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Research Article

Review of the methods for evaluating congestion in DEA and computing output losses due to congestion

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Abstract

Data Envelopment Analysis (DEA) is a branch of management, concerned with evaluating the performances of homogeneous Decision Making Units (DMUs). The performances of DMUs are affected by the amount of sources that DMUs used. Usually increases in inputs cause increases in outputs. However, there are situations where increase in one or more inputs generate a reduction in one or more outputs. In such situations there is congestion in inputs or production process. In this study, we review the approaches that are available in the DEA literature for evaluating congestion. Also we introduce a model to compute output losses due to congestion. Then, we present the results of the mentioned models on an empirical example and interpret the results.

Keywords : Data Envelopment Analysis; Congestion; Inefficiency; Decision Making Unit.

1 Introduction

DAta Envelopment Analysis (DEA) is a branch of management concerned with evaluating the performances of homogeneous Decision Making Units (DMUs). Charnes, Cooper and Rhodes (CCR) developed data envelopment analysis in 1978 by their famous article [1]. Since 1978 there has been a spurt of extensive investigations on DEA. Today, many scholars all over the world are working in this domain. The performances of DMUs are affected by the amount of sources that DMUs used. Usually increases in inputs cause increases in outputs. But there are situations where increase in one or more inputs generate a reduction in one or more outputs. For example, in an underground coal mine, too many men decreases the output of coal. In such situations there is congestion in inputs or production process. The definition we use is as follows:

Definition 1.1 *Congestion is said to occur when the output that is maximally possible can be increased by reducing one or more inputs without improving any other inputs or outputs. Conversely congestion is said to occur when some of the outputs that are maximally possible are reduced by increasing one or more inputs without improving any other inputs or outputs [19].*

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The first paper that studied congestion was the one by Fare and Svensson [10] in 1980. In that paper, three forms of congestion were defined and described for a production function of single output. Later, Fare and Grosskopf [11] and Fare et al. [12] expanded a data envelopment analysis (DEA) model to compute the impact of congestion. Their model is a radial approach that calculates the congestion impact as ratio of the observed amounts to the expected amounts. It shows only existence or non-existence of congestion but it can not identify congestion correctly in all cases. Because it focuses attention on efficiency computation while congestion is a kind of inefficiency.

Another approach originally studied by Cooper et al. [16] is a slack-based approach that calculates the congestion impact as the difference between the observed amounts and the expected amounts. This approach has some strong points to the previous method. It determines the congested inputs and provides a measure for the amount of congestion in each input. Later, using additive models, Cooper et al. [18] expanded a unified additive model for determining congestion too. However, in both economics and OR studies the speed of progress of investigations into congestion has accelerated after the Fare et al. studies [12].

Jahanshahloo and Khodabakhshi [4] introduced an input relaxation model for improving outputs and calculated the input congestion based on the proposed model.

In addition to the above, another studies done separately by Wei and Yan [8] and Tone and Sahoo [5]. The previous two study, declare the congestion impact in terms of immoderate inputs. According to the definition, congestion occurs when increases in some inputs results in decreases in some outputs. So, congestion can also be determined as shortfalls in outputs. In this way, it is easier to declare the congestion in terms of outputs. The models introduced by Wei and Yan [8] and Tone and Sahoo [5] are expanded from the output viewpoint. Wei and Yan in an another work [9] studied simultaneously the problems of congestion and different kinds of returns to scale by output oriented DEA models and recognized the necessary and sufficient conditions for the evidence of congestion and different kinds of returns to scale.

Sueyoshi and Sekitani [15] presented an approach which is able to assess congestion under the occurrence of multiple solutions.

Jahanshahloo et al. [2] in 2010 presented a new method that considerably reduce the computational effort required for calculating congestion. According to the Definition 1.1, we discover that congestion occurs in large sizes. The idea of this approach is to select the maximum amounts of each input between efficient DMUs. Then, it compares this amounts with inputs of other DMUs. Because efficient DMUs are not congested.

This paper discusses the approaches that are available in the DEA literature for measuring and evaluating congestion. In Section 2, we specify the congestion models. In Section 3, we will discuss calculating output losses due to congestion. Section 4 presents the results of the mentioned methods on an example adopted from Tone and Sahoo [5]. Section 5 provides conclusions and a summary of the review.

2 The congestion models

Suppose we have n DMUs. Each DMU_j $(j = 1, ..., n)$ produces s different outputs, y_{rj} $(r = 1, ..., s)$, using m different inputs, x_{ij} $(i = 1, ..., m)$. As given in Charnes et al. [1] the efficiency of a specific DMU_o can be evaluated by either of the following two DEA models:
Input orientation model:

$$\begin{aligned} \theta^* = \ & \min \theta \\ s.t. \ & \sum_{j=1}^{n} \lambda_j \, x_{ij} \leq \theta x_{io} \quad (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j \, y_{rj} \geq y_{ro} \quad (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j = 1 \,, \lambda_j \geq 0 \quad (j = 1, ..., n). \end{aligned} \tag{2.1}$$

Output orientation model:

$$\begin{aligned} \varphi^* = \ & \max \varphi \\ s.t. \ & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} \quad (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi y_{ro} \quad (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 \quad (j = 1, ..., n). \end{aligned} \tag{2.2}$$

Where x_{io} and y_{ro} are respectively the ith input and the rth output for the DMU_o under evaluation. Corresponding to the $m+s$ input and output constraints in (2.1) or (2.2), some non-zero input and output slacks, s_i^- and s_r^+, may exist in some multiple optimal solutions. Afterwards, the following models is applied:

$$\begin{aligned} \max \ & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\ s.t. \ & \sum_{j=1}^{n} \lambda_j x_{ij} - s_i^- = \theta^* x_{io} \quad (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j y_{rj} + s_r^+ = y_{ro} \quad (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 \quad (j = 1, ..., n), \\ & s_i^- \geq 0 \quad (i = 1, ..., m), \\ & s_r^+ \geq 0 \quad (r = 1, ..., s). \end{aligned} \tag{2.3}$$

$$\begin{aligned} \max \ & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\ s.t. \ & \sum_{j=1}^{n} \lambda_j x_{ij} - s_i^- = x_{io} \quad (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j y_{rj} + s_r^+ = \varphi^* y_{ro} \quad (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 \quad (j = 1, ..., n), \\ & s_i^- \geq 0 \quad (i = 1, ..., m), \\ & s_r^+ \geq 0 \quad (r = 1, ..., s). \end{aligned} \tag{2.4}$$

Definition 2.1 *An optimal solution of s_i^- and s_r^+ in (2.3) and (2.4) are respectively called DEA input and output slack values.*

Definition 2.2 *A DMU evaluated in the above manner is called DEA efficient if and only if the following conditions are satisfied:*

(a) $\theta^* = 1$ (*or* $\varphi^* = 1$)

(b) $s_i^{-*} = s_r^{+*} = 0 \quad (\forall i, r)$

According to the input disposability postulate, we have:

$$(x, y) \in PPS, \ \bar{x} \geq x \ \Rightarrow \ (\bar{x}, y) \in PPS$$

When there exists congestion, we have a point that increases in its inputs reduce the outputs. Therefore, the above postulate is not consistent. This is the reason that the models studied congestion usually apply T_{NEW}, where:

$$T_{NEW} = \{(x, y) \mid \ x = \sum_{j=1}^{n} \lambda_j x_j, \ y \leq \sum_{j=1}^{n} \lambda_j y_j,$$

$$\sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geq 0; \ j = 1, ..., n\}$$

2.1 The FGL model

Models (2.1) and (2.2) satisfies strong disposability. It means that input disposability and output disposability, both are consistent. If we suppose only input disposability, according to FGL [12] we have the following models to apply in a two-stage evaluation of congestion.

Input orientation:

$$\begin{aligned} \beta^* = \ & \min \beta \\ s.t. \ & \sum_{j=1}^{n} \lambda_j \ x_{ij} \ = \ \beta x_{io} \quad (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j \ y_{rj} \ \geq \ y_{ro} \quad (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j \ = \ 1 \ , \lambda_j \geq 0 \quad (j = 1, ..., n). \end{aligned} \tag{2.5}$$

Output orientation:

$$\begin{aligned} \hat{\beta}^* = \ & \min \hat{\beta} \\ s.t. \ & \sum_{j=1}^{n} \lambda_j \ x_{ij} \ = \ \tau x_{io} \quad (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j \ y_{rj} \ = \ \hat{\beta} y_{ro} \quad (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j \ = \ 1 \ , \lambda_j \geq 0 \quad (j = 1, ..., n) \\ & 0 \leq \tau \leq 1. \end{aligned} \tag{2.6}$$

The distinction between models (2.1) and (2.5) is that input inequalities are altered into input equalities. Hence, non-zero slack can not be correspond to any input. Then, the input congestion measure is defined in input and output orientation [12] via the following fractions:

$$C(\theta^*, \beta^*) = \frac{\theta^*}{\beta^*} \quad , \quad C(\varphi^*, \hat{\beta}^*) = \frac{\varphi^*}{\hat{\beta}^*}$$

Notice that $\theta^* \leq \beta^*$ or $\varphi^* \geq \hat{\beta}^*$. If $C(\theta^*, \beta^*) = 1$ or $C(\varphi^*, \hat{\beta}^*) = 1$, then, input is not congested and if $C(\theta^*, \beta^*) < 1$ or $C(\varphi^*, \hat{\beta}^*) > 1$, then, input is congested. The above congestion measure is strongly dependent on the orientation of DEA models applied. The FGL approach has two essential deficiency:

(a) It can show congestion to be present when this is not consistent with the observed behavior.

(b) It can also fail to exhibit congestion when the data show it to be present.

The reasons of these deficiencies is that the procedure followed by FGL focuses attention on efficiency measurement and assumes identification of sources and amounts of inefficiencies as an redundant duty [22].

2.2 The BCSW approach

This approach first was published by Cooper et al. [16] in 1996. Then, Brockett et al. [7] in 1998 examined it on real data and expanded it to check tradeoffs between employment and output which could be used to increase employment or increase output (or both) in Chinese production. This approach also progresses in a two-stage way. It's idea is, initially, to define the projection point on the efficiency frontier via (2.1) and (2.3) (or (2.2) and (2.4)). The projection point of DMU_o in input and output oriented is respectively:

$$\hat{x}_o = \sum_{j=1}^{n} \lambda_j^* \ x_j = \theta^* x_o - s^{-*},$$
$$\hat{y}_o = \sum_{j=1}^{n} \lambda_j^* \ y_j = y_o + s^{+*}.$$

and

$$\hat{x}_o = \sum_{j=1}^{n} \lambda_j^* \ x_j = x_o - s^{-*},$$
$$\hat{y}_o = \sum_{j=1}^{n} \lambda_j^* \ y_j = \varphi^* y_o + s^{+*}.$$

Afterwards, the outputs are fixed to those of the projection point and the maximum amount of inputs that can be augmented to the projection's inputs are computed by the following model:

$$\begin{aligned} \max \ & \sum_{i=1}^{m} \delta_i^{\ +} \\ s.t. \ & \sum_{j=1}^{n} \lambda_j \ x_{ij} - \delta_i^{\ +} = \hat{x}_{io} \quad (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j \ y_{rj} = \hat{y}_{ro} \quad (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j = 1 \ , \lambda_j \geq 0 \quad (j = 1, ..., n), \\ & 0 \leq \delta_i^{\ +} \leq s_i^{-*} \quad (i = 1, ..., m). \end{aligned} \tag{2.7}$$

Consequently, the amount of congestion in each input can be determined by the difference between each pair of s_i^{-*} and δ_i^{+*}, where δ_i^{+*} are optimal values in (2.7). That is:

$$s_i^c = s_i^{-*} - \delta_i^{+*} \quad (i = 1, ..., m) \tag{2.8}$$

We substitute the above equality by the following equation: $s_i^{-*} = \delta_i^{+*} + s_i^c$. The "total slack" obtained in stage 1, represented by s_i^{-*}, is separated into a value δ_i^{+*}, indicating a "technical inefficiency" component, and a value s_i^c , indicating a "congesting" component in input i.

It should be noted that using an input oriented BCSW approach usually is not resultful. The reason is that in this case, moving in a surface with fixed outputs causes the input slacks reach to its maximum value, So, the output value that is calculated for the projection point is equal to the output of DMU_o and output slacks become zero. This indicating pure technical inefficiency and no congestion, because input reduction do not alter the output.

2.3 The unified additive model approach

The BCSW approach fail to provide an "overall" measure of congestion. Therefore, Cooper et al. [18] in 2000 presented an approach in which additive models are used for both congestion and inefficiency analysis. This approach progresses in a two-stage way too. At first, the following model is applied:

$$\begin{array}{lll}
\max & \frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}} + \varepsilon\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{io}} & \\
s.t. & \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io} & (i = 1, ..., m), \\
& \sum_{j=1}^{n}\lambda_j y_{rj} - s_r^+ = y_{ro} & (r = 1, ..., s), \\
& \sum_{j=1}^{n}\lambda_j = 1 \ , \ \lambda_j \geq 0 & (j = 1, ..., n), \\
& s_i^- \geq 0 & (i = 1, ..., m), \\
& s_r^+ \geq 0 & (r = 1, ..., s).
\end{array} \tag{2.9}$$

Where $\varepsilon > 0$ is a non-Archimedean element and gives priority to maximizing $\sum_{r=1}^{s} s_r^+/y_{ro}$. It means that the model first maximizes outputs after which it tries to recognize all input congestion that may be exist. This is done as follows using the results from model (2.9).

$$\begin{array}{lll}
\max & \frac{1}{m}\sum_{i=1}^{m}\frac{\delta_i^+}{x_{io}} & \\
s.t. & \sum_{j=1}^{n}\lambda_j \ x_{ij} - \delta_i^+ = \hat{x}_{io} & (i = 1, ..., m), \\
& \sum_{j=1}^{n}\lambda_j \ y_{rj} = \hat{y}_{ro} & (r = 1, ..., s), \\
& \sum_{j=1}^{n}\lambda_j = 1 \ , \lambda_j \geq 0 & (j = 1, ..., n), \\
& 0 \leq \delta_i^+ \leq s_i^{-*} & (i = 1, ..., m), \\
& s_i^- \geq 0 & (i = 1, ..., m), \\
& s_r^+ \geq 0 & (r = 1, ..., s).
\end{array} \tag{2.10}$$

where $\hat{x}_{io} = x_{io} - s_i^{-*}$ and $\hat{y}_{ro} = y_{ro} + s_r^{+*}$ and s_i^{-*} and s_r^{+*} are the optimal slacks in (2.9). Then, equation (2.8) determines (s_i^c) the amount of congestion in input i.

2.4 The one-model approach of Cooper

Cooper et al. [23] replaced the two-model approach with a single model. They used $s_i^c = s_i^{-*} - \delta_i^{+*}$ in input constraint of model (2.7) and wrote the following model:

$$\begin{array}{lll}
\max & \varphi + \varepsilon(\sum_{r=1}^{s} s_r^+ - \sum_{i=1}^{m} s_i^c) & \\
s.t. & \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^c = x_{io} & (i = 1, ..., m), \\
& \sum_{j=1}^{n}\lambda_j y_{rj} - s_r^+ = \varphi y_{ro} & (r = 1, ..., s), \\
& \sum_{j=1}^{n}\lambda_j = 1, \ \lambda_j \geq 0 & (j = 1, ..., n), \\
& s_i^c \geq 0 & (i = 1, ..., m), \\
& s_r^+ \geq 0 & (r = 1, ..., s).
\end{array} \tag{2.11}$$

Afterwards, based on the following theorem [23], the amount of congestion in input i (s_i^{c*}) is recognized:

Theorem 2.1 *Congestion is present if and only if in an optimal solution* $(\varphi^*, \lambda^*, s^{+*}, s^{c*})$ *of (2.11) at least one of the following two conditions is satisfied:*

(a) $\varphi^* > 1$ *and there is at least one* $s_i^{c*} > 0$ $(1 \le i \le m)$,

(b) There exist at least one s_r^{+*} $(1 \le r \le s)$ *and at least one* $s_i^{c*} > 0$ $(1 \le i \le m)$.

2.5 Jahanshahloo and Khodabakhshi approach

In DEA models the changes which are done in inputs proportions, are often based on input reduction. Apparently, this subject is a reasonable justification economically because of reducing the costs of the reduced inputs. However, in some cases, input reduction such as labor may be faced with social tensions. Therefore, causing suitable changes to determining an input combination which is fitting with the condition of a society is necessary toward increasing outputs. Jahanshahloo and Khodabakhshi [4] with attending to this subject, proposed the following model which is called "input relaxation model":

$$\begin{array}{lll} \max & \varphi + \varepsilon(\sum_{i=1}^{m} s_{i1}^- + \sum_{r=1}^{s} s_r^+ - \sum_{i=1}^{m} s_{i2}^+) & \\ s.t. & \sum_{j=1}^{n} \lambda_j x_{ij} + s_{i1}^- - s_{i2}^+ = x_{io} & (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j \, y_{rj} - s_r^+ = \varphi y_{ro} & (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \ge 0 & (j = 1, ..., n), \\ & s_{i1}^- , s_{i2}^+ \ge 0 & (i = 1, ..., m), \\ & s_r^+ \ge 0 & (r = 1, ..., s). \end{array} \tag{2.12}$$

Here, s_{i1}^{-*} and s_{i2}^{+*} are respectively the slacks for decrement and increment of ith input. The objective function is defined so that at most possible amount is decreased from ith input or at least useful amount is increased to it. After solving model (2.12), the projection point of DMU_o is computed as follows:

$$\hat{x}_o = \sum_{j=1}^{n} \lambda_j^* \, x_j = x_o - s_1^{-*} + s_2^{+*},$$

$$\hat{y}_o = \sum_{j=1}^{n} \lambda_j^* \, y_j = \varphi^* y_o + s^{+*}.$$

Subsequently, the following model is solved for determining input congestion:

$$\begin{array}{lll} \max & \sum_{i=1}^{m} \delta_i^+ & \\ s.t. & \sum_{j=1}^{n} \lambda_j \, x_{ij} - \delta_i^+ = \hat{x}_{io} & (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j \, y_{rj} = \hat{y}_{ro} & (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j = 1 \ , \ \lambda_j \ge 0 & (j = 1, ..., n), \\ & 0 \le \delta_i^+ \le s_{i1}^{-*} & (i = 1, ..., m). \end{array} \tag{2.13}$$

Finally, congestion amount is defined as below:

$$s_i^c = s_{i1}^{-*} - \delta_i^{+*} \qquad (i = 1, ..., m)$$

2.6 The one-model approach of Khodabakhshi

Khodabakhshi [6] provided a one model approach of input congestion based on input relaxation model that reduces computation time in practical applications. The model is:

$$\begin{array}{lll} \max & \varphi + \varepsilon(\sum_{i=1}^{m} s_r^+ - \sum_{r=1}^{s} s_i^c - \sum_{i=1}^{m} s_{i2}^+) & \\ s.t. & \sum_{j=1}^{n} \lambda_j \, x_{ij} + s_i^c - s_{i2}^+ = x_{io} & (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j \, y_{rj} - s_r^+ = \varphi y_{ro} & (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \ge 0 & (j = 1, ..., n), \\ & s_i^c , s_{i2}^+ \ge 0 & (i = 1, ..., m), \\ & s_r^+ \ge 0 & (r = 1, ..., s). \end{array} \tag{2.14}$$

Then, the following theorem [6] determines the congestion amount of ith input:

Theorem 2.2 *Congestion is present if and only if for an optimal solution* $(\varphi^*, \lambda^*, s^{c*}, s_2^{+*}, s^{+*})$ *of (2.14), there is at least one* $s_i^{c*} > 0$ $(1 \leq i \leq m)$.

2.7 The Wei and Yan approaches

Wei and Yan [8] studied necessary and sufficient condition for the existence of congestion and returns to scale simultaneously with regards to DEA efficiency under four ordinary output-oriented DEA models and NEW model. They restricted the study on congestion to the technically efficient DMUs, which are all on the boundary of a production possibility set, because an internal point often shows congestion. First, they presented the following definition:

Definition 2.3 *Let* DMU_o *be weakly NEW efficient. If there exists* $(\hat{x}, \hat{y}) \in T_{NEW}$*, such that* $\hat{x} \leq x_o$, $\hat{x} \neq x_o$, $\hat{y} > y_o$*, then,* DMU_o *is said to have evidence of strong congestion.*

Assuming that a DMU is weakly NEW efficient, they discussed the necessary and sufficient conditions for a DMU to have returns to scale or evidence of congestion as follows:

(a) A DMU evidences congestion if and only if it is neither weakly FG nor weakly ST efficient.

(b) A DMU has constant returns to scale if and only if it is weakly CCR efficient, if and only if it is both weakly FG and weakly ST efficient.

(c) A DMU has increasing returns to scale if and only if it is weakly ST efficient but not weakly FG efficient.

(d) A DMU has decreasing returns to scale if and only if it is weakly FG efficient but not weakly ST efficient.

From the above results, they said that congestion is a particular type of returns to scale. A DMU which has increasing or constant, or decreasing returns to scale still has increasing output when the input increase. But the congestion implies that the output decrease as the input increase, or the output increases as the the input decreases, and so can be identified as a "negative" returns to scale [8]. Then, Wei and Yan expanded preceding definition [9] to a more general form by defining "weak congestion", wherein congestion occurs when the reduction of some inputs result in some, not necessarily all, outputs to increase without worsening others. In real activities, production changes often include only some, not necessarily all, inputs and outputs. Then, they define output efficiency for DMUs under a series of ordinary DEA output additive models that maximizes sum of shortfall in outputs. They applied BCC, FG and ST type of these models. Assuming that a DMU is weakly NEW output efficient, they presented necessary and sufficient conditions for the existence of weak congestion under each of these three models as follows:

(a) A DMU evidences weak congestion if and only if it is not BCC output efficient.

(b) A DMU evidences congestion if and only if it is neither FG output efficient nor ST output efficient.

2.8 Tone and Sahoo method

Tone and Sahoo [5] have investigated a relation between returns to scale and congestion, because these two economic concepts are strictly related to each other. To continue the discussion, they assumed the strongly efficient DMUs in T_{NEW} and for inefficient DMUs, they projected them onto the efficient frontier of T_{NEW} and presented the following definition:

Definition 2.4 *A* DMU_o *is strongly congested if there exists an activity* $(\tilde{x}_o, \tilde{y}_o) \in T_{NEW}$ *such that* $\tilde{x}_o = \alpha x_o$ $(0 < \alpha < 1)$ *and* $\tilde{y}_o \geq \beta y_o$ $(\beta > 1)$.

Then, they prove that "strong congestion" is identified when upper scale elasticity $(\bar{\rho})$, that is measured by the following model, is negative.

$$\begin{array}{rl} \bar{\rho} = & \max \ 1 + w \\ s.t. & u\,Y - v\,X + ew \leq 0\ , \\ & u\,y_o - v\,x_o + w = 0\ , \\ & u\,y_o = 1\ , \\ & u \geq 0. \end{array} \tag{2.15}$$

The definition of strong congestion is too limited because in this definition a balanced reduction in all inputs guarantees an increase in all outputs. Therefore, Tone and Sahoo [5] presented a new definition of weak congestion:

Definition 2.5 *A DMU_o is weakly congested if there exist an activity in T_{NEW} that uses less resources in one or inputs for making more products in one or more outputs.*

Subsequently, they demonstrated the following theorem for identifying weak congestion of a DMU that is efficient with respect to the T_{NEW}.

Theorem 2.3 *DMU_o is weakly congested if and only if it has $\varphi^* > 1$ or $\varphi^* = 1$ and $s^{+*} \neq 0$) by the output orientated of BCC model.*

2.9 Sueyoshi and Sekitani method

The previous methods consider a unique optimal solution in evaluating congestion in DEA. When multiple solutions occur in models for identifying congestion, the results obtained from the previous studies are all troublesome from theoretical and applicable viewpoints. So, Sueyoshi and Sekitani [15] discussed how to handle the occurrence of multiple solutions in evaluating congestion and proposed a new approach for the congestion computation and compared the presented approach with Tone and Sahoo (TS) approach. The approach proposed by TS [5] projects all DMUs onto strongly efficient boundary of T_{NEW}. The problem is that Tone and Sahoo considered unique optimal solutions in their approach and did not assume an occurrence of multiple projections. Therefore, Sueyoshi and Sekitani (SS) omitted the hypothesis of being strongly efficient in T_{NEW} and presented a new definition:

Definition 2.6 *A DMU is widely congested if it exists on the boundary of T_{NEW} and there exists an activity in T_{NEW} that uses less resources in one or more inputs to make more products in one or more outputs.*

After that, assuming an occurrence of multiple projected points, Sueyoshi and Sekitani [15] proposed the following procedure for recognizing wide congestion that is composed of two steps each contains one linear programming problem.

Step 1: Choose $\delta > 0$ arbitrarily (where δ is a real number) and solve the following problems:

$$\begin{aligned}
\max \quad & \beta + \sum_{r=1}^{s} s_r^+ \\
s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + w \leq 0 \ (\forall j), \\
& \sum_{r=1}^{s} u_r y_{ro} = 1 \ , \\
& \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} \quad (i = 1, ..., m), \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \varphi y_{ro} \quad (r = 1, ..., s), \\
& \sum_{j=1}^{n} \lambda_j = 1 \ , \\
& \sum_{i=1}^{m} v_i x_{io} - w = \varphi \ , \\
& v_i x_{io} - \beta \geq 0 \quad (i = 1, ..., m), \\
& \beta \leq \delta \\
& s_r^+ \ , \ u_r \ , \ \lambda_j \geq 0.
\end{aligned} \tag{2.16}$$

Let $(\lambda^*, \varphi^*, s^{+*}, v^*, u^*, w^*, \beta^*)$ be an optimal solution of (2.16), then, wide congestion for the projected point $(x_o, \varphi^* \, y_o)$ id identified as follows:

(a) If $\beta^* < 0$, then, $(x_o, \varphi^* y_o)$ is widely congested.

(b) If $\beta^* > 0$, then, $(x_o, \varphi^* y_o)$ is not widely congested.

(c) If $\beta^* = 0$ and $\sum_{r=1}^{s} s_r^{+*} > 0$, then, $(x_o, \varphi^* y_o)$ is widely congested.

(d) If $\beta^* = 0$ and $\sum_{r=1}^{s} s_r^{+*} = 0$, then, go to step 2.

Table 1: Data set of chain stores.

DMU=Year	Number	Area	Sales
$SMU_1 = 1975$	2412	5480	41,091
$SMU_2 = 1976$	3163	6233	48,367
$SMU_3 = 1977$	3350	6798	56,000
$SMU_4 = 1978$	3371	7274	60,940
$SMU_5 = 1979$	3778	7992	69,046
$SMU_6 = 1980$	4020	8500	77,347
$SMU_7 = 1981$	5029	9246	85,805
$SMU_8 = 1982$	5164	9639	90,433
$SMU_9 = 1983$	5285	9981	95,640
$SMU_{10} = 1984$	5618	10,276	100,257
$SMU_{11} = 1985$	5981	10,521	105,944
$SMU_{12} = 1986$	6217	10,766	109,857
$SMU_{13} = 1987$	6455	11,144	116,114
$SMU_{14} = 1988$	6674	11,418	125,404
$SMU_{15} = 1989$	6829	11,717	131,862
$SMU_{16} = 1990$	6995	11,987	140,817
$SMU_{17} = 1991$	7338	12,463	150,583
$SMU_{18} = 1992$	7946	13,426	152,943
$SMU_{19} = 1993$	8236	14,147	155,128
$SMU_{20} = 1994$	7722	15,014	158,714
$SMU_{21} = 1995$	7727	15,022	161,739
$SMU_{22} = 1996$	7822	16,191	169,786
$SMU_{23} = 1997$	7531	16,969	167,195
$SMU_{24} = 1998$	7201	17,627	167,187
$SMU_{25} = 1999$	7281	18,364	165,480
$SMU_{26} = 2000$	7053	19,698	162,847
$SMU_{27} = 2001$	6067	16,176	154,671

Step 2: Solve the following model:

$$
\begin{aligned}
\max \quad & \alpha \\
s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + w \le 0 \ (\forall j), \\
& \sum_{r=1}^{s} u_r y_{ro} = 1 \ , \\
& \sum_{i=1}^{m} \ v_i x_{io} - w = \varphi^*, \\
& u_r y_{ro} - \alpha \ge 0 \quad (r = 1, ..., s), \\
& v_i \ge 0 \quad (i = 1, ..., m) \ , \\
& u_r \ge 0 \quad (r = 1, ..., s), \ \alpha \ge 0.
\end{aligned} \tag{2.17}
$$

Based on the optimal objective value of (2.17), the wide congestion of DMU_o is identified as follows:

(a) If $\alpha^* > 0$, then, $(x_o, \varphi^* \ y_o)$ is not widely congested.

(b) If $\alpha^* = 0$, then, $(x_o, \varphi^* \ y_o)$ is widely congested.

2.10 The new method

Noura, Jahanshahloo, Hosseinzade Lotfi, Rashidi and Parker [2] present a new method that requires fewer calculations and compare its performance with those of existing methodologies.

In this method, first the models (2.2) and (2.4) is solved for each DMU_j $(j = 1, ..., n)$ and the optimal solution $(\varphi^*, \lambda^*, s^{-*}, s^{+*})$ is obtained. Afterwards, denoting the φ^* corresponding to DMU_j by φ_j^*, the set E is defined as follows:

$$E = \{j \mid \varphi_j^* \ = \ 1\}$$

Among the DMUs in set E, there exists at least one that has the highest consumption in its first

Table 2: Results of the methods for existence or non-existence of congestion.

DMU	FGL Input O	FGL output O	Wei-Yan Strong C	Wei-Yan Weak C	Tone-Sahoo Strong C	Tone-Sahoo Weak C	Sueyoshi Sekitani
1975							
1976	√	√	√	√		√	√
1977	√		√	√		√	√
1978							
1979							
1980							
1981	√						
1982							
1983							
1984							
1985	√		√	√		√	√
1986	√	√	√	√		√	√
1987	√	√	√	√		√	√
1988	√	√	√	√		√	√
1989	√		√	√		√	√
1990							
1991							
1992	√	√	√	√	√	√	
1993	√	√	√	√		√	√
1994			√	√		√	√
1995			√	√		√	√
1996							
1997			√	√	√	√	√
1998							
1999	√	√	√	√	√	√	√
2000	√	√	√	√		√	√
2001							

input component compared with the first input component of the remaining DMUs of set E. That is to say,

$$\exists\, l,\ l \in E\ ;\ \forall j \quad (j \in E)\ \Rightarrow\ x_{1l} \geq x_{1j}$$

We denote x_{1l} by x_1^*. Then, we find, among the DMUs in E, a DMU that has the highest consumption in its second input component compared to the remaining DMUs in E. In other words,

$$\exists\, t,\ t \in E\ ;\ \forall j \quad (j \in E)\ \Rightarrow\ x_{2t} \geq x_{2j}$$

We denote x_{2t} by x_2^*. In a similar manner, for all input components, i=1,...,m, we can identify a DMU in E whose input consumption is higher than that of all other DMUs in the set E. Such an input is denoted by x_i^* $(i = 1, ..., m)$, and the the input congestion is determined as follows:

Definition 2.7 *Congestion is present if and only if, in an optimal solution* $(\varphi^*, \lambda^*, s^{-*}, s^{+*})$ *of (2.4) for evaluating* DMU_o, *at least one of the following two conditions is satisfied:*

(a) $\varphi^* > 1$, *and there is at least one* $x_{io} > x_i^*$ $(i = 1, ..., m)$.

(b) There exists at least one $s_r^{+*} > 0$ $(r = 1, ..., s)$ *and at least one* $x_{io} > x_i^*$ $(i = 1, ..., m)$.

Finally, the amount of congestion in the ith input of DMU_o is denoted by $s_i^{c'}$ and is defines as:

$$s_i^{c'} = x_{io} - x_i^* \quad (i = 1, ..., m)$$

Congestion is not present when $x_{io} \leq x_i^*$. Also, sum of all $s_i^{c'}$ $(i = 1, ..., m)$ is the total amount of congestion in DMU_o.

Table 3: Results of Cooper et al. methods and Jahanshahloo et al. methods.

DMU	BCSW Input O	BCSW Output O	Additive	One-model Cooper et al.	Jahanshahloo Khodabakhshi	One-model Khodabakhshi	New method
1975							
1976	√	√	√	√			
1977	√	√	√	√			
1978							
1979							
1980							
1981	√						
1982							
1983							
1984							
1985	√	√	√	√			
1986	√	√	√	√			
1987	√	√	√	√			
1988	√	√	√	√			
1989	√	√	√	√			
1990	√						
1991							
1992	√	√	√	√	√	√	√
1993	√	√	√	√	√	√	√
1994		√	√	√			
1995		√	√	√			
1996							
1997		√	√	√	√	√	
1998							
1999	√	√	√	√	√	√	√
2000	√	√	√	√	√	√	√
2001							

Then, they presented three theorems, In the first theorem they defined DMU_o^* as: $DMU_o^* = (x_1^*,\ x_2^*,\ ...,\ x_m^*,\ \varphi^* y_{1o} + s_1^{+*},\ \varphi^* y_{2o} + s_2^{+*},\ ...,\ \varphi^* y_{so} + s_s^{+*})$ and proved that this virtual DMU is in the production possibility set.

In the second theorem they demonstrated that the congestion calculated by Cooper et al. [23], (s_i^{c*}), is equal to that calculate by this new method, $(s_i^{c'})$, where $x_{io} > x_i^*$.

In the third theorem they showed that congestion is not present in DMU_o when $x_{io} \leq x_i^*,\ \forall i\ (i = 1, ..., m)$.

3 Output reduction due to input congestion

Identifying and eliminating congestion has two advantages:

1. Congestion there exists in inputs and inputs have costs, hence, eliminating congestion minimizes the cost of production.

2. According to the Definition 1.1, congestion causes reduction in outputs, therefore, eliminating congestion increases outputs.

Suppose that using the approaches declared in previous section, we have identified congested inputs and the amount of congestion in each input. Now we want to calculate the outputs losses due to input congestion.

Table 4: The amounts of congestion in different models.

DMU	BCSW-O		J-Kh		New method	
	s_1^c	s_2^c	s_1^c	s_2^c	s_1^c	s_2^c
1975						
1976	219.81					
1977	8. 25					
1978						
1979						
1980						
1981						
1982						
1983						
1984						
1985	12.94					
1986	76.11					
1987	47.46					
1988	73.17					
1989	17.25					
1990						
1991						
1992	482.98		124		124	
1993	679.37		414		414	
1994	52.81					
1995	56.77					
1996						
1997		105.09		778		
1998				1436		
1999		921.99		2173		737
2000		2260.37		3507		2071
2001						

That is, we are going to see that eliminating congestion causes how much increases in outputs.

The congested DMU may have some output losses due to inefficiency. Since we are going to calculate output losses due to congestion, therefore, we project DMU_o on the boundary of T_{NEW} with output oriented NEW model:

$$\begin{aligned} \max \quad & \varphi \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j \ x_{ij} = x_{io} \qquad (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j \ y_{rj} \geq \varphi \ y_{ro} \quad (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j = 1 \ , \ \lambda_j \geq 0 \quad (j = 1, ..., n). \end{aligned} \tag{3.18}$$

Projection point $(\hat{x}_o, \hat{y}_o) = (x_o, \varphi^* y_o)$ is on the boundary of T_{NEW} and all of its output losses is due to congestion. The amount of congestion in ith input (s_i^c) is definite, so, subtracting this value from ith input, we find the maximum amount that augmented to rth output. Subsequently, we

Table 5: Results of output reduction.

DMU	Cooper et al. methods	New method
1975		
1976	4530.82	
1977	169.94	
1978		
1979		
1980		
1981		
1982		
1983		
1984		
1985	266.72	
1986	4827.12	
1987	1162.97	
1988	8793.64	
1989	355.53	
1990		
1991		
1992	2600.39	667.64
1993	4129.31	2700.52
1994	284.31	
1995	305.65	
1996		
1997	226.17	
1998		
1999	1984.28	1586.15
2000	2706.52	2479.77
2001		

use the following model:

$$\begin{array}{lll} \max & \sum_{r=1}^{s} \xi_r & \\ s.t. & \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - s_i^c & (i = 1, ..., m), \\ & \sum_{j=1}^{n} \lambda_j y_{rj} = \varphi^* y_{ro} + \xi_r & (r = 1, ..., s), \\ & \sum_{j=1}^{n} \lambda_j = 1 \ , \ \lambda_j \geq 0 & (j = 1, ..., n), \\ & \xi_r \geq 0 & (r = 1, ..., s). \end{array} \tag{3.19}$$

After solving model (3.19), ξ_r^* indicates the rth output losses due to congestion and the optimum value of the objective function $\sum_{r=1}^{s} \xi_r^*$, indicates the total amount of output losses due to congestion. To more clear the discussion, we apply the proposed approach to a simple example in the case of one input and one output.

Example 3.1 *Consider following DMUs:* $A(1,1),\ B(2,3),\ C(4,3),\ D(5,\frac{3}{2}),\ E(\frac{13}{3},2)$

Using the new method we find that D and E are congested. The amount of congestion in D's input is 1 and in E's input is $\frac{1}{3}$. *Thus,* $s_D^c = 1$ *and* $s_E^c = \frac{1}{3}$. *D is on the frontier of* T_{NEW}, *so all output losses of D is due to congestion. Applying model (3.19) for D we have:*
$\lambda_A^* = \lambda_B^* = \lambda_D^* = \lambda_E^* = 0\ ,\ \lambda_C^* = 1\ ,\ \xi^* = \frac{3}{2}$, *that* $\xi^* = 1.5$ *shows correctly output losses of D due to congestion.*

But E is not on the frontier of T_{NEW}, *so it has some output losses due to inefficiency. First using (3.18), we project E on the frontier of* T_{NEW}.

Then, we have $\varphi^* = \frac{5}{4}$ *and the projection point is* $(\hat{x}_E, \hat{y}_E) = (\frac{13}{3}, \frac{5}{2})$*. Now we evaluate* $(\hat{x}_E, \hat{y}_E)$ *with model (3.19), the optimal solution is:*
$\lambda_A^* = \lambda_B^* = \lambda_D^* = \lambda_E^* = 0\ ,\ \lambda_C^* = 1\ ,\ \xi^* = \frac{1}{2}$,
that $\xi^* = \frac{1}{2}$ *shows correctly output losses of E due to congestion.*

4 Empirical application

In this Section, we evaluate congestion in the performance of a set of chain stores in Japan for a period of 27 years from 1975 through 2001. This data that adopted from [5] have one output: annual sales (unit: hundred million yen), and two inputs: the number of stores and the total area of stores (unit: 1000 m^2). Table 1 reports such data.

As is seen in Table 1, there is a slow but steady rise in the number of chain stores until 1993, after which the trend continues declining consistently. Except the last year, the total area is consistently rising throughout. The output, annual sales has an increasing trend until 1996 after which the trend is consistently declining.

Table 2 shows the results of the following models respectively: FGL input oriented, FGL output oriented, strong congestion of Wei and Yan, weak congestion of Wei and Yan, strong congestion of Tone and Sahoo, weak congestion of Tone and Sahoo and wide congestion of Sueyoshi and Sekitani. These approaches show only existence or non-existence of congestion. The sign $\surd$ indicates that related DMU is congested.
As before said, the results of FGL method is dependent on the orientation of DEA models applied. In Table 2, results of input and output oriented FGL model is different from each other too. Also, these results is different from the results of the other models. The results of weak and strong congestion of Wei and Yan, weak congestion of Tone and Sahoo and wide congestion of Sueyoshi and Sekitani is similar to each other.

Table 3 shows the results of existence or non-existence of congestion using the following models respectively; input and output oriented of BCSW model, additive model, the one-model approach introduced by Cooper et al., Jahanshahloo and Khodabakhshi method, the one-model introduced by Khodabakhshi and the new method introduced by Jahanshahloo et al. The results of input and output oriented of BCSW model is different from each other. As said in Section 2.3, the input oriented of BCSW model have deficiency and the results of output oriented is more valid.

The DMUs which is recognized congested by output oriented of BCSW model, additive model and one-model is similar to each other. These models also recognize the congested input and the amount of congestion. Later, we will present these amounts.

Jahanshahloo and Khodabakhshi method and the one-model approach are applied for flexible inputs based on input relaxation model.

The new method is the most simplest approach. It also identifies the congested input and the amount of congestion with less computations compared with other ones. But the results of the new method and the other methods is different. Most of the models such as Cooper et al, Wei and Yan, Tone and Sahoo and sueyoshi and Sekitani recognize 14 DMUs to be congested while the new method recognizes only 4 DMUs to be congested.

As before said, some methods show the congested inputs and the amounts of congestion in each input. Table 4 shows this data. In this example, these amounts for Cooper et al. methods are equal to each other, but the results of the new method is different from them. Here, we ignore Jahanshahloo-Khodabakhshi method and the one-model approach introduced by Khodabakhshi, because they was introduced for flexible inputs. According to Definition 1.1, when a DMU is congested, it can not obtain more output by increasing the input. Now, suppose the years 1976 and 1991. If we compare the inputs and the output of these two years, we have: inputs of DMU_{22} = (7822,16191) > (3163,6233) = inputs of DMU_2, and output of DMU_{22} = 169786 > 48367 = output of DMU_2.

Therefore, DMU_{22} has been able to obtain more output by consuming more inputs than DMU_2. Thus, decrease in the output of DMU_2 can be associated with inefficiency and no congestion. From another viewpoint, according to

Definition 1.1, congestion occurs in large sizes, so we should verify congestion in large sizes. The new method acts in this way. Consequently, it seems that the new method is more valid.

Table 5 shows the results of output reduction due to congestion. Here we use the amounts of input congestion calculated by Cooper et al methods and the new method and then apply equations (3.18) and (3.19).

5 Conclusion

In this paper, we reviewed the approaches that are available in the DEA literature for evaluating congestion. Some methods show only existence or non-existence of congestion. The models introduced by Fare et al., Wei-Yan, Tone-Sahoo and Sueyoshi-Sekitani are so.

These models can not provide a value for measuring the amount of congestion. The models introduced by Cooper et al. (BCSW, the one-model, unified additive model) determine the congested inputs and the amounts of congestion. These models, first determine the projection point of the DMU under evaluation and then by assessing the projection point, find the maximum value that can augmented to the projection's inputs and remain in T_{NEW}. However, these models necessarily do not produce the same results, because they have different strategies to get an efficient point.

The idea of the model introduced by Jahanshahloo and Khodabakhshi is like the idea of the Cooper's methods. But the difference is that Jahanshahloo and Khodabakhshi apply input relaxation model for finding projection point. All the methods except the model introduced by Sueyoshi and Sekitani and the new method suffer from an occurrence of multiple solutions.

The new method introduced by Jahanshahloo et al. identifies the congested inputs and the amount of congestion with less computations compared with other methods. It seems that the results of the new method is more valid than other ones. So, we suggest more serious researches on the new method as an future work. Also this model do not have the problem of multiple solutions.

Then, we introduced a model to compute output losses due to congestion and then applied the mentioned methods on a numerical example and presented the results.

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