Available online at http://ijim.srbiau.ac.ir/
**Int. J. Industrial Mathematics (ISSN 2008-5621)**
Vol. 9, No. 3, 2017 Article ID IJIM-001021, 10 pages
Research Article

# The Solution of Coupled Nonlinear Burgers' Equations Using Interval Finite-difference Method

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Received Date: 2015-12-28 Revised Date: 2016-11-01 Accepted Date: 2017-01-11

**Abstract**

In this paper an coupled Burgers' equation is considered and then a method entitled interval finite-difference method is introduced to find the approximate interval solution of interval model in level wise cases. Finally for more illustration, the convergence theorem is confirmed and a numerical example is solved.

*Keywords* : Interval methods; Finite difference methods; Coupled burgers' equation.

## 1 Introduction

THe partial differential equations have an important role in many scientific fields. One of the models that is a special form of incompressible Navier-Stokes equation without having pressure term and continuity equation is the nonlinear coupled Burgers' equation. The Burgers' equation as an important member of family of partial differential equations is derived from fluid dynamics, and is widely used for various physical applications, such as traffic flow, gas dynamics and shock waves and Coupled Burgers equation is a simple model of physical flows that could be used in many physical fields like interaction between two viscous fluids. Several numerical methods are introduced to solve the differential equations which one of them is interval method [10]. An interval method based on the theory of backward finite difference method to find the approximate solution of one-dimensional heat conduction equation was proposed by Jankowska [7]. Then the first approach to an interval version of Crank-Nicolson method to solve above mentioned equation with Dirichlet boundary conditions was introduced in [9]. A similar method for constructing an interval method to solve a partial differential equation was introduced by Hoffmann [6]. In this research the conventional central-difference method is used with interval method to solve the Poisson equation. An overview of the researches on interval methods for solving the initial and boundary value problems is presented in [10, 12]. Also the following researchers have been published about the related interval arithmetic models [1, 2, 8, 4, 5, 13, 14, 11]. In this paper we will consider one dimensional interval coupled nonlinear Burgers' equations (1.1), (1.2)

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in generalized form:

$$\frac{\partial u}{\partial t}+\delta\frac{\partial^2 u}{\partial x^2}+\eta u\frac{\partial u}{\partial x}+\alpha\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)=0, \tag{1.1}$$

$$\frac{\partial v}{\partial t}+\mu\frac{\partial^2 v}{\partial x^2}+\xi v\frac{\partial v}{\partial x}+\beta\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)=0, \tag{1.2}$$

subject to the initial conditions ($x \in \Omega$):

$$\begin{aligned} u(x,0)&=[\underline{a}_1(x),\bar{a}_1(x)],\\ v(x,0)&=[\underline{a}_2(x),\bar{a}_2(x)],\end{aligned} \tag{1.3}$$

and interval Dirichlet boundary conditions ($t > 0$):

$$\begin{aligned} u(x,t)&=[\underline{b}_1(x,t),\bar{b}_1(x,t)],\\ v(x,t)&=[\underline{b}_2(x,t),\bar{b}_2(x,t)], \quad x\in\Omega,\end{aligned} \tag{1.4}$$

where $\Omega = \{x : c \leq x \leq d\}$ is the computational domain, $\delta, \mu, \eta$ and $\xi$ are real constants, and $\alpha$ and $\beta$ are arbitrary constants depending on the system parameters. The interval functions $u(x,t)$ and $v(x,t)$ are undetermined velocity components that should be determined and their uncertainty are related to $\underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2, \bar{a}_1, \bar{a}_2, \bar{b}_1$ and $\bar{b}_2$. It is clear that the known function $\frac{\partial u}{\partial t}$ is unsteady interval term, $u\frac{\partial u}{\partial x}$ is the nonlinear convection interval term, $\frac{\partial^2 u}{\partial x^2}$ is the diffusion interval term and its uncertainty should be considered based on two types of interval differentiability. The rest of the paper is organized as follows. In section 2 some concepts, definitions, theorems and lemmas are mentioned that are used in this study. In section 3, main research of the paper is introduced. In section 4 an example is solved and the subject ends by conclusion in section 5. .

## 2 Preliminaries

All the following required definitions in this section are referred to [3, 15].

**Definition 2.1** *An interval number(IN) $U$ is defined as the set of real numbers such that*

$$U=[\underline{u},\bar{u}]=\{u' \in : \underline{u}\leq u' \leq \bar{u}\}. \tag{2.5}$$

**Definition 2.2** *We define distance between two interval numbers $U = [\underline{u}, \bar{u}]$ and $V = [\underline{v}, \bar{v}]$ as:*

$$d(U,V)=\sqrt{\frac{(\underline{u}-\underline{v})^2+(\bar{u}-\bar{v})^2}{2}} \tag{2.6}$$

*Indeed it is a modified version of Euclidean distance between two interval numbers. We know that the Euclidean distance on the interval numbers is as follow:*

$$d_E(U,V)=\sqrt{(\underline{u}-\underline{v})^2+(\bar{u}-\bar{v})^2}$$

*Obviously, if interval numbers $U = [\underline{u}, \bar{u}]$ and $V = [\underline{v}, \bar{v}]$ are real numbers, i.e., $u = \underline{u} = \bar{u}$ and $v = \underline{v} = \bar{v}$, then we conclude $d_E(U,V) \neq |u - v|$ whereas $d(U,V) = |u - v|$.*

Therefore, the function $d(\cdot,\cdot)$ preserve the traditional distance in real space. Whereas the function has $d_E(.,.)$ not such property, we call it "modified Euclidean".

**Definition 2.3** *The distance between two interval number vectors $U = (u_1, u_2, \ldots, u_n)^T$ and $V = (v_1, v_2, \ldots, v_n)^T$ is as follow:*

$$D(U,V)=\max_{1\leq i\leq n} d(U_i,V_i), \tag{2.7}$$

*where the function $d(\cdot,\cdot)$ is defined in Definition 2.2.*

**Definition 2.4** *The $n\times n$ linear system*

$$\begin{cases} a_{11}U_1+a_{12}U_2+ & \ldots + a_{1n}U_n = V_1,\\ a_{21}U_1+a_{22}U_2+ & \ldots + a_{2n}U_n = V_2,\\ \vdots & \vdots\\ a_{n1}U_1+a_{n2}U_2+ & \ldots + a_{nn}U_n = V_n,\end{cases} \tag{2.8}$$

*where the coefficient matrix $A = (a_{ij})_{n\times n}$ is an $n \times n$ real valued matrix and $V_i = [\underline{v}_i, \bar{v}_i]$, $1 \leq i \leq n$ are interval numbers is called an interval linear system (ILS). We denote the ILS in compact form as*

$$AU=V, \tag{2.9}$$

*where $U = (u_1, u_2, \ldots, u_n)^T$ and $V = (v_1, v_2, \ldots, v_n)^T$ are the interval number vectors.*

Let $\mathrm{conv}(R)$ be a space of all nonempty closed intervals $U = [u_1, u_2] \subset R$ with the following Hausdorff metric:

$$h([u_1,u_2],[v_1,v_2])=\max\{|v_1,u_1|,|v_2,u_2|\}.$$

**Definition 2.5** *Let $U, V \in IN$, an interval number $Z$ such that $U = V + Z$ is called a Hukuhara difference of the intervals $U$ and $V$ is denoted by $U \overset{h}{-} V$.*

**Lemma 2.1** *Let diameter of $U \in IN$ is $(U) = \bar{u} - \underline{u}$. The Hukuhara difference of the sets $U = [\underline{u}, \bar{u}]$ and $V = [\underline{v}, \bar{v}]$ exists iff $(U) \geq (V)$ and is equal to*
$[\underline{u} - \underline{v}, \bar{u} - \bar{v}]$.

**Proof.**

$$\begin{aligned}
(U) &= \bar{u} - \underline{u}, (V) = \bar{v} - \underline{v} \\
(U) &\geq (V) \\
\bar{u} - \underline{u} &\geq \bar{v} - \underline{v} \\
-\underline{u} + \underline{v} &\geq -\bar{u} + \bar{v} \\
-(-\underline{u} + \underline{v} &\geq -\bar{u} + \bar{v}) \\
\underline{u} - \underline{v} &\leq \bar{u} - \bar{v} \\
&\underline{z} \leq \bar{z} \Rightarrow [\underline{z}, \bar{z}] \Rightarrow [\underline{u} - \underline{v}, \bar{u} - \bar{v}] \\
&\Rightarrow \underline{u} = \underline{v} + \underline{z}, \bar{u} = \bar{v} + \bar{z}
\end{aligned}$$

Conversely:

$$\begin{aligned}
Z = U \overset{h}{-} V = [\underline{u} - \underline{v}, \bar{u} - \bar{v}] &= [\underline{z}, \bar{z}] \\
\underline{z} &\leq \bar{z} \\
\underline{u} - \underline{v} &\leq \bar{u} - \bar{v} \\
\underline{u} - \bar{u} &\leq \underline{v} - \bar{v} \\
-(\underline{u} - \bar{u} &\leq \underline{v} - \bar{v}) \\
\bar{u} - \underline{u} &\geq \bar{v} - \underline{v} \\
U &\geq V
\end{aligned}$$

Let $U : I \to \text{conv}(R)$ be an interval-valued mapping; $(t_0 - \Delta, t_0 + \Delta) \subset I$ be a $\Delta$-neighborhood of a point $t_0 \in I; \Delta > 0$. For any $t \in (t_0 - \Delta, t_0 + \Delta)$ consider the following Hukuhara differences if these differences exist.

$$U(t) \overset{h}{-} U(t_0), t \geq t_0, \tag{2.10}$$

$$U(t_0) \overset{h}{-} U(t), t \geq t_0, \tag{2.11}$$

$$U(t_0) \overset{h}{-} U(t), t \leq t_0, \tag{2.12}$$

$$U(t) \overset{h}{-} U(t_0), t \leq t_0, \tag{2.13}$$

The differences (2.10) and (2.11) [(2.12) and (2.13)] are called the right [left] differences. From the definition of the Hukuhara difference it follows that both one-sided differences exist only in the case when $U(t) \equiv F + \{f(t)\}$ for $t \in [t_0, t_0 + \Delta)$ or $t \in (t_0 - \Delta, t_0]$. If all differences (2.10)-(2.13) exist, then $U(t) \equiv F + \{f(t)\}$ in $\Delta$-neighborhood of the point $t_0$. If for all $t \in (t_0 - \Delta, t_0 + \Delta)$ there exists only one of the one-sided differences, then using the properties of the Hukuhara difference, we get that the mapping diam $U : I \to R_+$ in the $\Delta$-neighborhood of the point $t_0$ can be:

[a)]non-decreasing on $(t_0 - \Delta, t_0 + \Delta)$; non-increasing on $(t_0 - \Delta, t_0 + \Delta)$; non-decreasing on $t \in (t_0 - \Delta, t_0)$ and non-increasing on $(t_0, t_0 + \Delta)$; non-increasing on $t \in (t_0 - \Delta, t_0)$ and non-decreasing on $(t_0, t_0 + \Delta)$;

Hence, for each of the above mentioned cases only one of combinations of differences is possible:

1. (2.10) and (2.12);
2. (2.11) and (2.13);
3. (2.11) and (2.12);
4. (2.10) and (2.13).

Consider four types of limits corresponding to one of the difference types:

$$\lim_{t \to t_0} \frac{1}{t - t_0} (U(t) \overset{h}{-} U(t_0)) \tag{2.14}$$

$$\lim_{t \to t_0} \frac{1}{t - t_0} (U(t_0) \overset{h}{-} U(t)) \tag{2.15}$$

$$\lim_{t_0 \to t} \frac{1}{t_0 - t} (U(t_0) \overset{h}{-} U(t)) \tag{2.16}$$

$$\lim_{t_0 \to t} \frac{1}{t_0 - t} (U(t) \overset{h}{-} U(t_0)) \tag{2.17}$$

So it is possible to say that in the point $t_0$ not more than two limits can exist (as we assumed that there exist only two of four Hukuhara differences). Considering all above, only following combinations of limits exist:

1. (2.14) and (2.16);
2. (2.15) and (2.17);
3. (2.15) and (2.16);
4. (2.14) and (2.17).

**Definition 2.6** *The generalized Hukuhara difference of two interval numbers $U$ and $V$ is defined as follows:*

$$\textit{I- type}: \begin{array}{l} Z = U - V \\ [\underline{z}, \bar{z}] = [\underline{u} - \underline{v}, \bar{u} - \bar{v}] \end{array}$$

$$\textit{II- type}: \begin{array}{l} -Z = V - U \\ [\underline{z}, \bar{z}] = [\bar{u} - \bar{v}, \underline{u} - \underline{v}] \end{array}$$

## 3 Numerical Scheme (II-FDM)

In this section, we illustrate the interval implicit finite-difference method (II-FDM). To this end suppose $u_i^n$ and $v_i^n$ denote the discrete approximations of $u(x,t)$ and $v(x,t)$, respectively, at the grid point $(i\Delta x, n\Delta t)$ for $i = 0,1,2,\dots,n_x$, $n = 0,1,2,\dots$, $\Delta x = 1/n_x$, $\Delta x$ is the grid size in x-direction, and $\Delta t$ represents the time step. In this method we will approximate the derivatives using forward and central differences as follow:

$$\begin{cases} u_t^{n+1} = \left(\dfrac{u_i^{n+1}-u_i^n}{\Delta t}\right), \\ u_x^{n+1} = \left(\dfrac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2\Delta x}\right), \\ u_{xx}^{n+1} = \left(\dfrac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(\Delta x)^2}\right) \\ v_t^{n+1} = \left(\dfrac{v_i^{n+1}-v_i^n}{\Delta t}\right), \\ v_x^{n+1} = \left(\dfrac{v_{i+1}^{n+1}-v_{i-1}^{n+1}}{2\Delta x}\right), \\ v_{xx}^{n+1} = \left(\dfrac{v_{i+1}^{n+1}-2v_i^{n+1}+v_{i-1}^{n+1}}{(\Delta x)^2}\right) \end{cases} \tag{3.18}$$

By substituting equation (3.18) in (1.1), (1.2) we have:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} + \delta\left(\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(\Delta x)^2}\right) + (\eta u_i^{n+1}+\alpha v_i^{n+1})\left(\frac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2\Delta x}\right) + a u_i^{n+1}\left(\frac{v_{i+1}^{n+1}-v_{i-1}^{n+1}}{2\Delta x}\right) = 0 \tag{3.19}$$

$$\frac{v_i^{n+1}-v_i^n}{\Delta t} + \mu\left(\frac{v_{i+1}^{n+1}-2v_i^{n+1}+v_{i-1}^{n+1}}{(\Delta x)^2}\right) + (\xi v_i^{n+1}+\beta u_i^{n+1})\left(\frac{v_{i+1}^{n+1}-v_{i-1}^{n+1}}{2\Delta x}\right) + a v_i^{n+1}\left(\frac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2\Delta x}\right) = 0 \tag{3.20}$$

The nonlinear systems of interval equations obtained from equations (3.19) and (3.20) can be written in the form:

$$\beta(\omega) = 0 \tag{3.21}$$

Where $\beta = \binom{\beta_1}{\beta_2}, \omega = \binom{\omega_1}{\omega_2}$

$$\begin{pmatrix}\beta_1\omega_1\\ \beta_2\omega_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}, \tag{3.22}$$

$$\begin{aligned} \beta_1 &= (\underline{\beta}_{1\ell}, \underline{\beta}_{2\ell}, \dots, \underline{\beta}_{2n\ell})^T, \\ \beta_2 &= (\bar{\beta}_{1r}, \bar{\beta}_{2r}, \dots, \bar{\beta}_{2nr})^T, \\ \omega_1^{n+1} &= (\underline{u}_1^{n+1}, \underline{v}_1^{n+1}, \underline{u}_2^{n+1}, \underline{v}_2^{n+1}, \dots, \\ &\qquad \underline{u}_{n\ell}^{n+1}, \underline{v}_{n\ell}^{n+1})^T \\ \omega_2^{n+1} &= (\bar{u}_1^{n+1}, \bar{v}_1^{n+1}, \bar{u}_2^{n+1}, \bar{v}_2^{n+1}, \dots, \\ &\qquad \bar{u}_{nr}^{n+1}, \bar{v}_{nr}^{n+1})^T \end{aligned}$$

and

$$\beta = (\underline{\beta}_{1\ell}, \underline{\beta}_{2\ell}, \dots, \underline{\beta}_{2n\ell}, \bar{\beta}_{1r}, \bar{\beta}_{2r}, \dots, \bar{\beta}_{2nr})^T$$

are the nonlinear equations containing interval parameters. With applying the Newton's method on (3.22), the following steps are being taken:

1. Set $\omega^{(0)} = \binom{\omega_1^{(0)}}{\omega_2^{(0)}}$ an initial approximation.
2. While for $k = 0,1,2,\dots$ until convergence do:
   - solve

$$J(\omega^{(k)})\Delta\omega^{(k)} = -\beta(\omega^{(k)}),$$

$$\begin{pmatrix}j_1\\j_2\end{pmatrix}\cdot\begin{pmatrix}\omega_1^{(k)}\\ \omega_2^{(k)}\end{pmatrix}\begin{pmatrix}\Delta\omega_1^{(k)}\\ \Delta\omega_2^{(k)}\end{pmatrix} = -\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\begin{pmatrix}\omega_1^{(k)}\\ \omega_2^{(k)}\end{pmatrix},$$

$$\begin{cases} j_1\omega_1^{(k)}\cdot\Delta\omega_1^{(k)} = -\beta_1\omega_1^{(k)} \\ j_2\omega_2^{(k)}\cdot\Delta\omega_2^{(k)} = -\beta_2\omega_2^{(k)} \end{cases}$$

   - set $\omega^{(k+1)} = \omega^{(k)} + \Delta\omega^{(k)}$,

$$\begin{pmatrix}\omega_1^{(k+1)}\\ \omega_2^{(k+1)}\end{pmatrix} = \begin{pmatrix}\omega_1^{(k)}\\ \omega_2^{(k)}\end{pmatrix} + \begin{pmatrix}\Delta\omega_1^{(k)}\\ \Delta\omega_2^{(k)}\end{pmatrix},$$

$$\begin{cases} \omega_1^{(k+1)} = \omega_1^{(k)} + \Delta\omega_1^{(k)} \\ \omega_2^{(k+1)} = \omega_2^{(k)} + \Delta\omega_2^{(k)} \end{cases}$$

In general form the Jacobian matrix $J(\omega^{(k)}) = \binom{j_1(\omega_1^{(k)})}{j_2(\omega_2^{(k)})}$ is as follows:

$$\begin{bmatrix} A_{2n\ell} & B_{2n\ell} \\ C_{2nr} & D_{2nr} \end{bmatrix}$$

where $A$ and $B$ are $2nl$ square matrix and $C$, $D$ are $2nr$ square matrix and $\Delta(\omega^{(k)})$ is the correction vector. Newton's iteration at each time-step is stopped when $\|\beta(\omega^{(k)})\| \le \xi$. Considering different kinds of differentiability in interval arithmetic, we have the following theorems. In the all cases two types of differentiability of $u_t, u_x, u_{xx}, v_x, v_t$ and $v_{xx}$ are considered. . Then using definitions of finite difference methods based on forward and central differences, we have many cases of FDMs (many cases) where in this research only four cases of them are considered and the others are similar.

**Theorem 3.1** *Let $u, v \in IN, (u = [\underline{u}, \bar{u}], v = [\underline{v}, \bar{v}])$ and suppose that $u_t, u_x, u_{xx}, v_x, v_t, v_{xx}$ are I-type differentiable. Then the equations (1.1), (1.2) convert to:*

$$\begin{cases} \underline{u}_t + \delta \underline{u}_{xx} + (\eta \underline{u} + \alpha \underline{v})\underline{u}_x + \alpha \underline{u}\ \underline{v}_x = 0 \\ \underline{v}_t + \mu \underline{v}_{xx} + (\xi \underline{u} + \beta \underline{v})\underline{v}_x + \alpha \underline{v}\ \underline{u}_x = 0 \\ \bar{u}_t + \delta \bar{u}_{xx} + (\eta \bar{u} + \alpha \bar{v})\bar{u}_x + \alpha \bar{u}\bar{v}_x = 0 \\ \bar{v}_t + \mu \bar{v}_{xx} + (\xi \bar{u} + \beta \bar{v})\bar{v}_x + \alpha \bar{v}\bar{u}_x = 0 \end{cases}$$

**Proof.** Lets suppose that $u_t, u_x, u_{xx}, v_x, v_t$ and $v_{xx}$ are I-type differentiable, then:

$$\begin{aligned} &u = [\underline{u}, \bar{u}], u_t = [\underline{u}_t, \bar{u}_t], u_x = [\underline{u}_x, \bar{u}_x], \\ &u_{xx} = [\underline{u}_{xx}, \bar{u}_{xx}], v = [\underline{v}, \bar{v}], v_t = [\underline{v}_t, \bar{v}_t], \\ &v_x = [\underline{v}_x, \bar{v}_x] \quad \text{and} \quad v_{xx} = [\underline{v}_{xx}, \bar{v}_{xx}]. \end{aligned}$$

By substituting in equations (1.1), (1.2) we have:
Lower:

$$\begin{aligned} &\underline{u}_t + \delta \underline{u}_{xx} + (\eta \underline{u} + \alpha \underline{v})\underline{u}_x + \alpha \underline{u}\ \underline{v}_x = 0 \\ &\underline{v}_t + \mu \underline{v}_{xx} + (\xi \underline{u} + \beta \underline{v})\underline{v}_x + \alpha \underline{v}\ \underline{u}_x = 0 \end{aligned}$$

Upper:

$$\begin{aligned} &\bar{u}_t + \delta \bar{u}_{xx} + (\eta \bar{u} + \alpha \bar{v})\bar{u}_x + \alpha \bar{u}\bar{v}_x = 0 \\ &\bar{v}_t + \mu \bar{v}_{xx} + (\xi \bar{u} + \alpha \bar{v})\bar{v}_x + \alpha \bar{v}\bar{u}_x = 0 \end{aligned}$$

Therefore, the proof of Theorem is completed.

Now using equation (3.18) we have the following results and the finite-differences for the derivatives are given as:

$$\begin{aligned} u_t^{n+1} &= \frac{u_i^{n+1} - u_i^n}{\Delta t} \\ &= \left[ \frac{\underline{u}_i^{n+1} - \underline{u}_i^n}{\Delta t}, \frac{\bar{u}_i^{n+1} - \bar{u}_i^n}{\Delta t} \right] \\ &= [\underline{u}_t^{n+1}, \bar{u}_t^{n+1}], \end{aligned}$$

$$\begin{aligned} u_x^{n+1} &= \left( \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} \right) \\ &= \left[ \left( \frac{\underline{u}_{i+1}^{n+1} - \underline{u}_{i-1}^{n+1}}{2\Delta x} \right), \left( \frac{\bar{u}_{i+1}^{n+1} - \bar{u}_{i-1}^{n+1}}{2\Delta x} \right) \right] \\ &= [\underline{u}_x^{n+1}, \bar{u}_x^{n+1}] \end{aligned}$$

$$\begin{aligned} u_{xx}^{n+1} &= \left( \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} \right) \\ &= \left[ \left( \frac{\underline{u}_{i+1}^{n+1} - 2\underline{u}_i^{n+1} + \underline{u}_{i-1}^{n+1}}{(\Delta x)^2} \right), \right. \\ &\quad \left. \left( \frac{\bar{u}_{i+1}^{n+1} - 2\bar{u}_i^{n+1} + \bar{u}_{i-1}^{n+1}}{(\Delta x)^2} \right) \right] \\ &= [\underline{u}_{xx}^{n+1}, \bar{u}_{xx}^{n+1}] \end{aligned}$$

$$\begin{aligned} v_t^{n+1} &= \frac{v_i^{n+1} - v_i^n}{\Delta t} \\ &= \left[ \left( \frac{\underline{v}_i^{n+1} - \underline{v}_i^n}{\Delta t} \right), \left( \frac{\bar{v}_i^{n+1} - \bar{v}_i^n}{\Delta t} \right) \right] \\ &= [\underline{v}_t^{n+1}, \bar{v}_t^{n+1}] \end{aligned}$$

$$\begin{aligned} v_x^{n+1} &= \left( \frac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{2\Delta x} \right) \\ &= \left[ \left( \frac{\underline{v}_{i+1}^{n+1} - \underline{v}_{i-1}^{n+1}}{2\Delta x} \right), \left( \frac{\bar{v}_{i+1}^{n+1} - \bar{v}_{i-1}^{n+1}}{2\Delta x} \right) \right] \\ &= [\underline{v}_x^{n+1}, \bar{v}_x^{n+1}] \end{aligned}$$

$$\begin{aligned} v_{xx}^{n+1} &= \left( \frac{v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}}{(\Delta x)^2} \right) \\ &= \left[ \left( \frac{\underline{v}_{i+1}^{n+1} - 2\underline{v}_i^{n+1} + \underline{v}_{i-1}^{n+1}}{(\Delta x)^2} \right), \right. \\ &\quad \left. \left( \frac{\bar{v}_{i+1}^{n+1} - 2\bar{v}_{i-1}^{n+1}}{(\Delta x)^2} \right) \right] \\ &= [\underline{v}_{xx}^{n+1}, \bar{v}_{xx}^{n+1}] \end{aligned}$$

Lower:

$$\frac{\underline{u}_i^{n+1}-\underline{u}_i^n}{\Delta t}+\delta\left(\frac{\underline{u}_{i+1}^{n+1}-2\underline{u}_i^{n+1}+\underline{u}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\eta\underline{u}_i^{n+1}+\alpha\underline{v}_i^{n+1})\left(\frac{\underline{u}_{i+1}^{n+1}-\underline{u}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\underline{u}_i^{n+1}\left(\frac{\underline{v}_{i+1}^{n+1}-\underline{v}_{i-1}^{n+1}}{2\Delta x}\right)=0$$
$$\frac{\underline{v}_i^{n+1}-\underline{v}_i^n}{\Delta t}+\mu\left(\frac{\underline{v}_{i+1}^{n+1}-2\underline{v}_i^{n+1}+\underline{v}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\xi\underline{v}_i^{n+1}+\beta\underline{u}_i^{n+1})\left(\frac{\underline{v}_{i+1}^{n+1}-\underline{v}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\underline{v}_i^{n+1}\left(\frac{\underline{u}_{i+1}^{n+1}-\underline{u}_{i-1}^{n+1}}{2\Delta x}\right)=0$$

Upper:

$$\frac{\bar{u}_i^{n+1}-\bar{u}_i^n}{\Delta t}+\delta\left(\frac{\bar{u}_{i+1}^{n+1}-2\bar{u}_i^{n+1}+\bar{u}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\eta\bar{u}_i^{n+1}+\alpha\bar{v}_i^{n+1})\left(\frac{\bar{u}_{i+1}^{n+1}-\bar{u}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\bar{u}_i^{n+1}\left(\frac{\bar{v}_{i+1}^{n+1}-\bar{v}_{i-1}^{n+1}}{2\Delta x}\right)=0$$
$$\frac{\bar{v}_i^{n+1}-\bar{v}_i^n}{\Delta t}+\mu\left(\frac{\bar{v}_{i+1}^{n+1}-2\bar{v}_i^{n+1}+\bar{v}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\xi\bar{v}_i^{n+1}+\beta\bar{u}_i^{n+1})\left(\frac{\bar{v}_{i+1}^{n+1}-\bar{v}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\bar{v}_i^{n+1}\left(\frac{\bar{u}_{i+1}^{n+1}-\bar{u}_{i-1}^{n+1}}{2\Delta x}\right)=0$$

Now case 2:

**Theorem 3.2** *Let* $u,v \in IN, (u = [\underline{u},\bar{u}], v = [\underline{v},\bar{v}])$ *and let's suppose that* $u_t, u_x, u_{xx}, v_x, v_t$ *and* $v_{xx}$ *are II-type differentiable. Then the equations (1.1), (1.2) convert to:*

$$\begin{cases}\bar{u}_t+\delta\bar{u}_{xx}+(\eta\underline{u}+\alpha\underline{v})\bar{u}_x+\alpha\underline{u}\bar{v}_x=0\\ \bar{v}_t+\mu\bar{v}_{xx}+(\xi\underline{u}+\beta\underline{v})\bar{v}_x+\alpha\underline{v}\bar{u}_x=0\\ \underline{u}_t+\delta\underline{u}_{xx}+(\eta\bar{u}+\alpha\bar{v})\underline{u}_x+\alpha\bar{u}\ \underline{v}_x=0\\ \underline{v}_t+\mu\underline{v}_{xx}+(\xi\bar{u}+\beta\bar{v})\underline{v}_x+\alpha\bar{v}\ \underline{u}_x=0\end{cases}$$

**Proof.** Lets suppose that $u_t, u_x, u_{xx}, v_x, v_t$ and $v_{xx}$ are II-type differentiable, then:

$$u=[\underline{u},\bar{u}], u_t=[\bar{u}_t,\underline{u}_t], u_x=[\bar{u}_x,\underline{u}_x],$$
$$u_{xx}=[\bar{u}_{xx},\underline{u}_{xx}], v=[\underline{v},\bar{v}], v_t=[\bar{v}_t,\underline{v}_t],$$
$$v_x=[\bar{v}_x,\underline{v}_x] \quad \text{and} \quad v_{xx}=[\bar{v}_{xx},\underline{v}_{xx}].$$

For this case suppose that $u_t, u_x, u_{xx}, v_x, v_t$ and $v_{xx}$ are II-type differentiable. In this case the lower and upper forms of the equations (1.1), (1.2) are as follows:

Lower:

$$\bar{u}_t+\delta\bar{u}_{xx}+(\eta\underline{u}+\alpha\underline{v})\bar{u}_x+\alpha\underline{u}\bar{v}_x=0$$
$$\bar{v}_t+\mu\bar{v}_{xx}+(\xi\underline{u}+\beta\underline{v})\bar{v}_x+\alpha\underline{v}\bar{u}_x=0$$

Upper:

$$\underline{u}_t+\delta\underline{u}_{xx}+(\eta\bar{u}+\alpha\bar{v})\underline{u}_x+\alpha\bar{u}\ \underline{v}_x=0$$
$$\underline{v}_t+\mu\underline{v}_{xx}+(\xi\bar{u}+\beta\bar{v})\underline{v}_x+\alpha\bar{v}\ \underline{u}_x=0$$

Therefore, the proof of Theorem is completed.

Now again using equation (3.18) we have the following results:

$$u_t^{n+1}=[\bar{u}_t^{n+1},\underline{u}_t^{n+1}], u_x^{n+1}=[\bar{u}_x^{n+1},\underline{u}_x^{n+1}],$$
$$u_{xx}^{n+1}=[\bar{u}_{xx}^{n+1},\underline{u}_{xx}^{n+1}], v=[\underline{v},\bar{v}],$$
$$v_t^{n+1}=[\bar{v}_t^{n+1},\underline{v}_t^{n+1}], v_x^{n+1}=[\bar{v}_x^{n+1},\underline{v}_x^{n+1}]$$
$$\text{and} \quad v_{xx}^{n+1}=[\bar{v}_{xx}^{n+1},\underline{v}_{xx}^{n+1}].$$

Lower:

$$\frac{\bar{u}_i^{n+1}-\bar{u}_i^n}{\Delta t}+\delta\left(\frac{\bar{u}_{i+1}^{n+1}-2\bar{u}_i^{n+1}+\bar{u}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\eta\underline{u}_i^{n+1}+\alpha\underline{v}_i^{n+1})\left(\frac{\bar{u}_{i+1}^{n+1}-\bar{u}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\underline{u}_i^{n+1}\left(\frac{\bar{v}_{i+1}^{n+1}-\bar{v}_{i-1}^{n+1}}{2\Delta x}\right)=0$$
$$\frac{\bar{v}_i^{n+1}-\bar{v}_i^n}{\Delta t}+\mu\left(\frac{\bar{v}_{i+1}^{n+1}-2\bar{v}_i^{n+1}+\bar{v}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\xi\underline{v}_i^{n+1}+\beta\underline{u}_i^{n+1})\left(\frac{\bar{v}_{i+1}^{n+1}-\bar{v}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\underline{v}_i^{n+1}\left(\frac{\bar{u}_{i+1}^{n+1}-\bar{u}_{i-1}^{n+1}}{2\Delta x}\right)=0$$

Upper:

$$\frac{\underline{u}_i^{n+1}-\underline{u}_i^n}{\Delta t}+\delta\left(\frac{\underline{u}_{i+1}^{n+1}-2\underline{u}_i^{n+1}+\underline{u}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\eta\bar{u}_i^{n+1}+\alpha\bar{v}_i^{n+1})\left(\frac{\underline{u}_{i+1}^{n+1}-\underline{u}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\bar{u}_i^{n+1}\left(\frac{\underline{v}_{i+1}^{n+1}-\underline{v}_{i-1}^{n+1}}{2\Delta x}\right)=0$$
$$\frac{\underline{v}_i^{n+1}-\underline{v}_i^n}{\Delta t}+\mu\left(\frac{\underline{v}_{i+1}^{n+1}-2\underline{v}_i^{n+1}+\underline{v}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\xi\bar{v}_i^{n+1}+\beta\bar{u}_i^{n+1})\left(\frac{\underline{v}_{i+1}^{n+1}-\underline{v}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\bar{v}_i^{n+1}\left(\frac{\underline{u}_{i+1}^{n+1}-\underline{u}_{i-1}^{n+1}}{2\Delta x}\right)=0$$

case3

**Theorem 3.3** *Let $u, v \in IN, (u = [\underline{u}, \bar{u}], v = [\underline{v}, \bar{v}])$ and suppose that $u_x$ is I-type differentiable and the others are II-type differentiable. Then the equations (1.1), (1.2) convert to:*

$$\begin{cases} \bar{u}_t+\delta\bar{u}_{xx}+(\eta\underline{u}+\alpha\underline{v})\underline{u}_x+\alpha\underline{u}\ \bar{v}_x=0 \\ \bar{v}_t+\mu\bar{v}_{xx}+(\xi\underline{u}+\beta\underline{v})\bar{v}_x+\alpha\underline{v}\ \underline{u}_x=0 \\ \underline{u}+\delta\underline{u}_{xx}+(\eta\bar{u}+\alpha\bar{v})\underline{u}_x+\alpha\bar{u}\underline{v}_x=0 \\ \underline{v}_t+\mu\underline{v}_{xx}+(\xi\bar{u}+\beta\bar{v})\underline{v}_x+\alpha\bar{v}\underline{u}_x=0 \end{cases}$$

**Proof.** Suppose that $u_x$ is I-type differentiable and the others are II-type differentiable, then:

$$u=[\underline{u},\bar{u}], u_t=[\bar{u}_t,\underline{u}_t], u_x=[\underline{u}_x,\bar{u}_x],$$
$$u_{xx}=[\bar{u}_{xx},\underline{u}_{xx}], v=[\underline{v},\bar{v}], v_t=[\bar{v}_t,\underline{v}_t],$$
$$v_x=[\bar{v}_x,\underline{v}_x] \quad \text{and} \quad v_{xx}=[\bar{v}_{xx},\underline{v}_{xx}].$$

In this case we have a system of equations where in each equation two endpoints are appeared.

Lower:

$$\bar{u}_t+\delta\bar{u}_{xx}+(\eta\underline{u}+\alpha\underline{v})\underline{u}_x+\alpha\underline{u}\ \bar{v}_x=0$$
$$\bar{v}_t+\mu\bar{v}_{xx}+(\xi\underline{u}+\beta\underline{v})\bar{v}_x+\alpha\underline{v}\ \underline{u}_x=0$$

Upper:

$$\underline{u}+\delta\underline{u}_{xx}+(\eta\bar{u}+\alpha\bar{v})\underline{u}_x+\alpha\bar{u}\underline{v}_x=0$$
$$\underline{v}_t+\mu\underline{v}_{xx}+(\xi\bar{u}+\beta\bar{v})\underline{v}_x+\alpha\bar{v}\underline{u}_x=0$$

Therefore, the proof of Theorem is completed.

By approximation of the derivatives using equation (3.18) we have : Lower:

$$\frac{\bar{u}_i^{n+1}-\bar{u}_i^n}{\Delta t}+\delta\left(\frac{\bar{u}_{i+1}^{n+1}-2\bar{u}_i^{n+1}+\bar{u}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\eta\underline{u}_i^{n+1}+\alpha\underline{v}_i^{n+1})\left(\frac{\underline{u}_{i+1}^{n+1}-\underline{u}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\underline{u}_i^{n+1}\left(\frac{\bar{v}_{i+1}^{n+1}-\bar{v}_{i-1}^{n+1}}{2\Delta x}\right)=0$$
$$\frac{\bar{v}_i^{n+1}-\bar{v}_i^n}{\Delta t}+\mu\left(\frac{\bar{v}_{i+1}^{n+1}-2\bar{v}_i^{n+1}+\bar{v}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\xi\underline{v}_i^{n+1}+\beta\underline{u}_i^{n+1})\left(\frac{\bar{v}_{i+1}^{n+1}-\bar{v}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\underline{v}_i^{n+1}\left(\frac{\underline{u}_{i+1}^{n+1}-\underline{u}_{i-1}^{n+1}}{2\Delta x}\right)=0$$

Upper:

$$\frac{\underline{u}_i^{n+1}-\underline{u}_i^n}{\Delta t}+\delta\left(\frac{\underline{u}_{i+1}^{n+1}-2\underline{u}_i^{n+1}+\underline{u}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\eta\bar{u}_i^{n+1}+\alpha\bar{v}_i^{n+1})\left(\frac{\bar{u}_{i+1}^{n+1}-\bar{u}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\bar{u}_i^{n+1}\left(\frac{\underline{v}_{i+1}^{n+1}-\underline{v}_{i-1}^{n+1}}{2\Delta x}\right)=0$$
$$\frac{\underline{v}_i^{n+1}-\underline{v}_i^n}{\Delta t}+\mu\left(\frac{\underline{v}_{i+1}^{n+1}-2\underline{v}_i^{n+1}+\underline{v}_{i-1}^{n+1}}{(\Delta x)^2}\right)$$
$$+(\xi\bar{v}_i^{n+1}+\beta\bar{u}_i^{n+1})\left(\frac{\underline{v}_{i+1}^{n+1}-\underline{v}_{i-1}^{n+1}}{2\Delta x}\right)$$
$$+\alpha\bar{v}_i^{n+1}\left(\frac{\bar{u}_{i+1}^{n+1}-\bar{u}_{i-1}^{n+1}}{2\Delta x}\right)=0$$

case4

**Theorem 3.4** *Lets $u, v \in IN, (u = [\underline{u}, \bar{u}], v = [\underline{v}, \bar{v}])$ and suppose that $u_t, u_x, u_{xx}$ are I-type differentiable and the others are II-type differentiable. Then the equations (1.1), (1.2) convert to:*

$$\begin{cases} \underline{u}_t+\delta\underline{u}_{xx}+(\eta\underline{u}+\alpha\underline{v})\underline{u}_x+\alpha\underline{u}\ \bar{v}_x=0 \\ \bar{v}_t+\mu\bar{v}_{xx}+(\xi\underline{u}+\beta\underline{v})\bar{v}_x+\alpha\underline{v}\ \underline{u}_x=0 \\ \bar{u}_t+\delta\bar{u}_{xx}+(\eta\bar{u}+\alpha\bar{v})\bar{u}_x+\alpha\bar{u}\underline{v}_x=0 \\ \underline{v}_t+\mu\underline{v}_{xx}+(\xi\bar{u}+\beta\bar{v})\underline{v}_x+\alpha\bar{v}\bar{u}_x=0 \end{cases}$$

**Proof.** Lets suppose that $u_t, u_x, u_{xx}$

are I-type differentiable and the others are II-type differentiable, then:

$$u = [\underline{u}, \bar{u}], u_t = [\underline{u}_t, \bar{u}_t], u_x = [\underline{u}_x, \bar{u}_x],$$
$$u_{xx} = [\underline{u}_{xx}, \bar{u}_{xx}], v = [\underline{v}, \bar{v}], v_t = [\bar{v}_t, \underline{v}_t],$$
$$v_x = [\bar{v}_x, \underline{v}_x] \quad \text{and} \quad v_{xx} = [\bar{v}_{xx}, \underline{v}_{xx}].$$

In this case we have a system of equations where in each equation two endpoints are appeared. Lower:

$$\underline{u}_t + \delta \underline{u}_{xx} + (\eta \underline{u} + \alpha \underline{v})\underline{u}_x + \alpha \underline{u}\ \bar{v}_x = 0$$
$$\bar{v}_t + \mu \bar{v}_{xx} + (\xi \underline{u} + \beta \underline{v})\bar{v}_x + \alpha \underline{v}\ \underline{u}_x = 0$$

Upper:

$$\bar{u}_t + \delta \bar{u}_{xx} + (\eta \bar{u} + \alpha \bar{v})\bar{u}_x + \alpha \bar{u}\underline{v}_x = 0$$
$$\underline{v}_t + \mu \underline{v}_{xx} + (\xi \bar{u} + \beta \bar{v})\underline{v}_x + \alpha \bar{v}\bar{u}_x = 0$$

Therefore, the proof of Theorem is completed.

By approximation of the derivatives using equation (3.18) we have : Lower:

$$\frac{\underline{u}_i^{n+1} - \underline{u}_i^n}{\Delta t} + \delta \left( \frac{\underline{u}_{i+1}^{n+1} - 2\underline{u}_i^{n+1} + \underline{u}_{i-1}^{n+1}}{(\Delta x)^2} \right)$$
$$+ (\eta \underline{u}_i^{n+1} + \alpha \underline{v}_i^{n+1}) \left( \frac{\underline{u}_{i+1}^{n+1} - \underline{u}_{i-1}^{n+1}}{2\Delta x} \right)$$
$$+ \alpha \underline{u}_i^{n+1} \left( \frac{\bar{v}_{i+1}^{n+1} - \bar{v}_{i-1}^{n+1}}{2\Delta x} \right) = 0$$
$$\frac{\bar{v}_i^{n+1} - \bar{v}_i^n}{\Delta t} + \mu \left( \frac{\bar{v}_{i+1}^{n+1} - 2\bar{v}_i^{n+1} + \bar{v}_{i-1}^{n+1}}{(\Delta x)^2} \right)$$
$$+ (\xi \underline{v}_i^{n+1} + \beta \underline{u}_i^{n+1}) \left( \frac{\bar{v}_{i+1}^{n+1} - \bar{v}_{i-1}^{n+1}}{2\Delta x} \right)$$
$$+ \alpha \underline{v}_i^{n+1} \left( \frac{\underline{u}_{i+1}^{n+1} - \underline{u}_{i-1}^{n+1}}{2\Delta x} \right) = 0$$

Upper:

$$\frac{\bar{u}_i^{n+1} - \bar{u}_i^n}{\Delta t} + \delta \left( \frac{\bar{u}_{i+1}^{n+1} - 2\bar{u}_i^{n+1} + \bar{u}_{i-1}^{n+1}}{(\Delta x)^2} \right)$$
$$+ (\eta \bar{u}_i^{n+1} + \alpha \bar{v}_i^{n+1}) \left( \frac{\bar{u}_{i+1}^{n+1} - \bar{u}_{i-1}^{n+1}}{2\Delta x} \right)$$
$$+ \alpha \bar{u}_i^{n+1} \left( \frac{\underline{v}_{i+1}^{n+1} - \underline{v}_{i-1}^{n+1}}{2\Delta x} \right) = 0$$
$$\frac{\underline{v}_i^{n+1} - \underline{v}_i^n}{\Delta t} + \mu \left( \frac{\underline{v}_{i+1}^{n+1} - 2\underline{v}_i^{n+1} + \underline{v}_{i-1}^{n+1}}{(\Delta x)^2} \right)$$
$$+ (\xi \bar{v}_i^{n+1} + \beta \bar{u}_i^{n+1}) \left( \frac{\underline{v}_{i+1}^{n+1} - \underline{v}_{i-1}^{n+1}}{2\Delta x} \right)$$
$$+ \alpha \bar{v}_i^{n+1} \left( \frac{\bar{u}_{i+1}^{n+1} - \bar{u}_{i-1}^{n+1}}{2\Delta x} \right) = 0$$

## 4 Numerical Example

As an example, consider the following coupled Burgers' equation:

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} + 2\left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right) = 0,$$
$$\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} + v\frac{\partial v}{\partial x} + 2\left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right) = 0,$$

Subject to the initial conditions:

$$\begin{aligned} u(x,0) = \Big[ &\min\{(1+0.4(0.2))\sin x, \\ &(1-0.4(0.2))\sin x\}, \\ &\max\{(1+0.4(0.2))\sin x, \\ &(1-0.4(0.2))\sin x\}\Big] \\ v(x,0) = \Big[ &\min\{(1+0.4(0.2))\sin x, \\ &(1-0.4(0.2))\sin x\}, \\ &\max\{(1+0.4(0.2))\sin x, \\ &(1-0.4(0.2))\sin x\}\Big] \end{aligned}$$

To compute the numerical solution, the following parameters are used in this example:

$$\Delta x = 0.0635, \qquad \Delta t = 0.001$$

Due to the symmetry in the problem, $u$ and $v$ have similar graph which is displayed in the following figure.

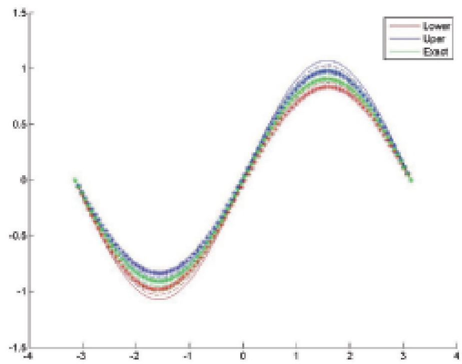


**Figure 1**

## 5 Conclusion

In this research for generalization, an interval difference method is introduced to solve interval coupled Bergers' equation. Based on interval arithmetics the interval difference method is used to transform the coupled equations to a system of interval numbers which is solved using Newton's method. Since the type of differentiability should be considered for the interval derivatives, many kinds of cases are appeared. For simplicity only four cases were considered and discussed. Finally a numerical example shows the accuracy and stability of the proposed method.

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