,w} \sim e_n^p = (e_{n-1}^R)^p = e_{n-1}^{Rp} \quad (3.21)$$

and

$$e_{n,z} \sim e_n^q = (e_{n-1}^R)^q = e_{n-1}^{Rq}. \quad (3.22)$$

By (3.22), (3.21), and Lemma 3.1, we obtain

$$1 + \gamma_n f'(\alpha) \sim e_{n-1}^{p+q+1}, \quad (3.23)$$

$$c_2 + p_n \sim e_{n-1}^{p+q+1}. \quad (3.24)$$

Substituting these into $e_{n,w}, e_{n,y}$, and e_{n+1} in Theorem 3.1, we have

$$e_{n,w} \sim \left(1 + \gamma_n f'(\alpha)\right) e_n = e_{n-1}^{(1+p+q)+R}, \quad (3.25)$$

$$\begin{aligned} e_{n,z} \sim c_2 \left(1 + \gamma_n f'(\alpha)\right) (c_2 + p_n) e_n^2 \\ = e_{n-1}^{2(1+p+q)+2R}, \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} e_{n+1} \sim A_4 \left(1 + \gamma_n f'(\alpha)\right)^2 (c_2 + p_n) \\ (\lambda + f'(\alpha)c_2(p + c_2) - f'(\alpha)c_3) e_n^4 \\ \sim e_{n-1}^{4(1+p+q)+4R}. \end{aligned} \quad (3.27)$$

Equating the powers of error exponents of e_{n-1} in pairs of relations (3.21)-(3.25), (3.22)-(3.26), and (3.20)-(3.27), we have

$$\begin{cases} Rp - R - (p + q + 1) = 0, \\ Rq - 2R - 2(p + q + 1) = 0, \\ R^2 - 4R - 4(p + q + 1) = 0. \end{cases} \quad (3.28)$$

This system has the solution $p = 1.8828, q = 3.7656$, and $R = 7.5311$ which specifies the R-order of convergence of the derivative-free scheme with memory (3.15). \square

3.2 A new family of three-step with memory methods

In a similar way to former section, in addition to mentioned parameters, due to three-point method error equation (2.12), we modify $\beta \rightarrow \beta_n$, too. Then, we have

$$\begin{cases} 1 + \gamma_n f'(\alpha) = 0, \\ p_n + c_2 = 0, \\ \lambda_n + f'(\alpha)c_2(p + c_2) - f'(\alpha)c_3 = 0, \\ -\beta_n + c_2(\lambda + c_2 f'(\alpha)(p + c_2) - c_3 f'(\alpha)) \\ + c_4 f'(\alpha) = 0. \end{cases} \tag{3.29}$$

So, we gain

$$\begin{aligned} \gamma_n &= -\frac{1}{f'(\alpha)}, & p_n &= -\frac{f''(\alpha)}{2!f'(\alpha)}, \\ \lambda_n &= \frac{f'''(\alpha)}{3!}, & \beta_n &= \frac{f^{(4)}(\alpha)}{4!}. \end{aligned} \tag{3.30}$$

Because α is unknown, therefore, we can not compute $f^{(j)}(\alpha), j = 1, 2, 3, 4$. Then, we use interpolation to approximate them as follows

$$\begin{aligned} \gamma_n &= -\frac{1}{N_4'(x_n)}, & p_n &= -\frac{N_5''(w_n)}{2!N_5'(w_n)}, \\ \lambda_n &= \frac{N_6'''(z_n)}{3!}, & \beta_n &= \frac{N_7^{(4)}(u_n)}{4!}, \end{aligned}$$

where $N_i(t), (i = 4, 5, 6, 7)$, are Newton's interpolatory polynomials of i degrees. As we state in previous section by estimating γ_n, p_n, λ_n , and β_n with Newton's interpolatory polynomials, we find out new three-point methods with memory as follows

$$\begin{cases} x_0, \gamma_0, p_0 \text{ are given, then } w_0 = x_0 + \gamma_0 f(x_0), \\ \gamma_n = -\frac{1}{N_4'(x_n)}, p_n = -\frac{N_5''(w_n)}{2N_5'(w_n)}, \\ w_n = x_n + \gamma_n f(x_n), \quad n = 1, 2, \dots, \\ z_n = x_n - \frac{f(x_n)}{f[x_n, w_n] + p_n f(w_n)}, \\ u_n = z_n - \frac{f(z_n)}{p_2'(z_n) + \lambda_n(z_n - x_n)(z_n - w_n)}, \\ x_{n+1} = u_n - \frac{f(u_n)}{p_3'(u_n) + \beta_n(u_n - w_n)(u_n - x_n)(u_n - z_n)}. \end{cases} \tag{3.31}$$

To demonstrate the convergence order of (3.31), we require this lemma

Lemma 3.2 *If $\gamma_n = -1/N_4'(x_n)$ and $p_n = -N_5''(w_n)/(2N_5'(w_n)), n = 1, 2, \dots$, then the es-*

timates

$$1 + \gamma_n f'(\alpha) \sim e_{n-1,u} e_{n-1,z} e_{n-1,w} e_{n-1}, \tag{3.32}$$

and

$$c_2 + p_n \sim e_{n-1,u} e_{n-1,z} e_{n-1,w} e_{n-1}, \tag{3.33}$$

$$\begin{aligned} \lambda_n + f'(\alpha)c_2(p_n + c_2) - f'(\alpha)c_3 \\ \sim e_{n-1,u} e_{n-1,z} e_{n-1,w} e_{n-1}, \end{aligned} \tag{3.34}$$

$$\begin{aligned} -\beta + c_2(\lambda + c_2 f'(\alpha)(p + c_2) - c_3 f'(\alpha)) \\ + c_4 f'(\alpha) \sim e_{n-1,u} e_{n-1,z} e_{n-1,w} e_{n-1}, \end{aligned} \tag{3.35}$$

hold.

Proof. Similar to Lemma 1 in [15] and Lemma 4 and 6 in [9].

The next theorem shows that the convergence order of the three-step iterative with memory method (3.31).

Theorem 3.2 *If an initial estimation x_0 is close enough to a simple root α of $f(x) = 0$, being f a real sufficiently differentiable function, then the R -order of convergence of the three-step method with memory (3.31) is at least 15.5156.*

Proof. Let $\{x_n\}$ have converged order R . Then, we can write

$$e_{n+1} \sim e_n^R, \quad e_n = x_n - \alpha, \tag{3.36}$$

Hence

$$e_{n+1} \sim e_n^R = (e_{n-1}^R)^R = e_{n-1}^{R^2}. \tag{3.37}$$

Assume sequences $\{w_n\}, \{z_n\}$, and $\{u_n\}$ have converged p, q , and s , respectively, that is

$$e_{n,w} \sim e_n^p = (e_{n-1}^R)^p = e_{n-1}^{Rp}, \tag{3.38}$$

$$e_{n,z} \sim e_n^q = (e_{n-1}^R)^q = e_{n-1}^{Rq}, \tag{3.39}$$

and

$$e_{n,u} \sim e_n^s = (e_{n-1}^R)^s = e_{n-1}^{Rs}. \tag{3.40}$$

By (3.38), (3.39), (3.40), and Lemma 3.2, we obtain

$$1 + \gamma_n f'(\alpha) \sim e_{n-1}^{p+q+s+1}, \tag{3.41}$$

$$c_2 + p_n \sim e_{n-1}^{p+q+s+1}. \tag{3.42}$$

Substituting these into $e_{n,w}, e_{n,z}, e_{n,u}$, and e_{n+1} in Theorem 3.2, we have

$$e_{n,w} \sim \left(1 + \gamma_n f'(\alpha)\right) e_n = e_{n-1}^{(1+p+q+s)+R}, \tag{3.43}$$

Table 1: Computational order of convergence of (3.31)

functions	$ x_1 - \alpha $	$ x_2 - \alpha $	$ x_3 - \alpha $	COC
$f_1(x)$	8.2290(-3)	1.0503(-30)	1.4474(-472)	15.848
$f_2(x)$	1.2212(-2)	7.8503(-36)	1.9533(-549)	15.475
$f_3(x)$	3.3663(-3)	5.6416(-46)	1.6379(-713)	15.605
$f_4(x)$	6.2164(-4)	7.3597(-43)	1.3871(-652)	15.664
$f_5(x)$	2.8111(-7)	9.0115(-107)	8.5433(-1644)	15.448

Table 2: Computational order of convergence of (3.15)

functions	$ x_1 - \alpha $	$ x_2 - \alpha $	$ x_3 - \alpha $	COC
$f_1(x)$	1.0831(-2)	1.1281(-13)	2.1218(-99)	7.7936
$f_2(x)$	4.6845(-2)	3.0552(-10)	2.1117(-74)	7.8469
$f_3(x)$	1.1972(-2)	4.8007(-19)	8.2100(-142)	7.4840
$f_4(x)$	1.4378(-3)	7.6488(-12)	4.8261(-85)	8.8502
$f_5(x)$	4.5624(-4)	3.6324(-26)	1.3751(-192)	7.5307

$$e_{n,z} \sim c_2 \left(1 + \gamma_n f'(\alpha)\right) (c_2 + p_n) e_n^2 = e_{n-1}^{2(1+p+q+s)+2R}, \tag{3.44}$$

$$e_{n,u} \sim a_{n,4} \left(1 + \gamma_n f'(\alpha)\right)^2 (c_2 + p_n) (\lambda + f'(\alpha)c_2(p + c_2) - f'(\alpha)c_3) e_n^4 = e_{n-1}^{4(1+p+q+s)+4R}, \tag{3.45}$$

and

$$e_{n+1} \sim a_{n,8} \left(1 + \gamma_n f'(\alpha)\right)^4 (c_2 + p_n)^2 (\lambda + f'(\alpha)c_2(p + c_2) - f'(\alpha)c_3)(-\beta + c_2(\lambda + c_2 f'(\alpha)(p + c_2) - c_3 f'(\alpha)) + c_4 f'(\alpha)) e_n^8 \sim e_{n-1}^{8(1+p+q+s)+8R}. \tag{3.46}$$

Equating the powers of error exponents of e_{n-1} in pairs of relations (3.38)-(3.43), (3.39)-(3.44), (3.40)-(3.45), and (3.37)-(3.46), we have

$$\begin{cases} Rp - R - (p + q + s + 1) = 0, \\ Rq - 2R - 2(p + q + s + 1) = 0, \\ R^s - 4R - 4(p + q + s + 1) = 0, \\ R^2 - 8R - 8(p + q + s + 1) = 0. \end{cases} \tag{3.47}$$

This system has the solution $p = 1.9394$, $q = 3.8789$, $s = 7.7578$, and $R = 15.5156$ which specifies the R-order of convergence of the derivative-free scheme with memory (3.31). \square

4 Numerical Results

Now we show the convergence behavior of developed the with memory methods in action. For

this purpose, ten test problems are chosen along with their initial approximations and the exact zeros in Table 1. The errors $|x_n - \alpha|$ denote approximations to the sought zeros, and $a(-b)$ stands for $a \times 10^{-b}$. Moreover, COC indicates the computational order of convergence [16] and is computed by

$$COC = \frac{\log|f(x_n)/f(x_{n-1})|}{\log|f(x_{n-1})/f(x_{n-2})|}. \tag{4.48}$$

To carry out the numerical results, the package Mathematica 9 with multi-precision arithmetic was used. We have used $\gamma_0 = 0.01$, $p_0 = -1$, $\lambda_0 = 0.1$, $\beta_0 = 5$ for all test problems. In Tables 1 and 2, we have examined some methods with different kinds of convergence order. It is observed that these methods support their theoretical aspects.

$$\left\{ \begin{array}{l} f_1(x) = x \log(x + 1) + e^{x^2+x \cos x-1} \sin x, \\ x_0 = 0.6, \quad \alpha = 0, \\ f_2(x) = e^{x^3-x} - \cos(x^2 - 1) + x^3 + 1, \\ x_0 = -1.65, \quad \alpha = -1, \\ f_3(x) = \frac{1}{2}(e^{x-2} - 1), \\ x_0 = 2.5, \quad \alpha = 2, \\ f_4(x) = (x - 1)(x^{10} + x^3 + 1) \sin x, \\ x_0 = 0.7, \quad \alpha = 1, \\ f_5(x) = e^{x^2-4} + \sin(x - 2) - x^4 + 15, \\ x_0 = 1.67, \quad \alpha = 2. \end{array} \right.$$

5 Conclusion

In this work, we have improved two kinds of optimal without memory methods so that they achieve convergence orders 7.53 and 15.51, respectively, using three and four functional evaluations. In other words, the efficiency indices of the optimal without memory methods have been increased from $4^{\frac{1}{3}} \simeq 1.5874$ and $8^{\frac{1}{4}} \simeq 1.6818$ to $7.53^{\frac{1}{3}} \simeq 1.9600$ and $15.51^{\frac{1}{4}} \simeq 1.9845$, which means that we were able to increase the convergence orders about 93% and 96%, respectively. Studying basin of attractions of the proposed methods can be considered for the future works.

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