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Topological Residuated Lattices

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Abstract

In this paper, we study the separtion axioms T_0, T_1, T_2 and $T_{5/2}$ on topological and semitopological residuated lattices and we show that they are equivalent on topological residuated lattices. Then we prove that for every infinite cardinal number α, there exists at least one nontrivial Hausdorff topological residuated lattice of cardinality α. In the follows, we obtain some conditions on (semi) topological residuated lattices under which this spaces will convert into regular and normal spaces. Finally by using of regularity and normality, we convert (semi)topological residuated lattices into metrizable topological residuated lattices.

Keywords : Topological residuated lattice; Filter; Regular space; Normal space; (locally) Compact space.

1 Introduction

Residuated lattices have been introduced by M.Ward and R.P. Dilworth [9] as generalization of ideal lattices of rings with identity. They are a common structure among algebras associated with logical systems. The main examples of residuated lattices related to logic are MV-algebras and BL-algebras. In recent years some mathematicians have endowed algebraic structures associated with logical systems with a topology and have found some their propertises. For example, Borzooei et.al. in [3] introduced (semi) topological BL-algebras and in [4] and [5] studied metrizability and separation axioms on them. In [11] Kouhestani and Borzooei defined the notion of (semi) topological residuated lattices and studied separation axioms T_0, T_1 and T_2 on them. For a topological space, metrizability is a highly desirable property, for the existence of a such a distance function gives one a valuable tool for proving theorems about the space. In the section 3 of this paper, we deal with relations between T_i spaces, for $i = 0, 1, 2, 5/2$ and (semi)topological residuated lattices. We will prove that they are equivalent and will show that there are nontrivial Hausdorff topological residuated lattices of infinite cardinality. Then in section 4 we will find some conditions under which a (semi)topological residuated lattice convert into a regular or nomal space. Finally, metrizable topological residuated lattices will be obtained by using of regularity and normality.

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2 Preliminaries

Recall that a set A with a family $\mathcal{U}$ of its subsets is called a *topological space*, denoted by $(A, \mathcal{U})$, if $A, \emptyset \in \mathcal{U}$, the intersection of any finite numbers of members of $\mathcal{U}$ is in $\mathcal{U}$ and the arbitrary

union of members of $\mathcal{U}$ is in $\mathcal{U}$. The members of $\mathcal{U}$ are called *open sets* of A and the complement of $U \in \mathcal{U}$, that is $A \setminus U$, is said to be a *closed set*. If B is a subset of A, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_u B$). A subfamily $\{U_\alpha : \alpha \in I\}$ of $\mathcal{U}$ is said to be a *base* of $\mathcal{U}$ if for each $x \in U \in \mathcal{U}$, there exists an $\alpha \in I$ such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{U}$ is the union of members of $\{U_\alpha\}$. A subset P of A is said to be a *neighborhood* of $x \in A$, if there exists an open set U such that $x \in U \subseteq P$. Let $\mathcal{U}_x$ denote the totality of all neighborhoods of x in A. Then a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a *fundamental system* of neighborhoods of x, if for each U_x in $\mathcal{U}_x$, there exists a V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$. A *directed set* I is a partially ordered set such that, for any i and j of I, there is a $k \in I$ with $k \geq i$ and $k \geq j$. If I is a directed set, then the subset $\{x_i : i \in I\}$ of A is called a *net*. A net $\{x_i; i \in I\}$ *converges* to $x \in A$ if for each neighborhood U of x, there exists a $j \in I$ such that for all $i \geq j$, $x_i \in U$. If $B \subseteq A$ and $x \in \overline{B}$, then there is a net in B that is converges to x.
Topological space $(A, \mathcal{U})$ is said to be a:
(i) T_0-space if for each $x \neq y \in A$, there is at least one in an open neighborhood excluding the other,
(ii) T_1-space if for each $x \neq y \in A$, each has an open neighborhood not containing the other,
(iii) T_2-space if for each $x \neq y \in A$, there two disjoint open neighborhoods U, V of x and y, respectively.
A T_2-space is also known as a *Hausdorff* space.[see, [7]]

Definition 2.1 *[9] A* residuated lattice *is an algebra* $\mathcal{L} = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *of type* $(2,2,2,2,0,0)$ *such that* $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice,* $(L, \odot, 1)$ *is a commutative monoid and for any* $a, b, c \in L$,

$$c \leq a \rightarrow b \Leftrightarrow a \odot c \leq b.$$

A residuated lattice L *is* divisible *if for each* $a, b \in L$, $a \odot (a \rightarrow b) = a \wedge b$.

Let L be a residuated lattice. We set $a' = a \rightarrow 0$ and denote $(a')'$ by a''. We call the map $p : L \rightarrow L$ by $p(a) = a'$, for any $a \in L$, the *negation map*. Also, for each $a \in L$, we define $a^0 = 1$ and $a^n = a^{n-1} \odot a$, for each natural numbers n.

Example 2.1 *[9](i) Let* $\odot$ *and* $\rightarrow$ *on the real unit interval* $I = [0,1]$ *be defined as follows:*

$$x \odot y = \min\{x, y\} \;\&\; x \rightarrow y = \begin{cases} 1, & x \leq y, \\ y, & x > y. \end{cases}$$

Then $\mathcal{I} = (I, \min, \max, \odot, \rightarrow, 0, 1)$ *is a residuated lattice,*
(ii) Let $\odot$ *be the usual multiplication of real numbers on the unit interval* $I = [0,1]$ *and* $x \rightarrow y = 1$ *iff,* $x \leq y$ *and* y/x *otherwise. Then* $\mathcal{I} = (I, \min, \max, \odot, \rightarrow, 0, 1)$ *is a residuated lattice.*
(iii) Let $L = \{0, a, b, c, 1\}$. *Define* $\odot$ *and* $\rightarrow$ *as follows :*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Easily we can check that $(L, \odot, \rightarrow, 0, 1)$ *is a residuated lattice, whose lattice* $(L, \wedge, \vee, 0, 1)$ *is given by the partial order* $0 < a < c < 1$, $0 < b < c < 1$ *and* $x \wedge y = min\{x, y\}$, $x \vee y = max\{x, y\}$, $a \wedge b = 0$ *and* $a \vee b = 1$.

Proposition 2.1 *[9] Let* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *be a residuated lattice. The following properties hold.*
(R_1) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$,
(R_2) $x \leq y$ *iff,* $x \rightarrow y = 1$,
(R_3) $1 * x = x$, *where* $* \in \{\wedge, \odot, \rightarrow\}$,
(R_4) $x \odot 0 = 0$, $1' = 0$, $0' = 1$,
(R_5) $x \odot y \leq x \wedge y \leq x, y$, *and* $y \leq (x \rightarrow y)$,
(R_6) $(x \rightarrow y) \odot x \leq y$,
(R_7) $x \leq y \rightarrow (x \odot y)$,
(R_8) $x \leq y$ *implies* $x * z \leq y * z$, *where* $* \in \{\wedge, \vee, \odot\}$,
(R_9) $x \leq y$ *implies* $z \rightarrow x \leq z \rightarrow y$ *and* $x \rightarrow z \geq y \rightarrow z$,
(R_{10}) $x \rightarrow y = x \rightarrow (x \wedge y)$,
(R_{11}) $x \leq y$ *implies* $x \leq z \rightarrow y$,
(R_{12}) $z \odot (x \wedge y) \leq (z \odot x) \wedge (z \odot y)$,

(R_{13}) $x \to y \leq (x \odot z) \to (y \odot z)$,
(R_{14}) $(x \to y) \odot (y \to z) \leq x \to z$,
(R_{15}) $x \odot x' = 0$,
(R_{16}) $x \to y' = (x \odot y)'$,
(R_{17}) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
(R_{18}) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(R_{19}) $x \to (y \wedge z) = (x \to y) \vee (x \to z)$,
(R_{20}) $(x \vee y)' = x' \wedge y'$, *and* $(x \wedge y)' \geq x' \vee y'$,
(R_{21}) *if* $x \vee y = 1$, *then* $x \to y = y$ *and* $x \odot y = x \vee y$,
(R_{22}) $x''' = x'$,
(R_{23}) $(a \to x) \odot (b \to y) \leq (a * b) \to (x * y)$, *where* $* \in \{\wedge, \vee\}$.

Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice. Then a *filter* of L is a nonempty subset $F \subseteq L$ which satisfies the following conditions:
(a) $x, y \in F$ implies $x \odot y \in F$,
(b) if $x \in F$ and $x \leq y$, then $y \in F$.
Let F be a filter of L. Then the relation $x \equiv^F y$ iff, $x \to y, y \to x \in F$ is a congruence relation on L. Moreovere, if for each $x \in L$, $F(x) = \{y \in A : y \equiv^F x\}$, then for each $* \in \{\wedge, \vee, \odot, \to\}$, $F(x) * F(y) = F(x * y)$. Thus, the set $L/F = \{F(x) : x \in L\}$ is a residuated lattice which called quotient residuated lattice.[See, [9]]

Notation. From now on, in this paper we let L be a residuated lattice and $\mathcal{U}$ be a topology on L, unless otherwise state.

3 Hausdorff topological residuated lattice

Definition 3.1 *[11] Let $\mathcal{U}$ be a topology on residuated lattice L and $* \in \{\wedge, \vee, \odot, \to\}$. Then:*
(i) *the operation $*$ is* continuous in the first (second) variable *if for each $a \in L$, the mapping $x \hookrightarrow x * a (x \hookrightarrow a * x)$ from L into L is continuous.*
(ii) $(L, *, \mathcal{U})$ *is* semitopological residuated lattice *if $*$ is continuous in the first and second variable,*
(iii) $(L, *, \mathcal{U})$ *is* topological residuated lattice *if $*$ is continuous,*
(iv) $(L, \mathcal{U})$ *is* (semi) topological residuated lattice *if for any $* \in \{\wedge, \vee, \odot, \to\}$, $(L, *, \mathcal{U})$ is (semi) topological residuated lattice.*

Proposition 3.1 *Let $(L, , \mathcal{U})$ be a semitopological residuated lattice. Then $(L, \mathcal{U})$ is T_0 iff, it is T_1.*

Proof. Let $(L, \mathcal{U})$ be T_0 and $x \neq y$. Then $xy \neq 1$ or $yx \neq 1$. W.L.O.G let $xy \neq 1$. Then there exists $U \in \mathcal{U}$ such that $xy \in U$ and $1 \notin U$ or $xy \notin U$ and $1 \in U$. First suppose $xy \in U$ and $1 \notin U$. Since is continuous in each variable separately, there are open neighborhoods V and W of x and y, respectively, such that $Vy \subseteq U$ and $xW \subseteq U$. But $x \notin W$ because if $x \in W$, then $1 = xx \in xW \subseteq U$ which is a contradiction. Similarly, $y \notin V$. Now let $1 \in U$ and $xy \notin U$. Since $xx = yy = 1 \in U$, there are $V, W \in \mathcal{U}$ such that $x \in V$, $y \in W$, $xV \subseteq U$ and $Wy \subseteq U$. If $x \in W$, then $xy \in Wy \subseteq U$ which is a contradiction. So $x \notin W$. Similaly, $y \notin V$. Hence $(L, \mathcal{U})$ is a T_1 space. Convesely is clear.

Proposition 3.2 *Let $(L, , \mathcal{U})$ be a topological residuated lattice. Then $(L, \mathcal{U})$ is T_1 iff, it is Hausdorff.*

Proof. Let $(L, \mathcal{U})$ be T_1 and $x \neq y$. Then $xy \neq 1$ or $yx \neq 1$. W.L.O.G let $xy \neq 1$. Then there exists $U, V \in \mathcal{U}$ such that $xy \in U$, $1 \in V$, $xy \notin V$ and $1 \notin U$. Since is continuous, there are open neighborhoods V_0 and V_1 of x and y, respectively, such that $V_0V_1 \subseteq U$. We prove that $V_0 \cap V_1$ is empty. Suppose $z \in V_0 \cap V_1$, then $1 = zz \in V_0V_1 \subseteq U$ which is a contradiction. So $(L, \mathcal{U})$ is Hausdorff. Conversely is clear.

Corollary 3.1 *Let $(L, , \mathcal{U})$ be topological residuated lattice. Then $(L, \mathcal{U})$ is T_0 iff, it is T_1 iff, it is Hausdorff.*

Proof. By Propositions 3.1, 3.2, the proof is clear.

Theorem 3.1 *Let $(L, \odot, , \mathcal{U})$ be topological residuated lattice. Then $(L, \mathcal{U})$ is Hausdorff iff, $\{1\}$ or $\{0\}$ is closed.*

Proof. Let $\{1\}$ be closed. We show that for each $a \in L$, the set $\{a\}$ is closed. Suppose $a \in L$. Since $\odot$ is continuous, $\odot^{-1}(1) = \{(1, 1)\}$ is closed in $L \times L$. On the other hand, as is continuous, the map $h : L \hookrightarrow L \times L$ by $h(x) = (ax, xa)$ is continuous. Hence $h^{-1}\{(1, 1)\} = \{x : xa = ax = 1\} = \{a\}$ is closed in L. This implies that $(L, \mathcal{U})$ is T_1. By Proposition 3.2, it is Hausdorff. If $\{0\}$ is closed, since the negation map $p : L \hookrightarrow L$ by $p(x) = x'$ is continuous and $1 = p^{-1}(0)$, the set $\{1\}$ is closed and so $(L, \mathcal{U})$ is Hausdorff. Conversely is clear.

Definition 3.2 *Let L be a residuated lattice. Then for each $a \in L$ and any $V \subseteq L$, we define*

$$V(a) = \{x \in L : xa \in V, ax \in V\},$$

and

$$V[a] = \{x \in L : xa \in V\}.$$

Proof. Let $(L, ,\mathcal{U})$ be a semitopological residuated lattice. Then for each $a \in L$ and $V \in \mathcal{U}$ the sets $V(a)$ and $V[a]$, both, are open.

Proof. Let $a \in L$ and $V \in \mathcal{U}$. Suppose $x \in V(a)$, then $xa \in V$ and $ax \in V$. Since is continuous in each variable, there is $W \in \mathcal{U}$ such that x is in W and Wa and aW are the subsets of V. Clearly, $x \in W \subseteq V(a)$. Similarly, $V[a] \in \mathcal{U}$.

Proposition 3.3 *Let $(L,\mathcal{U})$ be a topological residuated lattice. Then it is Hausdorff iff, for each $0 \neq x \in L$ there is $V \in \mathcal{U}$ such that $1 \in V$ and $x' \notin V$.*

Proof. First let $(L,\mathcal{U})$ be Hausdorff and $x \neq 0$. Then $x' \neq 1$ and so there is an open set V such that $1 \in V$ and $x' \notin V$. Conversely, let $x \in \overline{\{0\}}$. If $x \neq 0$, then there is an open set V such that $1 \in V$ and $x' \notin V$. By Proposition 3, $V(x)$ is an open neighborhood of x, so $0 \in V(x)$. This implies that $x' \in V$, a contradiction. Hence $\{0\}$ is closed. By Theorem 3.1, $(L,\mathcal{U})$ is Hausdorff. Recall that a topological space $(X,\mathcal{U})$ is an *Uryshon space* if for each $x \neq y \in L$, there exist two open neighborhoods U and V of x and y, respectively, such that $\overline{U} \cap \overline{V} = \phi$. An Uryshon space is also known as a $T_{5/2}$ space.[See, [7]]

Proposition 3.4 *Let $(L,\rightarrow,\mathcal{U})$ be a topological residuated lattice. Then L is an Uryshon space iff, for each $x \neq 1$, there exist two open neighborhoods U and V of x and 1, respectively, such that $\overline{U} \cap \overline{V} = \phi$.*

Proof. If $(L,\mathcal{U})$ is an Uryshon space, the proof is clear. Conversely, let for each $x \neq 1$, there exist two open neighborhoods U and V of x and 1, respectively, such that $\overline{U} \cap \overline{V} = \phi$. Let $x, y \in L$ and $x \neq y$. Then $x \rightarrow y \neq 1$ or $y \rightarrow x \neq 1$. W.O.L.G, let $x \rightarrow y \neq 1$ and U and V be two open neighborhoods of $x \rightarrow y$ and 1 such that $\overline{U} \cap \overline{V} = \phi$. Since $(L,\rightarrow,\mathcal{U})$ is a topological residuated lattice, there are two open neighborhoods W_1 and W_2 of x and y, respectively, such that $W_1 \rightarrow W_2 \subseteq U$. We prove that $\overline{W_1}$ and $\overline{W_2}$ are disjoint. Let $z \in \overline{W_1} \cap \overline{W_2}$. Then there exist two nets $\{x_i\}$ and $\{y_i\}$ in W_1 and W_2, respectively, such that both converge to z. Since the operation $\rightarrow$ is continuous, the net $\{x_i \rightarrow y_i\}$ converges to $z \rightarrow z = 1$. Hence $1 \in \overline{W_1 \rightarrow W_2} \subseteq \overline{U}$, which is a contradiction.

Theorem 3.2 *Let $(L,\rightarrow,\mathcal{U})$ be a topological residuated lattice. Then the following statements are equivalent:*
(i) $(L,\mathcal{U})$ is a T_0 space,
(ii) $(L,\mathcal{U})$ is a T_1 space,
(iii) $(L,\mathcal{U})$ is a Hausdorff space,
(iv) $(L,\mathcal{U})$ is an Uryson space,
(v) $\bigcap_{U\in\mathcal{N}} U = 1$, where $\mathcal{N}$ is a fundamental system of neighborhoods of 1.

Proof. By Corollary 3.1, the statements (i), (ii) and $(iiii)$ are equivalent.
$(iii) \Longrightarrow (iv)$ Let $(L,\mathcal{U})$ be a Hausdorff space and $1 \neq x$. Then there are two disjoint open sets U and V such that $x \in U$ and $1 \in V$. Since $1x = x \in U$, there are two open neighborhoods W and W_1 of x and 1, respectively, such that $W_1 W \subseteq U$. We prove that $\overline{W} \cap \overline{W_1} = \phi$. Let $z \in \overline{W} \cap \overline{W_1}$. Then there are nets $\{x_i : i \in I\} \subseteq W_1$ and $\{y_i : i \in I\} \subseteq W$ such that both converges to z. Hence $\{x_i y_i : i \in I\}$ is a net in U which converges to $zz = 1$. This implies that $1 \in \overline{U}$. Since $1 \in V \in \mathcal{U}$, $V \cap U \neq \phi$, a conteradiction. Therefore, by Proposition 3.4, $(L,\mathcal{U})$ is an Uryson space.
$(iv) \Longrightarrow (v)$ Let $(L,\mathcal{U})$ be an Uryson space and $x \neq 1$. By Proposition 3.4, there are open sets U and V such that $1 \in U$, $x \in V$ and $\overline{U} \cap \overline{V} = \phi$. Hence $x \notin U$ and so $x \notin \bigcap_{U\in\mathcal{N}} U$.
$(v) \Longrightarrow (ii)$ Let $\bigcap_{U\in\mathcal{N}} U = 1$, and $1 \neq x$. Then there is a $V \in \mathcal{N}$ such that $x \notin V$. Hence $1 \notin V(x)$. By [[11], Proposition 4.6], $(L,\mathcal{U})$ is a T_1 space.

Proposition 3.5 *Let L be a noncountable set and $0, 1 \in L$. Then there are operations $\wedge, \vee, \odot$ and on L such that $(L, \wedge, \vee, \odot, , 0, 1)$ is a residuated lattice.*

Proof. Let $L_0 = \{x_0 = 0, x_1, x_2, ...\} \cup \{1\} \subseteq L$. Define $\rightsquigarrow, \oplus, \barwedge$ and $\veebar$ on L_0 as follows:

$$xy := \begin{cases} [0, x], & \text{if } x \leq y, \\ (0, y], & \text{if } x > y \neq 0, \\ \{x\}, & \text{if } y = 0 \end{cases}$$

$$1 \rightsquigarrow x_i = x_i, \;\; x_i \rightsquigarrow 1 = 1,$$

$$x_i \rightsquigarrow x_j = \begin{cases} 1, & \text{if } \ i \leq j, \\ x_j, & \text{if } \ otherwise \end{cases}$$

$$x \oplus y = min\{x, y\} = x \barwedge y, \quad x \veebar y = max\{x, y\},$$

where $x_0 \leq x_1 \leq x_2 \leq ... \leq 1$. It is easy to see that $(L_0, \barwedge, \veebar, \oplus, \rightsquigarrow, 0, 1)$ is a residuated lattice. Given $a \in L \setminus L_0$ and put $L_1 = L_0 \cup \{a\}$. Let $\curlywedge, \curlyvee, \otimes$ and $\hookrightarrow$ on the L_1 be defined as follows:

$$x \curlywedge y = \begin{cases} x \barwedge y, & \text{if } \ x, y \in L_0, \\ min\{x, y\}, & \text{if } \ otherwise \end{cases}$$

$$x \curlyvee y = \begin{cases} x \veebar y, & \text{if } \ x, y \in L_0, \\ max\{x, y\}, & \text{if } \ otherwise, \end{cases}$$

$$x \otimes y = \begin{cases} x \oplus y, & \text{if } \ x, y \in L_0, \\ min\{x, y\}, & \text{if } \ otherwise, \end{cases}$$

$$x \hookrightarrow y = \begin{cases} x \rightsquigarrow y, & \text{if } \ x, y \in L_0, \\ 1, & v \ \ x \in L_1 \setminus \{1\}, y = a, \\ a, & \text{if } \ x = 1, y = a, \\ y, & \text{if } \ x = a, y \in L_0, \end{cases}$$

where $x_0 \leq x_1 \leq x_2 \leq ... \leq a \leq 1$.Then $(L_1, \curlywedge, \curlyvee, \oplus, \hookrightarrow, 0, 1)$ is a residuated lattice. Now consider the operations $\wedge, \vee, \odot$ and on L as follows:

$$x \wedge y = x \odot y = \begin{cases} min\{x, y\}, & \text{if } \ x \ or \ y \in L_1, \\ a, & \text{if } \ otherwise, \end{cases}$$

$$x \vee y = \begin{cases} max\{x, y\}, & \text{if } \ x \ or \ y \in L_1, \\ 1, & \text{if } \ otherwise, \end{cases}$$

$$xy = \begin{cases} x \hookrightarrow y, & \text{if } \ x, y \in L_1, \\ 1, & \text{if } \ x \in L_1 \setminus \{1\}, y \notin L_1, \\ y, & \text{if } \ x = 1, y \notin L_1, \\ y, & \text{if } \ x \notin L_1, y \in L_1, \\ 1, & \text{if } \ x = y \notin L_1, \\ y, & \text{if } \ x \neq y \notin L_1, \end{cases}$$

where $x \leq y$ if and only if $xy = 1$. Then it is easy to claim that $(L, \wedge, \vee, \odot, , 0, 1)$ is a residuated lattice.

Theorem 3.3 *Let L_0 and L_1 be residuated lattices in Proposition 3.5. Then there are two topologies $\mathcal{V}$ and $\mathcal{W}$ on L_0 and L_1, respectively, such that $(L_0, \mathcal{V})$ and $(L_1, \mathcal{W})$ are Hausdorff topological residuated lattices.*

Proof. Let $F_k = \{x_k, x_{k+1}, ...\} \cup \{1\}$, for each $k \in \{0, 1, 2, 3, ...\}$. Then $\mathcal{F} = \{F_k : k = 0, 1, 2, ...\}$ is a family of filters in L_0 which is closed under finite intersections. Let $\mathcal{V} = \{V \subseteq L_0 : \forall x \in V \ \exists F_k \ s.t \ F_k(x) \subseteq V\}$. It is easy to prove that $\mathcal{V}$ is closed under finite intersections and arbitrary unions. Also, for each $x \in L$ and each $F_k \in \mathcal{F}$, $F_k(x)$ belongs to $\mathcal{V}$. Hence $\mathcal{V}$ is a nontrivial topology on L_0. We prove that $(L_0, \mathcal{V})$ is a topological resiuated lattice. For this, let $* \in \{\barwedge, \veebar, \oplus, \rightsquigarrow\}$, $x, y \in L_0$ and $F_k \in \mathcal{F}$. If $z_1 \in F_k(x)$ and $z_2 \in F_k(y)$, since $\equiv^F$ is a congruence relation, then $z_1 * z_2 \equiv^{F_k} x * y$. Hence $F_k(x) * F_k(y) \subseteq F_k(x * y)$. This proves that the operation $*$ is continuous and so $(L_0, \mathcal{V})$ is a topological residuated lattice. Now suppose $x \in L_0$ and $x \neq 1$. Then there is a $F_k \in \mathcal{F}$ such that $x \notin F_k$. This implies that $x \in F_k(x)$ and $1 \notin F_k(x)$. By [[11], Proposition 4.4] $(L_0, \mathcal{V})$ is a T_1 space. By Theorem 3.2, $(L_0, \mathcal{V})$ is a Hausdorff space.
In continue we will find a topology $\mathcal{W}$ on L_1 such that $(L_1, \mathcal{W})$ is a Hausdorff topological residuated lattice. Let $J_k = F_k \cup \{a\}$. We show that J_k is a filter in L_1. Let $x, y \in J_k$. Then $x \otimes y \in \{x, y, a\} \subseteq J_k$. Suppose $x \in J_k$ and $x \leq y$. If $y = a$, then $y \in J_k$. If $y \neq a$, then $x \in F_k$, since F_k is a filter, $y \in F_k \subseteq J_k$. Hence J_k is a filter for each k and $\mathcal{J} = \{J_k : k = 0, 1, ...\}$ is a family of filters in L_1 which is closed under finite intersections. In a similar way with the above paragraph we can prove that $\mathcal{W} = \{W \subseteq L_1 : \forall x \in W \ \exists J_k \ s.t \ J_k(x) \subseteq W\}$ is a nontrival topology on L_1 such that $(L_1, \mathcal{W})$ is a topological residuated lattice. Since $L_1 \setminus \{0\} = J_1 \in \mathcal{W}$, the set $\{0\}$ is closed in L_1. By Theorem 3.1, $(L_1, \mathcal{W})$ is Hausdorff.

Theorem 3.4 *Let α be an infinite cardinal number. Then there is a Hausdorff topological residuated lattice of cardinality α.*

Proof. Let L be a set of cardinality α and $0, 1 \in L$. If L is a countable set, then take $L = L_0$, where L_0 is the residuated lattice in Proposition 3.5. Then by Theorem 3.3, there is a nontrivial topology $\mathcal{V}$ on L such that $(L, \mathcal{V})$ is a Hausdorff topological residuated lattice. Let L be a noncountable set and $L_0 = \{x_0 = 0, x_1, ...\} \cup \{1\} \subseteq L$ and $L_1 = L_0 \cup \{a\}$, where $a \in L \setminus L_0$. As the proof of Proposition 3.5, there are operations $\barwedge, \veebar, \oplus \rightsquigarrow$ on L_0 and operations $\curlywedge, \curlyvee, \otimes, \hookrightarrow$ on L_1 and operations $\wedge, \vee, \odot,$ on L such that $(L_0, \barwedge, \veebar, \oplus \rightsquigarrow)$ and $(L_1, \curlywedge, \curlyvee, \otimes, \hookrightarrow)$ and $(L, \wedge, \vee, \odot,)$ are residuated lattices. Suppose for each $k \in \{0, 1, 2, ...\}$, $F_k = \{x_k, x_{k+1}, ...\}$, $J_k = F_k \cup \{a\}$ and $H_k = J_k \cup (L \setminus L_1)$, then F_k is a filter in L_0 and J_k is a filter in L_1. We prove that H_k is a filter in L. Let $x, y \in H_k$, then $x \odot y \in \{a, x, y\} \subseteq H_k$. Let $x \leq y$

and $x \in H_k$. If $y \in L \setminus L_1$, then clearly $y \in H_k$. If $y \in L_1$, then $x \in L_1$. Since J_k is a filter in L_1, $y \in J_k \subseteq H_k$. Since $\mathcal{H} = \{H_k : k = 0, 1, 2, ...\}$ is closed under finite intersections, as the proof of Theorem 3.3, there is a topology $\mathcal{U}$ on L such that $(L, \mathcal{U})$ is a topological residuated lattice. Since $L \setminus \{0\} = H_1 \in \mathcal{U}$, the set $\{0\}$ is closed in $(L, \mathcal{U})$. By Theorem 3.1, $(L, \mathcal{U})$ is Hausdorff.

4 Regular and normal (semi)topological residuated lattices

In this section we study regularity and normality on (semi)topological residuated lattices. First we show that for each residuated lattice L there exists a topology $\mathcal{U}$ such that $(L, \mathcal{U})$ is a regular and normal topological residuated lattice. In Example 4.1, we show that there exist residuated lattices that are not regular or normal. Then we find some conditions under which a (semi)topological residuated lattice becomes regular or normal.
Recall that a topological space $(X, \mathcal{U})$ is *regular* if for each $x \in U \in \mathcal{U}$, there exists an open set H such that $x \in H \subseteq \overline{H} \subseteq U$. Also, $(X, \mathcal{U})$ is *normal* if for each closed set S and each open set U contains S, there is an open set H such that $S \subseteq H \subseteq \overline{H} \subseteq U$. [See, [7]]

Theorem 4.1 *Let $\mathcal{F}$ be a familly of filters in L which is closed under intersections. Then there is a nontrivial topology $\mathcal{U}$ on L such that $(L, \mathcal{U})$ is a regular and normal topological residuated lattice.*

Proof. Let $\mathcal{U} = \{U \subseteq L : \forall x \in U\ \exists F \in \mathcal{F}\ s.t\ F(x) \subseteq U\}$. It is easy to prove that $\mathcal{U}$ is closed under arbitrary intersections and unions. Also, for each $x \in L$ and each $F \in \mathcal{F}$, $F(x)$ belongs to $\mathcal{U}$. Hence $\mathcal{U}$ is a nontrivial topology. We prove that $(L, \mathcal{U})$ is a topological resiuated lattice. For this, let $* \in \{\wedge, \vee, \odot, \rightarrow\}$, $x, y \in L$ and $F \in \mathcal{F}$. If $z_1 \in F(x)$ and $z_2 \in F(y)$, since $\equiv^F$ is a congruence relation, then $z_1 * z_2 \equiv^F x * y$. Hence $F(x) * F(y) \subseteq F(x * y)$. This proves that the operation $*$ is continuous and so $(L, \mathcal{U})$ is a topological residuated lattice. Now we will prove that $(L, \mathcal{U})$ is a regular and normal space. First we show that for each $x \in L$ and $F \in \mathcal{F}$, the set $F(x)$ is closed. Let $x \in L$, $F \in \mathcal{F}$ and $y \in \overline{F(x)}$. Since $F(y)$ is an open neighborhood of y, there is a $z \in L$ such that $y \equiv^F z \equiv^F x$. Hence $y \in F(x)$ and so $\overline{F(x)} = F(x)$. To prove regularity, let $x \in U \in \mathcal{U}$. Then there is a $F \in \mathcal{F}$ such that $F(x) \subseteq U$. Since $F(x)$ is clsoed, we get that $x \in F(x) \subseteq \overline{F(x)} \subseteq U$, which implies that $(L, \mathcal{U})$ is a regular space. To complete the proof, we have to show that $(L, \mathcal{U})$ is a normal space. Let S be a closed set, U be an open set and $S \subseteq U$. For each $x \in S$ there exits a $F \in \mathcal{F}$ such that $F(x) \subseteq U$. Let $H = \bigcup_{x \in S, F \in \mathcal{F}} F(x)$. Then H is a closed and open set because for each $x \in S$ and $F \in \mathcal{F}$, the set $F(x)$ is a closed and open set. Thus $S \subseteq H \subseteq \overline{H} \subseteq U$, which implies that $(L, \mathcal{U})$ is a normal space.

Example 4.1 (i) *Let L be a nontrivial residuated lattice and for each $a \in L$, $L_a = \{x \in L : a \leq x\}$. Then the set $\boldsymbol{B} = \{L_a : a \in L\}$ is a base for a topology $\mathcal{U}$ on L. We show that $(L, \mathcal{U})$ is not a regular space. If $(L, \mathcal{U})$ is a regular space and $1 \in L_1 = \{1\} \in \mathcal{U}$, then there is an open set H such that $1 \in H \subseteq \overline{H} \subseteq L_1$. Hence $\{1\} = H = \overline{H}$. On the other hand, $\overline{\{1\}} = L$, so $L = \overline{H} = \{1\}$, which is a contradiction.*
(ii) *Let L be the residuated lattice in Example 2.1(iii). Then $\mathcal{U} = \{\{0, 1, a\}, \{0, 1\}, \{0, 1, b, c\}, L, \phi\}$ is a topology on L. Clearly, $S = \{b, c\}$ is a closed set and $U = \{0, 1, b, c\}$ is an open set contains $\{b, c\}$ which is not a closed set because $a \in \overline{U}$. Obviously, there is not any open set H such that $S \subseteq H \subseteq \overline{H} \subseteq U$.*

Theorem 4.2 *For each $n \geq 3$ there exist a normal and regular topological residuated lattice of order n.*

Proof. Let $L_3 = \{0, a, 1\}$. Define the operations $, \odot, \wedge$ and $\vee$ on L_3 as follows:

$\odot$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	a

	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

$$x \wedge y = min\{x, y\} \qquad x \vee y = max\{x, y\}$$

It is easy to prove that $(L_3, \wedge, \vee, \odot, , 0, 1)$ is a residuated lattice and $F = \{a, 1\}$ is a filter in L_3. By Theorem 4.1, $\mathcal{U} = \{\phi, \{0\}, \{1, a\}, L_3\}$ is a topology on L_3 such that $(L_3, \mathcal{U})$ is a normal and regular topological residuated lattice. Now suppose $L_4 = L_3 \cup \{b\}$, where $b \notin L_3$. Consider the operations $\wedge, \vee, \odot$ and $\hookrightarrow$ on L as follows:

$$x \wedge y = x \odot y = \begin{cases} min\{x,y\}, & \text{if } \ x \ or \ y \in L_1, \\ a, & \text{if } \ otherwise, \end{cases}$$

$$x \vee y = \begin{cases} max\{x,y\}, & \text{if } \ x \ or \ y \in L_1, \\ 1, & \text{if } \ otherwise, \end{cases}$$

$$x \hookrightarrow y = \begin{cases} xy, & \text{if } \ x, y \in L_3, \\ 1, & \text{if } \ x \in L_4 \setminus \{1\}, y = b, \\ b, & \text{if } \ x = 1, y = b, \\ y, & \text{if } \ x = b, y \in L_3 \end{cases}$$

where $x \leq y$ if and only if $xy = 1$. Then it is easy to claim that $(L, \wedge, \vee, \odot, , 0, 1)$ is a residuated lattice. If $F = \{1, a, b\}$, then F is a filter in L_4. By Theorem 4.1, $\mathcal{U} = \{\phi, \{0\}, \{1, a, b\}, L_4\}$ is a topology on L_4 such that $(L_4, \mathcal{U})$ is a normal and regular topological residuated lattice of order 4. In a similar way we can make a normal and regular topological residuated lattice of order n, for each $n \geq 4$.

Theorem 4.3 *For every infinte cardinal number α there is a normal, regular and Hausdorff topological residuated lattice of order α.*

Proof. If α is countable cardinal number, then by Theorem 4.1, Hausdorff topological residuated lattice $(L_0, \mathcal{V})$ in Theorem 3.3, is a normal and regular space. If α is noncuontable cardinal number, then by Theorem 4.1, Hausdorff topological residuated lattice $(L, \mathcal{U})$ in Theorem 3.4, is a normal and regular space.

Theorem 4.4 *Let F be a nontrivial filter in L. Then:*
(i) there exists a nontrivial topology $\mathcal{U}$ on L such that $(L, \odot, \mathcal{U})$ is a T_0 topological residuated lattice and F is a closed and open set,
(ii) $(L, \mathcal{U})$ is a regular space iff, for each $x \in U \in \mathcal{U}$, the set $F \odot x$ is a closed set,
(iii) $(L, \mathcal{U})$ is a normal space iff, for each closed set S, the set $\bigcup_{x \in S} F \odot x$ is a closed set.

Proof. (i) By [[11], Proposition 4.1(ii)] the set $\mathcal{U} = \{U \subseteq L : \forall x \in U \ F \odot x \subseteq U\}$ is a nontrivial topology on L such that $(L, \mathcal{U})$ is a T_0 topological residuated lattice. We prove that F is closed and open. Since for each $x \in F$, $F \odot x \subseteq F$, we get that $F \in \mathcal{U}$. If $x \in \overline{F}$, then $F \odot x \cap F \neq \phi$. Hence there is a $f \in F$ such that $f \odot x \in F$. Since F is a filter by (R_5), we conclude that $x \in F$. So F is a closed set.

(ii) Let $(L, \mathcal{U})$ be regular and $x \in U \in \mathcal{U}$. Since $F \odot x$ is an open neighborhood of x, there is an open set H such that $x \in H \subseteq \overline{H} \subseteq F \odot x$. Since $x \in H$, the set $F \odot x$ is contained in H. Hence $F \odot x = \overline{H}$, which implies that $F \odot x$ is closed. Conversely, let $x \in U \in \mathcal{U}$. Since $F \odot x$ is closed, hence $x \in F \odot x = \overline{F \odot x} \subseteq U$, which implies that $(L, \mathcal{U})$ is a regular space.
(iii) Let $(L, \mathcal{U})$ be normal and S be a closed set in L. Since $\bigcup_{x \in S} F \odot x$ is an open set contains S, there is an open set H such that $S \subseteq H \subseteq \overline{H} \subseteq \bigcup_{x \in S} F \odot x$. Since $S \subseteq H$, the set $\bigcup_{x \in S} F \odot x$ is contained in H. Hence $\bigcup_{x \in S} F \odot x = \overline{H}$ is a closed set. Conversely, let S be a closed set in L and U be an open set contains S. Now $\bigcup_{x \in S} F \odot x$ is a closed and open set such that $S \subseteq \bigcup_{x \in S} F \odot x = \overline{\bigcup_{x \in S} F \odot x} \subseteq U$. Therefore, $(L, \mathcal{U})$ is a normal space.

Definition 4.1 *Let L be a residuated lattice. A nonempty subset V of L is a* prefilter *if for each $x, y \in L$, $x \leq y$ and $x \in V$ imply $y \in V$.*

Example 4.2 *(i) Let I be the residuated lattice in Example 2.1(ii). Then for each $a \in I$ the set $[a, 1]$ is a prefilter. It is clear that it is not filter.*
(ii) Let L be residuated lattice in Example 2.1(iii). Then $V = \{a, b, c, 1\}$ is a prefilter. It is not filter because $a \odot b = 0 \notin V$.

Proposition 4.1 *Let L be a residuated lattice and V be a prefilter in L. Then:*
(i) $F_V = \{x \in V : \forall y \in V \ x \odot y \in V\}$ is a filter,
(ii) for each $x \in V$, $F_V(x) \subseteq V(x)$,
(ii) for each $x, y \in L$, $F_V(x) * F_V(y) \subseteq V(x * y)$, when $* \in \{\odot, , \wedge, \vee\}$.

Proof. (i) Clearly, $1 \in F_V$. Let $x, y \in F_V$ and $z \in V$. Then $y \odot z \in V$ and so $x \odot (y \odot z) \in V$. This implies that $(x \odot y) \odot z \in V$. Hence $x \odot y \in F_V$. Now suppose $x \leq y$ and $x \in F_V$. If $z \in V$, then by (R_8), $x \odot z \leq y \odot z$. Since $x \odot z \in V$, $y \odot z \in V$. Hence $y \in F_V$.
(ii) The proof is easy.
(iii) First we prove that $F_V(x) \odot F_V(y) \subseteq V(x \odot y)$. For this, let $a \in F_V(x)$ and $b \in F_V(y)$. We have

$$\begin{aligned} & (xa)((x \odot y)(a \odot b)) \\ = \ & (xa)(x(y(a \odot b))) \\ = \ & (x \odot (xa))(y(a \odot b)) \\ \geq \ & a(y(a \odot b)) \\ = \ & y(aa \odot b) \geq yb \end{aligned}$$

Since yb and xa, both, belong to F_V, by (i), $(x \odot y)(a \odot b) \in F_V \subseteq V$. In a similar way, we can show that $(a \odot b)(x \odot y) \in V$. Hence $a \odot b \in V(x \odot y)$.

This implies that $F_V(x) \odot F_V(y) \subseteq V(x \odot y)$. Now we show that $F_V(x)F_V(y) \subseteq V(xy)$. Let $a \in F_V(x)$ and $b \in F_V(y)$. We have

$$\begin{aligned} & (yb)((xy)(ab)) \\ = \ & [(xy) \odot (yb)](ab) \\ \geq \ & (xb)(ab) \\ = \ & a((xb)b) \geq ax \end{aligned}$$

Since yb and ax, both, belong to F_V, $(xy)(ab) \in F_V \subseteq V$. In a similar way, we can show that $(ab)(xy) \in V$. Hence $ab \in V(xy)$. Thus $F_V(x)F_V(y) \subseteq V(xy)$. In continue we will show that $F_V(x) \wedge F_V(y) \subseteq V(x \wedge y)$. Let $a \in F_V(x)$ and $b \in F_V(y)$. By (R_5) and (R_9), $(x \wedge y)a \geq xa$ and $(x \wedge y)b \geq yb$. Hence $(x \wedge y)a$ and $(x \wedge y)b$, both, are in F_V and so $[(x \wedge y)a] \vee [(x \wedge y)b] \in F_V$. By (R_{19}),

$$(x \wedge y)(a \wedge b) = [(x \wedge y)a] \vee [(x \wedge y)b].$$

Hence

$$(x \wedge y)(a \wedge b) \in F_V \subseteq V.$$

By the similar way, we get that $(a \wedge b)(x \wedge y) \in V$. Therefore, $a \wedge b \in V(x \wedge y)$. To complete the proof we have to show that $F_V(x) \vee F_V(y) \subseteq V(x \vee y)$. Let $a \in F_V(x)$ and $b \in F_V(y)$. By (R_9), $ax \leq a(x \vee y)$ and $by \leq b(x \vee y)$. Hence $(ax) \wedge (by)$ is less than

$$(a(x \vee y)) \wedge (b(x \vee y)).$$

Since ax and by are in filter F_V, $(a(x \vee y)) \wedge (b(x \vee y)) \in F_V$. By (R_{18}), $(a \vee b)(x \vee y) \in F_V$. In a similar way, $(x \vee y)(a \vee b) \in F_V$. So $a \vee b \in F_V$.

Theorem 4.5 *Suppose Ω is a family of prefilters in a residuated lattice L such that Ω is closed under intersections, and for each $V \in \Omega$, $F_V \in \Omega$.*

Then there is a topology $\mathcal{U}$ on L such that $(L,\mathcal{U})$ is a regular topological residuated lattice. Moreover, Ω is a fundamental system of neighborhoods 1.

Proof. Let $\mathcal{U} = \{U \subseteq L : \forall x \in U \ \exists V \in \Omega \ \ s.t \ \ V(x) \subseteq U\}$. Then it is easy to prove that $\mathcal{U}$ is a topology on L. We show that for any $x \in L$ and $V \in \Omega$, the set $V(x)$ is open and closed. Let $y \in V(x)$. By (ii) of Proposition 4.1, $F_V(xy) \subseteq V$ and $F_V(yx) \subseteq V$. Let $z \in F_V(y)$. By (R_6) and (R_1),

$$yz \leq (zx)(yx)$$

and

$$zy \leq (yx)(zx).$$

Hence $zx \in F_V(yx) \subseteq V$. By the similar way, we can show that $xz \in V$. Hence $z \in V(x)$. This implies that $y \in F_V(y) \subseteq V(x)$, and so $V(x) \in \mathcal{U}$. Now let $y \in \overline{V(x)}$, where $V \in \Omega$. Then there is a $z \in F_V(y) \cap V(x)$. Since $zy \in F_V$ and $xz \in V$, by Proposition 4.1(i), $(xz) \odot (zy) \in V$. By (R_{14}), $xy \in V$. Similarly, we can prove that $yx \in V$. Hence $y \in V(x)$. Therefore $V(x)$ is closed. Since for each $x \in L$ and $V \in \Omega$, the set $V(x)$ is open, by (iii) of Proposition 4.1, the operations $\wedge, \vee, \odot$ and are continuous. Hence $(L,\mathcal{U})$ is a topological residuated lattice. Finally, $(L,\mathcal{U})$ is a regular space because for any $x \in U \in \mathcal{U}$, there is a $V \in \Omega$ such that $x \in V(x) = \overline{V(x)} \subseteq U$. Since for each $V \in \Omega$, $V(1) = V$, it is easy to see that Ω is a fundamental system of neighborhoods of 1.

Definition 4.2 *Let L be a residuated lattice. Then we call filter F in L satisfies* maximum condition *if for each $x \in L$, the set $F(x)$ has a maximum.*

If F is a filter in L which satisfies maximum condition, then we also call F is a *filter with maximum condition.*

Example 4.3 *Let $\mathcal{I}$ be the residuated lattices in Example 2.1 (i), and F be a filter in it. Then for each $x \in I$, $F(x) = \{x\}$ or $F(x) = F$, and so F satisfies the maximum condition.*

Lemma 4.1 *Let F be a filter with maximum condition in a divisible residuated lattice L. Then for each $x \in L$, there exists a $x^* \in L$ such that $F(x) = F \odot x^*$.*

Proof. Let $x \in L$. Since F is a filter with maximum condition, there exists $x^* \in L$ such that $x^* = maxF(x)$. For each $f \in F$, by (R_5), since $x^* \odot f \leq f$ and $f \leq x^* \to (x^* \odot f)$, it follows that $(x^* \odot f) \to f = 1 \in F$ and $x^* \to (x^* \odot f) \in F$. Hence $x^* \odot f \equiv^F x^*$. Since $x^* \in F(x)$, we conclude that $x^* \odot f \in F(x)$. Therefore, $F \odot x^* \subseteq F(x)$. If $z \in F(x)$, since $x^* \in F(x)$, we have $z \in F(x^*)$. Since $z = x^* \wedge z = x^* \odot (x^* \to z)$ and $x^* \to z \in F$, we conclude that $z \in x^* \odot F$. Thus, $F(x) \subseteq F \odot x^*$.

Theorem 4.6 *Let F be a compact filter in a Hausdorff semitopological divisible residuated lattice $(L, \odot, \mathcal{U})$ which satisfies maximum condition. If the set L/F is finite, then $(L, \mathcal{U})$ is a normal and regular space.*

Proof. First we prove that for each $x, y \in L$, $F \odot x = F \odot y$ iff, $x = y$. If $x = y$, clearly $F \odot x = F \odot y$. Let $F \odot x = F \odot y$. Since $x \in F \odot x$, for some $f \in F$, $x = f \odot y$. By (R_5), $x \leq y$. Similarity, $y \leq x$. Hence $x = y$. Now since the set L/F is finite, by Lemma 4.1, there exist $x_1, ..., x_n \in L$ such that $L = (F \odot x_1) \cup ... \cup (F \odot x_n)$. Since the operation $\odot$ is continuous in the first variable, for each $1 \leq i \leq n$, $F \odot x_i$ is compact and so L is compact. Since L is compact and Hausdorff space, we conclude that L is normal and regular space.

Theorem 4.7 *Let F be a fiter with maximum condition in topological divisible residuated lattice $(L, \mathcal{U})$. If the mapping $t_a : L \hookrightarrow L$ by $t_a(x) = a \odot x$ is a closed map and F is a normal subspace of $(L, \mathcal{U})$, then $(L, \mathcal{U})$ is a normal space.*

Proof. Since for each $a \in L$, the mapping $t_a(x) = a \odot x$ is a closed continuous map of F onto $F \odot a$, it follows that $F \odot a$ is a normal subspace of $(L, \mathcal{U})$. Also by Lemma 4.1, for each $x \in L$ there is a $x^* \in L$ such that $F(x) = F \odot x^*$. Thus $L = \cup F \odot x^*$ is a union of disjoint normal spaces which implies that L is normal.

Proposition 4.2 *Let $(L, , \mathcal{U})$ be a T_1 semitopological residuated lattice. Then $\{1\}$ is closed.*

Proof. Let $x \in \overline{\{1\}}$. If $x \neq 1$, then there is an open set U such that $1 \in U$ and $x \notin U$. By Proposition 3, $U(x)$ is an open neighborhood of x. Since $x \in \overline{\{1\}}$, $1 \in U(x)$. Hence $x = 1x \in U$ which is a contradiction.

Proposition 4.3 *Let $(L, \rightarrow, \mathcal{U})$ be a second countable semitopological residuated lattice. If J is an open nontrivial filter in L, then there is a nontrivial topology $\mathcal{V}$ on L caorser than $\mathcal{U}$ such that $(L, \mathcal{V})$ is a metrizable topological residuated lattice.*

Proof. Let $\mathcal{F}$ be a family of open fiters which contains J and is closed under intersection. It is easy to prove that the set $B = \{F(x) : F \in \mathcal{F}, x \in L\}$ is a base for the topology $\mathcal{V} = \{V \subseteq L : \forall x \in V\ \exists F \in \mathcal{F}\ s.t\ F(x) \subseteq V\}$ on L. Since J is a nontrivial filter and for each $x \in J$, $J(x) \subseteq J$, hence $J \in \mathcal{V}$ which implies that $\mathcal{V}$ is a nontrivial topology on L. By Proposition 3, $F(x)$ is in $\mathcal{U}$, for all $F \in \mathcal{F}$ and $x \in L$. Hence $\mathcal{V}$ is coarser than $\mathcal{U}$. As the proof of Proposition 4.1, we can show that $(L, \mathcal{V})$ is a regular topological residuated lattice. By Proposition 4.2, $\{1\}$ is closed and by Proposition 3.1, $(L, \mathcal{V})$ is Hausdorff. Since $(L, \mathcal{V})$ is a T_2 second countable regular topological space, it is metrizable.

Proposition 4.4 *Let Ω be a family of prefilters in residuated lattice L which is closed under intersection. If*
(i) for each $U \in \Omega$ there is a $W \in \Omega$ such that $W[W] \subseteq U$,
(ii) for each $x \neq 0$ there is a $V \in \Omega$ such that $x' \notin V$,
then there is a topology $\mathcal{U}$ on L such that $(L, \odot, , \mathcal{U})$ is a Hausdorff compact topological residuated lattice. Moreover, if $(L, \mathcal{U})$ is second countable, it is metrizable.

Proof. First we prove that the set $B = \{V[x] : V \in \Omega, x \in L\}$ is a base for a topology $\mathcal{U}$ on L. Clearly, $L \subseteq \bigcup B$. Let $V[x]$ and $U[y]$, both, be in B and $a \in V[x] \cap U[y]$. By (i), there is a $W \in \Omega$ such that $W[W] \subseteq V \cap U$. We show that $W[a] \subseteq V[x] \cap U[y]$. Suppose $z \in W[a]$. By (R_5), $(zx)(za) \geq za$, so $zx \in W[za] \subseteq W]W] \subseteq V$. Similarly, $zy \in U$. Hence $W[a] \subseteq V[x] \cap U[y]$. Therefore, B is a base for a topology $\mathcal{U}$ on L. Now we show that $V(x)$ is open, for any $V \in \Omega$ and $x \in L$. For this, suppose $y \in V(x)$ and $W[W] \subseteq V$, for some $W \in \Omega$. Let $z \in W[y]$. By (R_5), $zx \in W[zy] \subseteq W[W] \subseteq V$. Similarly, $xz \in V$. Hence $y \in W[y] \subseteq V(x)$. This shows that $V(x) \in \mathcal{U}$. Let $x, y \in L$, $V \in \Omega$. Then there exists a $W \in \Omega$ such that $W[W] \subseteq V$. As the proof of Proposition 4.1, we can prove that $W[x] \odot W[y] \subseteq V[x \odot y]$. We show that $W(x)W[y] \subseteq V[xy]$. For this, suppose $a \in W(x)$ and $b \in W[y]$. We have

$$\begin{aligned} & (by)((ab)(xy)) \\ = \ & (ab)[(by)](xy)] \\ = \ & (ab)[x((by)y)] \\ \geq \ & (ab)(xb) \\ = \ & x((ab)b) \\ \geq \ & xa \end{aligned}$$

Hence $(ab)(xy) \in W[by] \subseteq V$ and so $W(x)W[y] \subseteq V[xy]$. Therefore is continuous. Let $\{U_i : i \in I\}$

be an open cover of L. Then for some $V \in \Omega$ and $i \in I$, $1 \in V[1] \subseteq U_i$. It is easy to see that $L = V[1]$. Hence $L \subseteq U_i$, which shows that L is compact. Now, we show that $\{0\}$ is closed. Let $x \in \overline{\{0\}}$. If $x \neq 0$, by (ii) there is a $V \in \Omega$ such that $x' \notin V$. Since $V(x)$ is an open neighborhood of x, $0 \in V(x)$ which implies that $x' \in V$, a contradiction. Hence $\{0\}$ is closed. By Proposition 3.1, $(L,\mathcal{U})$ is Hausdorff. Finally, if $(L,\mathcal{U})$ is second countable, then it is metrizable because it is Hausdorff compact second countable.

Proposition 4.5 *Let F be a closed filter with maximum condition in a second countable topological Hausdorff divisible residuated lattice $(L,\mathcal{U})$. If F is a normal subspace of $(L,\mathcal{U})$ and the mapping $t_a(x) = a \odot x$ is a closed map of L into L, then $(L,\mathcal{U})$ is metrizable.*

Proof. Since $(L,\mathcal{U})$ is a topological residuated lattice, the mappings $l_a(x) = a \to x$ and $r_a(x) = x \to a$ of L into L are continuous and so for each $x \in L$, $F(x) = l_x^{-1}(F) \cap r_x^{-1}(F)$ is a closed set in L. Since L is divisible, by Lemma 4.1, for each $x \in L$, there is a $x^* \in L$ such that $F(x) = F \odot x^*$. On the other hand, for each $x \in L$, the mapping $\phi(y) = y \odot x^*$ is a closed continuous map of F onto $F \odot x^*$. Hence $F(x) = F \odot x^*$ is a normal subspace of $(L,\mathcal{U})$. Let $\mathcal{C}$ be the family of pairwise disjoint sets $\{F(x)\}$. Since $L = \cup\mathcal{C}$, residuated lattice L is normal. Since $(L,\mathcal{U})$ is Hausdorff, it is a regular space. Finally, since $(L,\mathcal{U})$ is regular second countable, it is metrizable.

Proposition 4.6 *Let $(L,\mathcal{U})$ be a second countable topological residuated lattice and let for each $a \in L$ the mapping $t_a(x) = a \odot x$ be an open and closed map of L into L. Then $(L,\mathcal{U})$ is metrizable iff, for each $x \neq 1$ and each open neighborhood U of 1, there exists an open set V such that $1 \in V \subseteq \overline{V} \subseteq U$ and $x \notin V$.*

Proof. $(\Leftarrow)$ Let for each $x \neq 1$ and each open neighborhood U of 1, there exists an open set V such that $1 \in V \subseteq \overline{V} \subseteq U$ and $x \notin V$. Then by [[11], Proposition 4.10,] $(L,\mathcal{U})$ is a T_1 space. By Theorem 3.2, $(L,\mathcal{U})$ is a Hausdorff space. We prove that it is regular. For this, let $x \in U \in \mathcal{U}$. Clearly, if $x = 1$, then there is a $V \in \mathcal{U}$ such that $1 \in V \subseteq \overline{V} \subseteq U$. Hence we suppose that $x \neq 1$. Since the operation $\odot$ is continuous, there is an open neighborhood W of 1 such that $x \odot W \subseteq U$. Let V be an open set such that $1 \in V \subseteq \overline{V} \subseteq W$. Since t_x is an open and closed map, the set $x \odot V$ is an open neighborhood of x and $x \odot \overline{V}$ is a closed set. So we have

$$x \in x \odot V \subseteq \overline{x \odot \overline{V}} = x \odot \overline{V} \subseteq x \odot W \subseteq U$$

which prove that $(L,\mathcal{U})$ is a regular space. Since $(L,\mathcal{U})$ is second countable, it is metrizable.
$(\Rightarrow)$ Let $(L,\mathcal{U})$ be metrizable, $\mathcal{U}$ be induced topology by metric d and $x \neq 1$. If U is an open neighborhood of 1, then there exists a $\epsilon > 0$ such that $\{x \in L : d(x,1) < \epsilon\} \subseteq U$. Let $V = \{x \in L : d(x,1) < \frac{\epsilon}{2}\}$. Then clearly, $1 \in V \subseteq \overline{V} \subseteq U$ and $x \notin V$.

Proposition 4.7 *Let $(L,\to,\mathcal{U})$ be a second countable topological residuated lattice and let for each $a \in L$ the mapping $r_a(x) = x \to a$ be an open map of L into L. Then $(L,\mathcal{U})$ is metrizable iff, for each $x \neq 1$ and each open neighborhood U of 1, there exists an open set V such that $1 \in V \subseteq \overline{V} \subseteq U$, $\overline{V}$ is compact and $x \notin V$.*

Proof. $(\Leftarrow)$ Let for each $x \neq 1$ and each open neighborhood U of 1, there exists an open set V such that $1 \in V \subseteq \overline{V} \subseteq U$, $\overline{V}$ is compact and $x \notin V$. Then by [[11], Proposition 4.11,] $(L,\mathcal{U})$ is a Hausdorff space. We prove that it is regular. For this, let $x \in U \in \mathcal{U}$. Clearly, if $x = 1$, then there is a $V \in \mathcal{U}$ such that $1 \in V \subseteq \overline{V} \subseteq U$. Hence we suppose that $x \neq 1$. Since the operation $\to$ is continuous, there is an open neighborhood W of 1 such that $W \to x \subseteq U$. Let V be an open set such that $1 \in V \subseteq \overline{V} \subseteq W$. Since r_x is an open and continuous map, the set $V \to x$ is an open neighborhood of x and $\overline{V} \to x$ is a compact set. Since $(L,\mathcal{U})$ is Hausdorff, the set $\overline{V} \to x$ is closed. So we have

$$V \to x \subseteq \overline{\overline{V} \to x} = \overline{V} \to x \subseteq W \to x \subseteq U$$

which prove that $(L,\mathcal{U})$ is a regular space. Since $(L,\mathcal{U})$ is second countable, it is metrizable.
$(\Rightarrow)$ The proof is similar to the proof of Theorem 4.6. Recall, a topological space $(X,\mathcal{U})$ is locally compact if for each $x \in X$ there is an open neighborhood U of x such that $\overline{U}$ is compact. If $(X,\mathcal{U})$ is Hasudorff, then X is locally compact iff, for each $x \in U \in \mathcal{U}$ there is an open set V such that $x \in V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact. It is easy to see that any compact set is locally compact.[See, [7]]

Proposition 4.8 *Let $(L,\mathcal{U})$ be a second countable Hausdorff residuated lattice and for each $a \in L$, $t_a(x) = a \odot x$ be an open map of L into L. Then $(L,\mathcal{U})$ is metrizable iff, there is a locally compact open neighborhood of* 1.

Proof. $(\Leftarrow)$ Let U be a locally compact open neighborhood of 1. Let $a \in L$. Since $t_a(x) = x \odot a$ is an open map of U onto $U \odot a$, by [[7], Theorem 3.3.15] $U \odot a$ is an open locally compact subset of L. As $L = \cup_{x \in L} U \odot x$ is a union of open locally compact subspaces of L, then by [[7], Exercise 3.3.A] it is a locally compact space. Since $(L,\mathcal{U})$ is Hausdorff and second countable, it is regular and so metrizable.
$(\Rightarrow)$ Let $(L,\mathcal{U})$ be metrizable and $\mathcal{U}$ be topology induced by metric d. Then clearly, the set $\{x \in L : d(x, 1) < r\}$ is a locally compact open neighborhood of 1.

Proposition 4.9 *Let $(L, \to, \mathcal{U})$ be a second countable Hausdorff semitopological residuated lattice. If F is a locally compact closed filter, then $(L,\mathcal{U})$ is metrizable.*

Proof. Let for each $a \in L$, $r_a(x) = x \to a$ and $l_a(x) = a \to x$ be two maps of L into L. Then r_a and l_a, both, are continuous. Since F is locally compact, $r_a^{-1}(F)$ is locally compact because if $x \in r_a^{-1}(F)$, then $r_a(x) \in F$ and so there is an open set V such that $r_a(x) \in V \cap F \subseteq \overline{V \cap F} \subseteq F$. Now

$$x \in r_a^{-1}(V \cap F) \subseteq r_a^{-1}(\overline{V \cap F}) \subseteq r_a^{-1}(F)$$

proves that $r_a^{-1}(F)$ is locally compact. Similarity $l_a^{-1}(F)$ is locally compact. Since $F(a) = r_a^{-1}(F) \cap l_a^{-1}(F)$, and F is locally compact closed, so $F(a)$ is locally compact closed for each $a \in L$. Now since $L = \cup_{a \in L} F(a)$ is a union of locally compact closed subspaces of L, by [[7], Exercise 3.3.B] $(L,\mathcal{U})$ is locally compact. Since $(L,\mathcal{U})$ is Hausdorff and second countable, $(L,\mathcal{U})$ is regular and then metrizable.

Definition 4.3 *Let $\mathcal{U}$ be a topology on a residuated lattice L and $W \subseteq L$. We call:*
(i) W is a n-quasifilter if $x \in W^n$ and $x \leq y$ imply $y \in W^n$,
(ii) W is a quasifilter if for each $n \geq 1$, W is a n-quasifilter,
(iii) W is a quasifilter neighborhood of 1 *if W is a neighborhood of* 1 *which is quasifilter,*
(iv) $\mathcal{W}$ is a QN-fundamental system, if $\mathcal{W}$ is a fundamental system of open neighborhoods 1 *which all are quasifilter.*

Example 4.4 *(i) Every open filter is a quasifilter neighborhood.*
(ii) In Example 2.1, $W = \{1, a, b, c\}$ is a quasifilter neighborhood of 1 *which is not a filter.*
(iii) Let $\mathcal{I}$ be residuated lattice in Example 2.1 (ii). Let $0 \neq a \in \mathcal{I}$ and $W = (a, 1]$. Then for each $n \geq 1$, $W^n = (a^n, 1]$. If $\mathcal{U}$ is subspace topology induced by real number, then W is a quasifilter neighborhood of 1 *which is not a filter. Moreover, $\mathcal{W} = \{(a, 1] : a \in \mathcal{I}\}$ is a QN-fundamental system.*

Proposition 4.10 *Let $\mathcal{W}$ be a QN-fundamental system in the second countable Hausdorff topological residuated lattice $(L,\mathcal{U})$. If for each $a \in L$, $t_a(x) = a \odot x$ is an open map of L into L, then $(L,\mathcal{U})$ is metrizable provided there exists a compact open neighborhood of* 1.

Proof. Let U be a compact open neighborhood of 1 and $x \in U$. Since the operation $\odot$ is continuous, there exist two open neighborhoods V_x and H_x of 1 such that $x \odot V_x \subseteq U$ and $H_x \odot H_x \subseteq V_x$. Clearly, $\{x \odot H_x : x \in U\}$ is an open covering of U. Since U is compact, the union of a finite number sets $\{x_i \odot H_{x_i} : 1 \leq i \leq n\}$ covers U. Let $V = \bigcap_{i=1}^{n} H_{x_i}$. Then V is an open neighborhood of 1 such that

$$U \odot V \subseteq (\bigcup_{i=1}^{n} x_i \odot H_{x_i}) \odot V \subseteq \bigcup_{i=1}^{n} x_i \odot V_{x_i} \subseteq U.$$

Since $U \cap V$ is an open neighborhood of 1, there is a $W \in \mathcal{W}$ such that $W \subseteq U \cap V$. Thus $W^2 \subseteq U \odot V \subseteq U$. But then by induction, $W^n = W^{n-1} \odot W \subseteq U \odot V \subseteq U$. Let $F = \cup_{n \geq 1} W^n$. If $x, y \in F$, then for some n and m, $x \in W^n$ and $y \in W^m$. Hence $x \odot y \in W^{n+m} \subseteq F$. If $x \in F$ and $x \leq y$, then for some $n \geq 1$, $x \in W^n$. Since W is a quasifilter, $y \in W^n$ and so $y \in F$. Thus F is a filter. Since for each $a \in L$, t_a is an open map, F is an open set. If $y \in \overline{F}$, then since $F \odot y$ is an open neighborhood of y, there is a $f \in F$ such that $f \odot y \in F$. Since F is a filter, $y \in F$. Thus F is a closed set in the compact Hausdorff space U and so F is compact. Now by Proposition 4.9, $(L,\mathcal{U})$ is metrizable.

5 Conclusion

In this paper, we studied separation axioms on topological residuated lattices. In Theorem 3.2

we showed that axioms T_0, T_1, T_2 and $T_{5/2}$ are equivalence. In Theorem 3.4 we proved that there is a Hausdorff topological residuated lattice of each infinite cardinality. Axioms regularity and normality were studied in section 4. For the future we suggest to study metrizability, uniformity, quasi-uniformity on this structure.

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