Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 6, No. 3, 2014 Article ID IJIM-00385, 8 pages
Research Article

Variational iteration method for solving nth-order fuzzy integro-differential equations

O. Sedaghatfar * †, S. Moloudzadeh ‡, P. Darabi §

Abstract

In this paper, the variational iteration method for solving nth-order fuzzy integro-differential equations (nth-FIDE) is proposed. In fact the problem is changed to the system of ordinary fuzzy integro-differential equations and then fuzzy solution of nth-FIDE is obtained. Some examples show the efficiency of the proposed method.

Keywords : Variational iteration method; nth-order fuzzy integro-differential equations (nth-FIDE); The system of ordinary fuzzy integro-differential equations.

1 Introduction

Many authors have been worked about variational iteration method (VIM), see [7, 8, 14, 9] for more details. VIM is an iterative method which used the Lagrange multipliers. Also several modifications of VIM can be found in [3, 4, 6]. Because of facility and easy to use, VIM widely employed to various problems. Very recently Abbasbandy et al. have been considered VIM for solving n-th order fuzzy differential equations [2]. In this manuscript, the VIM is extent to solve nth-FIDE and obtain approximate fuzzy solution.

The VIM is proposed by He [9, 10] as a modification of a general Lagrange multiplier method [11]. To illustrate its basic idea of the technique, we consider following general nonlinear system

$$L[u(t)] + N[u(t)] = g(t),$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a given construct a correction functional for the system, which reads

$$u^{[k+1]}(t) =$$

$$u^{[k]}(t) + \int_a^x \lambda[Lu^{[k]}(s) + N\widetilde{u}^{[k]}(s) - g(s)]ds,$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory [9, 10, 11], the subscript k denotes the nth-order approximation and $\widetilde{u}^{[k]}$ denotes a restricted variation, i.e., $\delta\widetilde{u}^{[k]} = 0$.

The structure of this paper is organized as follows. In Section 2, some basic definitions and notations which will be used are brought. In Section 3, the numerical method to solve nth-FIDE is proposed. In Section 4, convergency of VIM for this system is proved. In Section 5, the application of mentioned method VIM is brought by solving some numerical examples and finally the results are compared with exact solutions. Conclusion is drawn in Section 6.

*Corresponding author. sedaghatfar@yahoo.com
†Department of Mathematics, Shahr-e-rey Branch, Islamic Azad University, shahr-e-rey, Iran.
‡Department of Mathematics, Faculty of Science, University of Duhok, Kurdistan Region, Duhok, Iraq.
§Department of Mathematics, Borujerd Branch, Islamic Azad University, Borujerd, Iran.

2 Basic Definitions and Notations

Definition 2.1 *An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(\alpha), \overline{u}(\alpha))$ for all $\alpha \in [0,1]$, which satisfy the following requirements [5]*

- *$\underline{u}(\alpha)$ is a bounded left continuous nondecreasing function over $[0,1]$;*
- *$\overline{u}(\alpha)$ is a bounded left continuous non-increasing function over $[0,1]$;*
- *$\underline{u}(\alpha) \leq \overline{u}(\alpha), \quad 0 \leq \alpha \leq 1$.*

Remark 2.1 *[1] Let $u(\alpha) = (\underline{u}(\alpha), \overline{u}(\alpha)),\ 0 \leq \alpha \leq 1$ be a fuzzy number, we take*

$$u^c(\alpha) = \frac{\underline{u}(\alpha) + \overline{u}(\alpha)}{2}, \ u^d(\alpha) = \frac{\underline{u}(\alpha) - \overline{u}(\alpha)}{2}.$$

It is clear that $u^d(\alpha) \geq 0$ and $\underline{u}(\alpha) = u^c(\alpha) - u^d(\alpha)$ and $\overline{u}(\alpha) = u^c(\alpha) + u^d(\alpha)$ also a fuzzy number $u \in E$ is said symmetric if $u^c(\alpha)$ is independent of α for all $0 \leq \alpha \leq 1$.

Remark 2.2 *[1] Let $u(\alpha) = (\underline{u}(\alpha), \overline{u}(\alpha))$, $v(\alpha) = (\underline{v}(\alpha), \overline{v}(\alpha))$ and also k, s are arbitrary real numbers. If $w = ku + sv$ then*

$$w^c(\alpha) = ku^c(\alpha) + sv^c(\alpha),$$

$$w^d(\alpha) = |k|u^d(\alpha) + |s|v^d(\alpha).$$

Let E be the set of all upper semi-continuous normal convex fuzzy numbers with bounded α-level intervals. This means that if $\widetilde{v} \in E$ then the α-level set

$$[v]_\alpha = \{s | v(s) \geq \alpha\},$$

is a closed bounded interval which is denoted by $[v]_\alpha = [\underline{v}(\alpha), \overline{v}(\alpha)]$ for $\alpha \in (0,1]$, and $[v]_0 = \overline{\bigcup_{\alpha \in (0,1]} [v]_\alpha}$.

Two fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$ are called equal, $\widetilde{u} = \widetilde{v}$, if $u(s) = v(s)$ for all $s \in \mathbb{R}$ or $[u]_\alpha = [v]_\alpha$ for all $\alpha \in [0,1]$.

Lemma 2.1 *[12] If $\widetilde{u}, \widetilde{v} \in E$, then for $\alpha \in (0,1]$,*

$$[u+v]_\alpha = [\underline{u}(\alpha) + \underline{v}(\alpha), \overline{u}(\alpha) + \overline{v}(\alpha)],$$

$$[u.v]_\alpha = [\min k_\alpha, \max k_\alpha],$$

where

$$k_\alpha = \{\underline{u}(\alpha)\underline{v}(\alpha), \underline{u}(\alpha)\overline{v}(\alpha), \overline{u}(\alpha)\underline{v}(\alpha), \overline{u}(\alpha)\overline{v}(\alpha)\}.$$

Lemma 2.2 *[12] Let $[\underline{v}(\alpha), \overline{v}(\alpha)]$, $\alpha \in (0,1]$, be a given family of non-empty intervals. If*

$$(i) \quad [\underline{v}(\alpha), \overline{v}(\alpha)] \supset [\underline{v}(\beta), \overline{v}(\beta)] \quad for \quad 0 < \alpha \leq \beta,$$

and

$$(ii) \quad [\lim_{k \to \infty} \underline{v}(\alpha_k), \lim_{k \to \infty} \overline{v}(\alpha_k)] = [\underline{v}(\alpha), \overline{v}(\alpha)],$$

whenever (α_k) is a nondecreasing sequence converging to $\alpha \in (0,1]$, then the family $[\underline{v}(\alpha), \overline{v}(\alpha)]$, $0 < \alpha \leq 1$, represent the α-level sets of a fuzzy number v in E. Conversely if $[\underline{v}(\alpha), \overline{v}(\alpha)]$, $0 < \alpha \leq 1$, are the α-level sets of a fuzzy number $\widetilde{v} \in E$, then the conditions (i) and (ii) hold true.

Definition 2.2 *[13] Let I be a real interval. A mapping $\widetilde{v} : I \to E$ is called a fuzzy process and we denote the α-level set by $[v(t)]_\alpha = [\underline{v}(t,\alpha), \overline{v}(t,\alpha)]$. The Seikkala derivative $\widetilde{v}'(t)$ of $\widetilde{v}$ is defined by*

$$[v'(t)]_\alpha = [\underline{v}'(t,\alpha), \overline{v}'(t,\alpha)],$$

provided that is a equation defines a fuzzy number $\widetilde{v}'(t) \in E$.

Definition 2.3 *[13] The fuzzy integral of fuzzy process $\widetilde{v}$, $\int_a^b v(t)dt$ for $a, b \in I$, is defined by*

$$[\int_a^b v(t)dt]_\alpha = [\int_a^b \underline{v}(t,\alpha)dt, \int_a^b \overline{v}(t,\alpha)dt],$$

provided that the Lebesgue integrals on the right exist.

Definition 2.4 *Let $\widetilde{u} = (\underline{u}(\alpha)\ , \overline{u}(\alpha))$, $\widetilde{v} = (\underline{v}(\alpha)\ , \overline{v}(\alpha))$ be fuzzy numbers then the Hausdorff distance between $\widetilde{u}$ and $\widetilde{v}$ is*

$$d_H(\widetilde{u}, \widetilde{v}) =$$

$$sup_{\alpha \in [0,1]} max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\}.$$

3 Variational iteration method

In this section, we are going to investigate solution of nth-FIDE. Let

$$\begin{cases} \widetilde{y}^{(n)}(x) = \widetilde{g}(x) + f(x)\widetilde{y}(x) \\ \qquad + \int_a^b k(x,t)\widetilde{y}^{(m)}(t)dt, \\ \\ \widetilde{y}(a) = \widetilde{\alpha}_0, & a \leq x \leq b, \\ \widetilde{y}'(a) = \widetilde{\alpha}_1, \\ \quad \vdots \\ \widetilde{y}^{(n-1)}(a) = \widetilde{\alpha}_{n-1}, \end{cases} \tag{3.1}$$

where $\widetilde{\alpha}_i$, $i = 0, 1, ..., n-1$ are fuzzy constant numbers, m and n are integers and $m < n$, also $f(x) \geq 0$, $k(x,t)$ are real known functions, and $\widetilde{g}(x)$ is fuzzy known function, too. $\widetilde{y}(x)$ is the solution which to be determined.

Using the following assumptions

$$\widetilde{y} = \widetilde{y}_1,\ \widetilde{y}' = \widetilde{y}_2,\ \widetilde{y}'' = \widetilde{y}_3, ...,\ \widetilde{y}^{(n-1)} = \widetilde{y}_n,$$

then equation (3.1) is transformed to the following fuzzy integro-differential equations

$$\begin{cases} \widetilde{y}_1' = \widetilde{y}_2, \\ \widetilde{y}_2' = \widetilde{y}_3, \\ \widetilde{y}_3' = \widetilde{y}_4, \\ \quad\vdots & a \leq x \leq b, \\ \widetilde{y}_n' = \widetilde{g}(x) + f(x)\widetilde{y}_1(x) \\ \qquad + \int_a^b k(x,t)\widetilde{y}_{m+1}(t)dt, \end{cases} \tag{3.2}$$

with fuzzy initial conditions

$$\widetilde{y}_1(a) = \widetilde{\alpha}_0,\quad \widetilde{y}_2(a) = \widetilde{\alpha}_1, ...,\quad \widetilde{y}_n(a) = \widetilde{\alpha}_{n-1}.$$

Let $(\underline{g}(x;r), \overline{g}(x;r))$,$(\underline{y_1}(x;r), \overline{y_1}(x;r))$,$(\underline{y_2}(x;r), \overline{y_2}(x;r))$,...,$(\underline{y_n}(x;r), \overline{y_n}(x;r))$ for, $0 \leq r \leq 1$ and $a \leq x \leq b$ are parametric form of $\widetilde{g}(x), \widetilde{y}_1(x), \widetilde{y}_2(x), ..., \widetilde{y}_n(x)$, respectively.

Then, parametric form of (3.2) is

$$\begin{cases} \underline{y}_1' = \underline{y}_2, \\ \underline{y}_2' = \underline{y}_3, \\ \underline{y}_3' = \underline{y}_4, \\ \quad\vdots \\ \underline{y}_n' = \underline{g}(x) + f(x)\underline{y}_1(x) \\ \qquad + \int_a^b \underline{k(x,t)y_{m+1}(t)}dt, \\ \\ \overline{y}_1' = \overline{y}_2, \\ \overline{y}_2' = \overline{y}_3, \\ \overline{y}_3' = \overline{y}_4, \\ \quad\vdots \\ \overline{y}_n' = \overline{g}(x) + f(x)\overline{y}_1(x) \\ \qquad + \int_a^b \overline{k(x,t)y_{m+1}(t)}dt, \end{cases} \tag{3.3}$$

where

$$\underline{k(x,t)y_{m+1}(t)}$$

$$= \begin{cases} k(x,t)\underline{y}_{m+1}(t), & k(x,t) \geq 0, \\ k(x,t)\overline{y}_{m+1}(t), & k(x,t) \leq 0, \end{cases}$$

$$\overline{k(x,t)y_{m+1}(t)}$$

$$= \begin{cases} k(x,t)\overline{y}_{m+1}(t), & k(x,t) \geq 0, \\ k(x,t)\underline{y}_{m+1}(t), & k(x,t) \leq 0. \end{cases}$$

To solve this system by VIM the following formulas are obtained:

$$\underline{y}_j^{[k+1]}(x) = \underline{y}_j^{[k]}(x) + \int_a^x \lambda_j(x,t)[\underline{y}_j'^{[k]}(t)$$

$$-\underline{\widetilde{y}}_{j+1}^{[k]}(t)]dt, \qquad j = 1, 2, ..., n-1,$$

$$\underline{y}_n^{[k+1]}(x) = \underline{y}_n^{[k]}(x) + \int_a^x \lambda_n(x,t)[\underline{y}_n'^{[k]}(t)$$

$$-\underline{g}(t) - f(t)\underline{\widetilde{y}}_1^{[k]}(t) - \int_a^b k(t,s)\underline{\widetilde{y}}_{m+1}^{[k]}(s)ds]dt,$$

$$\overline{y}_j^{[k+1]}(x) = \overline{y}_j^{[k]}(x) + \int_a^x \lambda_j(x,t)[\overline{y}_j'^{[k]}(t)$$

$$-\overline{\widetilde{y}}_{j+1}^{[k]}(t)]dt, \qquad j = 1, 2, ..., n-1,$$

$$\overline{y}_n^{[k+1]}(x) = \overline{y}_n^{[k]}(x) + \int_a^x \lambda_n(x,t)[\overline{y}_n'^{[k]}(t)$$

$$-\overline{g}(t) - f(t)\overline{\widetilde{y}}_1^{[k]}(t) - \int_a^b k(t,s)\overline{\widetilde{y}}_{m+1}^{[k]}(s)ds]dt,$$

where $\lambda(x,t)$ is a general Lagrangian multiplier which can be identified optimally via variational theory, $\underline{\widetilde{y}}^{[k]}, \overline{\widetilde{y}}^{[k]}$denote a restricted variation, i.e. $\delta\underline{\widetilde{y}}^{[k]} = \delta\overline{\widetilde{y}}^{[k]} = 0$, and k is iteration step.

The variation is calculated with respect to $\underline{y}_j^{[k]}$ $(j = 1, 2, ..., n)$, respectively, and $\delta\underline{\widetilde{y}}^{[k]} = 0$, then we have

$$\delta\underline{y}_j^{[k+1]}(x) = \delta\underline{y}_j^{[k]}(x) + \delta\int_a^x \underline{\lambda}_j(x,t)[\underline{y}_j'^{[k]}(t)$$

$$-\underline{\widetilde{y}}_{j+1}^{[k]}(t)]dt = \delta\underline{y}_j^{[k]}(x) + \underline{\lambda}_j(x,t)\delta\underline{y}_j^{[k]}(t)|_{t=x}$$

$$-\int_a^x \frac{\partial\underline{\lambda}_j(x,t)}{dt}\ \delta\underline{y}_j^{[k]}(t)dt = (1 + \underline{\lambda}_j(x,x)$$

$$\delta\underline{y}_j^{[k]}(x) + \int_a^x (-\frac{\partial\underline{\lambda}_j(x,t)}{dt})\ \delta\underline{y}_j^{[k]}(t)dt = 0,$$

$$j = 1, 2, ..., n-1,$$

$$\delta\underline{y}_n^{[k+1]}(x) = \delta\underline{y}_n^{[k]}(x) + \delta\int_a^x \underline{\lambda}_n(x,t)[\underline{y}_n'^{[k]}(t)$$

$$-\underline{g}(t) - f(t)\underline{\widetilde{y}}_1^{[k]}(t) - \int_a^b k(t,s)\underline{\widetilde{y}}_{m+1}^{[k]}(s)ds]dt$$

$$= \delta\underline{y}_n^{[k]}(x) + \underline{\lambda}_n(x,t)\delta\underline{y}_n^{[k]}(t)|_{t=x} - \int_a^x \frac{\partial\underline{\lambda}_n(x,t)}{dt}$$

$$\delta\underline{y}_n^{[k]}(t)dt = (1+\underline{\lambda}_n(x,x)\delta\underline{y}_n^{[k]}(x)$$

$$+\int_a^x (-\frac{\partial\underline{\lambda}_n(x,t)}{dt})\delta\underline{y}_n^{[k]}(t)dt = 0.$$

$$For \quad j = 1,2,...,n$$

$$-\frac{\partial\underline{\lambda}_1(x,t)}{\partial t} = -\frac{\partial\underline{\lambda}_2(x,t)}{\partial t} = -\frac{\partial\underline{\lambda}_n(x,t)}{\partial t} = 0,$$

then

$$1+\underline{\lambda}_j(x,x) = 0, \qquad j = 1,2,...,n,$$

and therefor we have

$$\underline{\lambda}_j(x,t) = -1, \qquad j = 1,2,...,n.$$

Similar to above we have

$$\overline{\lambda}_j(x,t) = -1, \qquad j = 1,2,...,n,$$

and we have following iteration formulas

$$\begin{cases} \underline{y}_j^{[k+1]}(x) = \underline{y}_j^{[k]}(x) - \int_a^x [\underline{y}_j'^{[k]}(t) - \underline{\widetilde{y}}_{j+1}^{[k]}(t)]dt, \\ \qquad j = 1,2,...,n-1, \\ \underline{y}_n^{[k+1]}(x) = \underline{y}_n^{[k]}(x) - \int_a^x [\underline{y}_n'^{[k]}(t) - \underline{g}(t) - f(t) \\ \qquad \underline{\widetilde{y}}_1^{[k]}(t) - \int_a^b k(t,s)\underline{\widetilde{y}}_{m+1}^{[k]}(s)ds]dt, \\ \overline{y}_j^{[k+1]}(x) = \overline{y}_j^{[k]}(x) - \int_a^x [\overline{y}_j'^{[k]}(t) - \overline{\widetilde{y}}_{j+1}^{[k]}(t)]dt, \\ \qquad j = 1,2,...,n-1, \\ \overline{y}_n^{[k+1]}(x) = \overline{y}_n^{[k]}(x) - \int_a^x [\overline{y}_n'^{[k]}(t) - \overline{g}(t) - f(t) \\ \qquad \overline{\widetilde{y}}_1^{[k]}(t) - \int_a^b k(t,s)\overline{\widetilde{y}}_{m+1}^{[k]}(s)ds]dt. \end{cases} \tag{3.4}$$

4 Convergence Theorem

In this section we analyze the convergency of VIM for (3.1). Similar to Remark (2.1), let

$$y^c(r) = \frac{\underline{y}(r)+\overline{y}(r)}{2}, \; y^d(r) = \frac{\underline{y}(r)-\overline{y}(r)}{2},$$

then the fuzzy version of (3.1) can be written as

$$\begin{cases} y_j'^c(x;r) = y_{j+1}^c(x;r), \qquad (1 \le j \le n-1) \\ y_n'^c(x;r) = g^c(x) + f(x)y_1^c(x) + \int_a^b k(x,t) \\ \qquad y_{m+1}^c(t)dt, \\ y_j'^d(x;r) = y_{j+1}^d(x;r), \qquad (1 \le j \le n-1) \\ y_n'^d(x;r) = g^d(x) + f(x)y_1^d(x) + \int_a^b k(x,t) \\ \qquad y_{m+1}^d(t)dt, \end{cases} \tag{4.5}$$

and

$$\begin{cases} y_j^c(a;r) = \frac{\underline{y}_j(a;r)+\overline{y}_j(a;r)}{2}, \qquad (1 \le j \le n) \\ y_j^d(a;r) = \frac{\underline{y}_j(a;r)-\overline{y}_j(a;r)}{2}. \end{cases}$$

Similarly from (3.4) we can obtain the following formula

$$\begin{cases} y_j^{[k+1]c}(x,r) = y_j^{[k]c}(x,r) - \int_a^x [y_j'^{[k]c}(t,r) \\ \qquad - y_{j+1}^{[k]c}(t,r)]dt, \quad j = 1,2,...,n-1, \\ y_n^{[k+1]c}(x,r) = y_n^{[k]c}(x) - \int_a^x [y_n'^{[k]c}(t,r) \\ \qquad - g^c(t) - f(t)y_1^{[k]c}(t,r) - \int_a^b k(t,s) \\ \qquad y_{m+1}^{[k]c}(s,r)ds]dt, \\ y_j^{[k+1]d}(x,r) = y_j^{[k]d}(x;r) - \int_a^x [y_j'^{[k]d}(t,r) \\ \qquad - y_{j+1}^{[k]d}(t,r)]dt, \quad j = 1,2,...,n-1, \\ y_n^{[k+1]d}(x,r) = y_n^{[k]d}(x) - \int_a^x [y_n'^{[k]d}(t,r) \\ \qquad - g^d(t) - f(t)y_1^{[k]d}(t,r) - \int_a^b k(t,s) \\ \qquad y_{m+1}^{[k]d}(s,r)ds]dt. \end{cases} \tag{4.6}$$

Let

$$e_j^{[k]c}(x,r) = y_j^{[k]c}(x,r) - y_j^c(x,r),$$

obviously

$$\begin{cases} y_j^c(x,r) = y_j^c(x,r) - \int_a^x [y_j'^c(t,r) \\ \qquad - y_{j+1}^c(t,r)]dt, \quad j = 1,2,...,n-1, \\ y_n^c(x,r) = y_n^c(x,r) - \int_a^x [y_n'^c(t,r) - g(t) \\ -f(t)y_1^c(t,r) - \int_a^b k(t,s)y_{m+1}^c(s,r)ds]dt, \end{cases}$$

then

$$\begin{cases} e_j^{[k+1]c}(x,r) = e_j^{[k]c}(x,r) - \int_a^x [e_j'^{[k]c}(t,r) \\ \qquad - e_{j+1}^{[k]c}(t,r)]dt, \quad j = 1,2,...,n-1, \\ e_n^{[k+1]c}(x,r) = e_n^{[k]c}(x) - \int_a^x [e_n'^{[k]c}(t,r) \\ -f(t)e_1^{[k]c}(t,r) - \int_a^b k(t,s)e_{m+1}^{[k]c}(s,r)ds]dt. \end{cases} \tag{4.7}$$

The Eqs. (4.7) can be written as follow

$$\begin{cases} e_j^{[k+1]c}(x,r) = \int_a^x e_{j+1}^{[k]c}(t,r)dt, \\ \qquad\qquad j=1,2,...,n-1, \\ e_n^{[k+1]c}(x,r) = \int_a^x [f(t)e_1^{[k]c}(t,r) \\ \qquad + \int_a^b k(t,s)e_{m+1}^{[k]c}(s,r)ds]dt. \end{cases}$$

Suppose

$$|e_j^{[k]c}|=\max_{a\le t\le b}|e_j^{[k]c}(t,r)|,$$

$$|e^{[k]c}|=\max_j|e_j^{[k]c}|,$$

$$j=1,2,...,n,\ \ k=0,1,...,$$

and

$$K=\max_{a\le t,s\le b}|k(s,t)|,\ \ F=\max_{a\le t\le x}|F(t)|.$$

Then

$$\begin{cases} |e_j^{[1]c}(x,r)|\le \int_a^x|e_{j+1}^{[0]c}(t,r)|dt\le(x-a)|e^{[0]c}|, \\ \qquad\qquad j=1,2,...,n-1, \\ e_n^{[1]c}(x,r)\le\int_a^x[|f(t)||e_1^{[0]c}(t,r)|+\int_a^b|k(t,s)| \\ |e_{m+1}^{[0]c}(s,r)|ds]dt\le(x-a)|e^{[0]c}|(F+K(b-a)), \end{cases}$$

also

$$\begin{cases} |e_j^{[2]c}(x,r)|\le \int_a^x|e_{j+1}^{[1]c}(t,r)|dt\le\frac{(x-a)^2}{2!}|e^{[0]c}|, \\ \qquad\qquad j=1,2,...,n-1, \\ e_n^{[2]c}(x,r)\le\int_a^x[|f(t)||e_1^{[1]c}(t,r)|+\int_a^b|k(t,s)| \\ |e_{m+1}^{[1]c}(s,r)|ds]dt\le\frac{(x-a)^2}{2!}|e^{[0]c}|(F+K(b-a))^2, \end{cases}$$

and similarly we can obtain

$$\begin{cases} |e_j^{[k]c}(x,r)|\le\frac{(x-a)^k}{k!}|e^{[0]c}|,\quad j=1,2,...,n-1, \\ e_n^{[k]c}(x,r)\le\frac{(x-a)^k}{k!}|e^{[0]c}|(F+K(b-a))^k. \end{cases}$$

Thus

$$\begin{cases} e_j^{[k]c}(x,r)\to 0\ as\ k\to\infty,\ \ j=1,2,...,n-1, \\ e_n^{[k]c}(x,r)\to 0\ as\ k\to\infty. \end{cases} \tag{4.8}$$

In similar way, it can be proven that

$$\begin{cases} e_j^{[k]d}(x,r)\to 0\ as\ k\to\infty,\ \ j=1,2,...,n-1, \\ e_n^{[k]d}(x,r)\to 0\ as\ k\to\infty, \end{cases} \tag{4.9}$$

and (4.8), (4.9) imply the convergency of method.

5 Numerical Examples

In this section, four numerical examples are solved by MATLAB for illustration and the obtained solutions are compared with the exact solutions.

Example 5.1 *Consider the following third-order Fuzzy integro-differential equation*

$$\begin{cases} \widetilde{y}'''(x)=\widetilde{g}(x)+\int_0^1(x+t)\widetilde{y}'(t)dt, \\ \widetilde{g}=(60x^2(r+1)+x(1-3r)+(29/3)r \\ \quad -34/3,-1/6(r-3)(360x^2-6x-5)), \\ \qquad \widetilde{y}(0)=(0,0), \\ \qquad \widetilde{y}'(0)=(0,0), \\ \qquad \widetilde{y}''(0)=(0,0). \end{cases} \tag{5.10}$$

The exact solution for this problem is $\widetilde{y}(x) = ((r+1)x^5+(2r-2)x^3\ ,\ (3-r)x^5)$. See Fig. 1 and Table 1 for comparing the exact solution and obtained solution by the variational iteration method for different k and x.

Example 5.2 *Consider the following second-order Fuzzy integro-differential equation*

$$\begin{cases} \widetilde{y}''(x)=\widetilde{g}(x)+\int_0^1(e^x+e^t)\widetilde{y}(t)dt, \\ \widetilde{g}=(6(r-1)x+(1/4-(7/12)r)e^x \\ \quad +(e-2)r+6-2e,1/3(r-2) \\ \quad (-12+e^x+3e)), \\ \qquad \widetilde{y}(0)=(0,0), \\ \qquad \widetilde{y}'(0)=(0,0). \end{cases} \tag{5.11}$$

The exact solution for this problem is $\widetilde{y}(x) = ((r-1)x^3+rx^2\ ,\ (2-r)x^2)$. See Fig. 2 and Table 2 K for comparing the exact solution and obtained solution by the variational iteration method for different k and x.

Table 1: numerical result for Example 5.1.

$d_H(\widetilde{y}^{(k)}, \widetilde{y}_{exact})$			
x	$k=5$	$k=10$	$k=15$
0.2	5.8611e-004	1.6689e-005	8.0161e-008
0.4	0.0050	1.4173e-004	6.8076e-007
0.6	0.0178	5.0607e-004	2.4308e-006
0.8	0.0444	0.0013	6.0776e-006
1	0.0913	0.0026	1.2487e-005

Table 2: numerical result for Example 5.2.

$d_H(\widetilde{y}^{(k)}, \widetilde{y}_{exact})$			
x	$k=10$	$k=20$	$k=30$
0.2	0.0048	3.1510e-004	1.9237e-005
0.4	0.0198	0.0013	8.2562e-005
0.6	0.0457	0.0030	1.8732e-004
0.8	0.0836	0.0054	3.6093e-004
1	0.1346	0.0088	5.6461e-004

Example 5.3 *Consider the following third-order Fuzzy integro-differential equation*

$$\begin{cases} \widetilde{y}'''(x) = \widetilde{g}(x) + \int_0^1 (x+t)^2 \widetilde{y}'(t)dt, \\ \widetilde{g} = (-1/15(r+1)(15x^2 - 366x + 10), \\ \qquad 1/15(r-3)(15x^2 - 366x + 10)), \\ \widetilde{y}(0) = (0,0), \\ \widetilde{y}'(0) = (0,0), \\ \widetilde{y}''(0) = (0,0). \end{cases} \tag{5.12}$$

The exact solution of this problem is $\widetilde{y}(x) = ((r+1)x^4 \ , \ (3-r)x^4)$. See Fig. 3 and Table 3 for comparing the exact solution and obtained solution by the variational iteration method for different k and x.

Example 5.4 *Consider the following third-order Fuzzy integro-differential equation*

$$\begin{cases} \widetilde{y}'''(x) = \widetilde{g}(x) + \int_0^{\pi/2} (x\cos(t))\widetilde{y}'(t)dt \\ \widetilde{g} = (1/2(r-1)(2\sin(x) + x), 1/2(1-r) \\ \qquad (2\sin(x) + x)) \\ \widetilde{y}(0) = (r-1, 1-r), \\ \widetilde{y}'(0) = (0,0), \\ \widetilde{y}''(0) = (1-r, r-1). \end{cases} \tag{5.13}$$

The exact solution of this problem is $\widetilde{y}(x) = ((r-1)\cos(x) \ , \ (1-r)\cos(x))$. See Fig.4 and Table 4 for comparing the exact solution and obtained solution by the variational iteration method for different k and x.

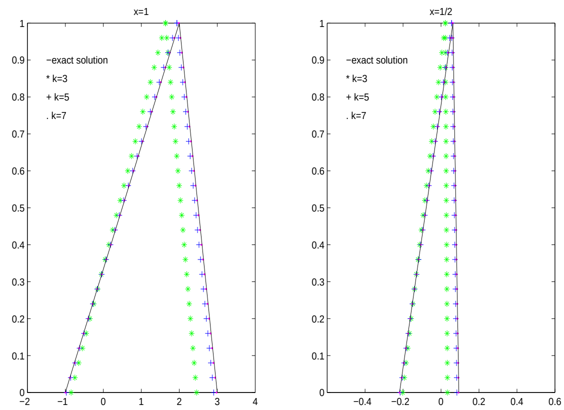

Figure 1: Comparing of exact solution and obtained solution in Example 5.1.

6 Conclusions

In this paper, we used He's variational iteration method to obtain fuzzy solution of the nth-order fuzzy integro-differential equations. Convergency of VIM for this system is proved. Since choosing initial approximations are free so without un-

Table 3: numerical result for Example 5.3.

$d_H(\widetilde{y}^{(k)}, \widetilde{y}_{exact})$			
x	$k = 5$	$k = 10$	$k = 15$
0.2	5.5540e-004	1.9377e-005	1.2624e-007
0.4	0.0050	1.7442e-004	1.1363e-006
0.6	0.0189	6.6002e-004	4.2999e-006
0.8	0.0501	0.0017	1.1385e-005
1	0.1089	0.0038	2.4741e-005

Table 4: numerical result for Example 5.4.

$d_H(\widetilde{y}^{(k)}, \widetilde{y}_{exact})$			
x	$k = 5$	$k = 10$	$k = 15$
$\pi/8$	3.7242e-005	2.9794e-008	8.9369e-011
$\pi/4$	5.9587e-004	4.7671e-007	1.4299e-009
$3\pi/4$	0.0483	3.8613e-005	1.1582e-007
$\pi/2$	0.0095	7.6274e-006	2.2879e-008

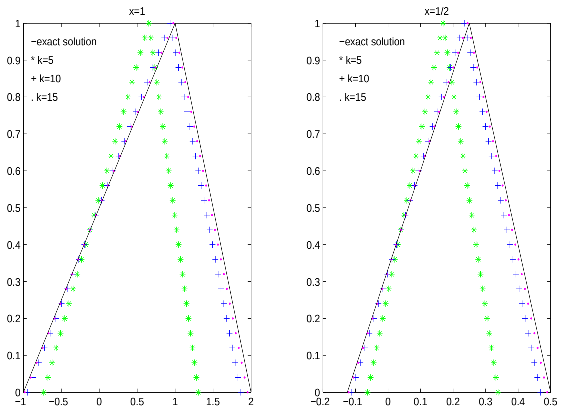

Figure 2: Comparing of exact solution and obtained solution in Example 5.2.

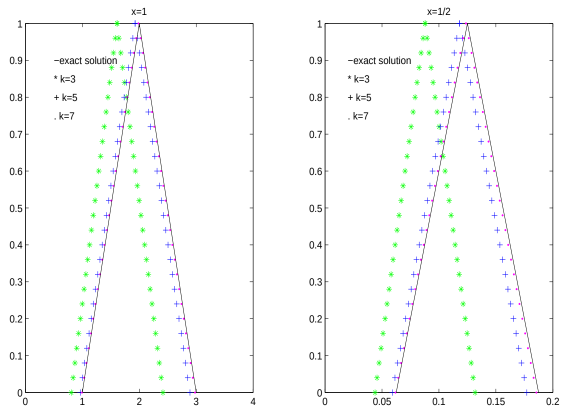

Figure 3: Comparing of exact solution and obtained solution in Example 5.3.

known initial values were constructed. Convergency of VIM for this system is proved. The effectiveness of the method was shown by different examples with separable and inseparable kernels.

References

[1] S. Abbasbandy, E. Babolian, M. Alavi, *Nunerical method for solving linear fredholm fuzzy integral equations of the second kind*, Chaos, Solitons & Fractals 31 (2007) 138-146.

[2] S. Abbasbandy, T. Allahviranloo, P. Darabi, O. Sedaghatfar, *Variational iteration method for solving n-th order fuzzy differential equations*, Math. Comput. Appl., in press.

[3] T. A. Abassy, *Modified variational iteration method (nonlinear homogeneous initial value problem*, Comput. Math. Appl. 59 (2010) 912-918.

[4] T. A. Abassy, M. A. El-Tawil, H. El-Zoheiry, *Modified variational iteration method for boussinesq equation*, Comput. Math. Appl. 54 (2007) 955-965.

[5] M. Friedman, M. Ming, A. Kandel, *Fuzzy linear systems*, Fuzzy Set and Systems 96 (1998) 201-209.

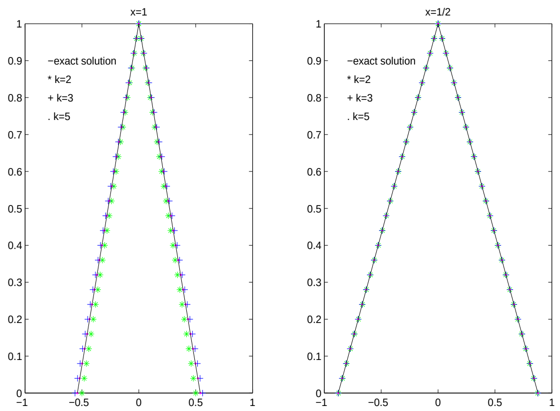

Figure 4: Comparing of exact solution and obtained solution in Example 5.4.

[6] A. Ghorbani, J. Saberi-Nadjafi, *An effective modification of He's variational iteration method*, Nonlinear Anal. Real World Appl. 10 (2009) 2828-2833.

[7] J. H. He, *Variational iteration method-a kind of non-linear analytical technique:Some examples*, Int. J. Nonlinear Mech. 34 (1999) 699-708.

[8] J. H. He, *Variational iteration method Some recent result and new interpretations*, J. Comput. Appl. Math. 207 (2007) 3-17.

[9] J. H. He, Variational iteration method for delay differential equations, Commun. Nonlinear Sci. Numer. Simul. 2 (1997) 235-236.

[10] J. H. He, *A new approach to nonlinear partial differential equations*, Commun. Nonlinear Sci. Numer. Simul. 2 (1997) 230-235.

[11] M. Inokuti, H. Sekine, T. Mura, *General use of the Lagrange multiplier in nonlinear mathematical physics*, in: Variational method in the Mechanics of solids, Pergamon press, New York 1978, 156-162.

[12] D. Ralescu, *A survey of the representation of fuzzy concepts and its applications*, in: M.M. Gupta, R.K. Ragade, R.R. Yager, Eds., Advances in Fuzzy Set Theory and Applications (North-Holland, Amsterdam (1979) 77-91.

[13] S. Seikkala, *On the fuzzy initial value problem*, Fuzzy Sets and Systems 24 (1987) 319-330.

[14] X. Shang, D. Han, *Application of the variational iteration method for solving nth-order integro-differential equations*, J. Comput. Appl. Math. In press.

Omolbanin Sedaghatfar was born in the Tehran, Shahr-e-rey in 1982. She received B.Sc and M.Sc degree in applied mathematics from Zahedan Branch, Tehran - Science and Research Branch, *IAU*, respectively, and her Ph.D degree in applied mathematics from science and research Branch, IAU, Tehran, Iran in 2011. She is a Assistant Prof. in the department of mathematics at Islamic Azad University, Shahr-e-rey, Tehran in Iran. She is research interests include numerical solution of fuzzy integro-differential equations and fuzzy differential equation and similar topics.

Saeid Moloudzadeh was born in the Naghadeh-Iran in 1976. He received B.Sc degree in mathematics and M.Sc degree in applied mathematics from Payam-e-Noor University of Naghadeh, science and research Branch, *IAU* to Tehran, respectively, and his Ph.D degree in applied mathematics from Science and Research Branch, IAU, Tehran, Iran in 2011. He is research interests include fuzzy mathematics, especially, on numerical solution of fuzzy linear systems, fuzzy differential equations and similar topics.

noindent Pejman Darabi was born in Tehran-Iran in 1978. He received B.Sc and M.Sc degrees in applied mathematics from Sabzevar Teacher Education University, science and research Branch, *IAU* to Tehran, respectively, and his Ph.D degree in applied mathematics from science and research Branch, IAU, Tehran, Iran in 2011. Main research interest include numerical solution of Fuzzy differential equation systems, ordinary differential equation, partial differential equation, fuzzy linear system and similar topics.