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Research Article

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On J-C-Numerical Range and Its Generalizations

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Abstract
In this paper, we study *J*-*C*-numerical range of a set or a tuple of matrices and investigate their basic properties. Also, we give the conditions of star-shapeness of *J*-*C*-numerical range. Finally, we generalize these results to a set of matrices.

Keywords: Joint numerical range, *C*-numerical range, *J*-*C*-numerical range, star-shaped, star-center.

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1. Introduction

Let M_n be the set of all $n\times n$ complex matrices. Toeplitz [1] defined the concept of the numerical range of $A\in M_n$ by

$$W(A)=\{x^*Ax \mid x\in\mathbb{C}^n, x^*x=1\}$$
$$=\{Tr(Axx^*) \mid x\in\mathbb{C}^n, x^*x=1\}.$$

For a nonempty set $\mathfrak{A}$ of matrices in M_n, Lau et al. [2] considered and investigated

$$W(\mathfrak{A})=\cup\{W(A) \mid A\in\mathfrak{A}\}.$$

Let

$$J=I_r\oplus(-I_{n-r})=diag\Big(\underbrace{1,\dots,1}_{r},\underbrace{-1,\dots,-1}_{n-r}\Big).$$

Clearly, *J* has *r* positive and *n*-*r* negative eigenvalues. The *J*-adjoint of $A\in M_n$ is defined by

$$\left[A^{\#}a,b\right]=\left[a,JAJb\right],\qquad a,b\in\mathbb{C}$$

or equivalently, $A^{\#}=JA^*J$. A matrix *A* is called *J*-scalar, *J*-normal, *J*-unitary and *J*-Hermitian if it satisfies $A=mJ\ (m\in\mathbb{C}), A^{\#}A=AA^{\#}, A^{\#}A=AA^{\#}=I_n$, and $A=A^{\#}$, respectively. We denote by $\mathfrak{U}_{r,n-r}$ the group of all *J*-unitary matrices. If $A\in M_n$ is similar to a diagonal matrix, then *A* is said to be diagonalizable. For a matrix $C\in M_n$, Goldberg and Straus [3] defined the *C*-numerical range of $A\in M_n$ by

$$W_C(A)=\{Tr(CU^*AU) \mid U \text{ is unitary}\}.$$

For the standard basis $\{E_{11},\dots,E_{nn}\}$ of M_n, if $C=E_{11}$, then $W_C(A)=W(A)$. For a nonempty set $\mathfrak{A}$ of matrices in M_n, Lau et al.[2] introduced $W_C(\mathfrak{A})$ as follows:

$$W_C(\mathfrak{A})=\cup\{W_C(A) \mid A\in\mathfrak{A}\}.$$

Let $J=I_r\oplus(-I_{n-r}), 0<r<n,$ be a Hermitian involutive, that is, $J^2=I$ and $J^*=J^{-1}=J$. Bebiano et al. [4] defined *J*-*C*-numerical range (or *J*-*C*-tracial range) as

$$W_C^J(A)=\{Tr(CU^{-1}AU) \mid U\in M_n, U^*JU=J\},$$

Where $A,C\in M_n(\mathbb{C})$. For $J=I_n$, we have $W_C^J(A)=W_C(A)$.

Following Lau et al. [2], we state the following definition.

Definition 1. Let $J = I_r \oplus (-I_{n-r}), 0 < r < n,$ and $A, C \in M_n(\mathbb{C})$. For a nonempty set $\mathfrak{A}$ of matrices in M_n, we have

$$W_C^J(\mathfrak{A}) = \bigcup \{W_C^J(A) \mid A \in \mathfrak{A}\}.$$

Definition 2. If $A \in M_n$ is not *J*-Hermitian, then one may consider the *J*-Hermitian decomposition

$$A = Re^J(A) + iIm^J(A) = A_1 + iA_2,$$

Where

$$A_1 = Re^J(A) = \frac{1}{2}\left(A + A^{\#}\right)$$

and

$$A_2 = Im^J(A) = \frac{1}{2i}\left(A - A^{\#}\right)$$

are *J*-Hermitian. Now, we consider $W_C^J(A_1, A_2)$ as the joint *J*-*C*-numerical range of (A_1, A_2) defined by

$$W_C^J(A_1, A_2) = \left\{\left(Tr(CU^{-1}A_1U), Tr(CU^{-1}A_2U)\right) \mid U \in M_n, U^*JU = J\right\}.$$

Also, we define the joint *J*-*C*-numerical range of $(A_1, \ldots, A_k) \in M_n^k$ by

$$W_C^J(A_1, \ldots, A_k) = \left\{\left(Tr(CU^{-1}A_1U), \ldots, Tr(CU^{-1}A_kU)\right) \mid U \in M_n, U^*JU = J\right\} \subseteq \mathbb{C}^k.$$

If $m \in \mathbb{C}$ and $C = mI$, then $W_C^J(A) = \{mTr(A)\}$ and

$$W_C^J(A_1, \ldots, A_k) = \left\{m\left(Tr(A_1), \ldots, Tr(A_k)\right)\right\}.$$

So, we consider *C* to be not a scalar matrix.

This paper is organized as follows. In Section 2, we survey the elementary properties of $W_C^J(\mathfrak{A})$. In Section 3, we give the geometric properties of $W_C^J(\mathfrak{A})$ and generalize the conditions of star-shapeness for that. Finally, in Sections 4, we extend $W_C^J(\mathfrak{A})$ to joint *J*-*C*-numerical range and give some properties that can be concluded from $W_C^J(\mathfrak{A})$.

2. Elementary properties of *J*-*C*-numerical range

In this section, we give some basic results about *J*-*C*-numerical range and then investigate their generalization.

Proposition 1. Let $A,C \in M_n$. then the following properties hold:

a) For every $U \in \mathfrak{U}_{r,n-r}, W_C^J(A) = W_C^J(U^{-1}AU)$.

b) For every $a,b \in \mathbb{C}, W_C^J(aI + bA) = aTr(CI) + bW_C^J(A)$.

c) $W_{C^*}^J(A^*) = \overline{W_C^J(A)}$.

d) $W_C^J(A) = W_A^J(C)$.

e) $W_C^J(A)$ is a connected set.

f) If A and C are J-Hermitian matrices, then $W_C^J(A) \subseteq \mathbb{R}$.

Proof. (a), (b), (c) and (d) immediately follow from definition of *J*-*C*-numerical range.

e) As $\mathfrak{U}_{r,n-r}$ is connected and $W_C^J(A)$ is the range of the continuous map from $\mathfrak{U}_{r,n-r}$ to $\mathbb{C}$, so $W_C^J(A)$ is a connected set in the complex plane.

f) For any $U \in \mathfrak{U}_{r,n-r}$, it follows from [5] that $\overline{Tr(CU^{-1}AU)} = Tr(CU^{-1}AU)$.□

Now, we generalize these properties to $W_C^J(\mathfrak{A})$, where $\varnothing \neq \mathfrak{A} \subseteq M_n$.

Theorem 2. Let $C \in M_n$ is a nonscalar matrix and let $\varnothing \neq \mathfrak{A} \subseteq M_n$.

a) For every $U \in \mathfrak{U}_{r,n-r}$, we have $W_C^J(\mathfrak{A}) = W_C^J(U^{-1}\mathfrak{A}U)$.

b) For every $a,b \in \mathbb{C}$, if

$$\mathrm{a}\mathfrak{A} + \mathrm{bI} = \{aA + bI \mid A \in \mathfrak{A}\},$$

then

$$W_C^J(\mathrm{a}\mathfrak{A} + \mathrm{bI}) = aW_C^J(\mathfrak{A}) + bTr(C)$$
$$= \{aw + bTr(C) \mid w \in W_C^J(\mathfrak{A})\}.$$

c) If $\mathfrak{A}$ is bounded, then so is $W_C^J(\mathfrak{A})$.

d) If $\mathfrak{A}$ is compact, then so is $W_C^J(\mathfrak{A})$.

Proof. (a) and (b) follows from Proposition 1(a) and (b), respectively.

c) If $\mathfrak{A}$ is bounded, then there is $B > 0$ such that for every $A \in M_n$, we have $\|A\| < B$. Hence,

$$\left|Tr\left(CU^{-1}AU\right)\right| \le n\|C\|\|A\| < n\|C\|B.$$

Therefore, $W_C^J(\mathfrak{A})$ is bounded.

d) Since $\mathfrak{A}$ is compact, so is bounded and closed. Hence, $W_C^J(\mathfrak{A})$ is also bounded, from (c). To prove that $W_C^J(\mathfrak{A})$ is closed, we suppose that $\left\{Tr\left(CU_i^{-1}A_iU_i\right) \mid i=1,2,\ldots\right\}$ is a sequence in $W_C^J(\mathfrak{A})$ converging to $w \in \mathbb{C}$, where $A_i \in \mathfrak{A}$ and $U_i \in \mathfrak{U}_{r,n-r}$, for each *i*. Because $\mathfrak{A}$ is compact, there is a subsequence $\left\{A_{k_i} \mid k=1,2,\ldots\right\}$ of $\left\{A_i \mid i=1,2,\ldots\right\}$ converging to $A_0 \in \mathfrak{A}$. Furthermore, We can consider a subsequence $\left\{U_{k_i} \mid k=1,2,\ldots\right\}$ of $\left\{U_i \mid i=1,2,\ldots\right\}$ converging to U_0. Therefore, $\left\{Tr\left(CU_{k_i}^{-1}A_{k_i}U_{k_i}\right) \mid k=1,2,\ldots\right\}$ is converged to

$$Tr\left(CU_0^{-1}A_0U_0\right) = w_0 \in W_C^J(\mathfrak{A}).$$

Thus, $W_C^J(\mathfrak{A})$ is closed, forcing $W_C^J(\mathfrak{A})$ is compact. □

Part (a) of the following example shows that the converse of (c) and (d) of the above theorem is not true in general.

Example 1. a) Let $C \in M_n$ is a nonscalar matrix whose trace is zero and let $\mathfrak{A} = \left\{mI \mid m \in \mathbb{C}\right\}$. Then $W_C^J(\mathfrak{A}) = \{0\}$ is compact and bounded, but $\mathfrak{A}$ is not bounded.

b) Let

$$\mathfrak{A} = \left\{diag\left(0, a+\frac{i}{a}\right) \mid a>0\right\} \cup \left\{diag(0,0)\right\}.$$

Then $\mathfrak{A}$ is closed, but

$$W(\mathfrak{A}) = \left\{a+ib \mid a,b>0, ab \le 1\right\} \cup \{0\}$$

is not closed.

Remark 1. a) For every $B \subseteq \mathbb{C}$, if $Tr(C) \ne 0$ and

$$\mathfrak{A} = \left\{\frac{mI}{Tr(C)} \mid m \in B\right\},$$

then $W_C^J(\mathfrak{A}) = B$. Therefore, the geometrical shape of $W_C^J(\mathfrak{A})$ may be quite arbitrary.

b) If $C = mI$ and $m \in \mathbb{C}$, then $W_C^J(\mathfrak{A}) = \left\{mTr(A) \mid A \in \mathfrak{A}\right\}$.

In both cases, we see that we do not have information about the matrices in $\mathfrak{A}$ and the geometrical properties of $W_C^J(\mathfrak{A})$, but the following theorem provides conditions for the simultaneous description of the geometric properties of $W_C^J(\mathfrak{A})$ and the matrices in $\mathfrak{A}$.

Theorem 3. Let $C \in M_n$ is a nonscalar matric and let $\emptyset \neq \mathfrak{A} \subseteq M_n$. Then the following conditions hold:

a) $W_C^J(\mathfrak{A}) = \{m\}$, $m \in \mathbb{C}$ if and only if $\mathfrak{A} = \{lI \mid lTr(C) = m\}$.

b) The set $W_C^J(\mathfrak{A})$ is a subset of a straight-line L if and only if the following conditions hold:

i) $\mathfrak{A} \subseteq \{lI \mid l \in \mathbb{C}, lTr(C) \in L\}$.

ii) $C = diag(c_1, \ldots, c_n) \in \mathbb{R}^n$ with the $c_i J_i$ pairwise distinct, where J_i denote the ith diagonal element of J, $i = 1, \ldots, n$ and $\mathfrak{A}$ is a set of J-Hermitian matrices.

Proof. Condition (a) follows from the fact that

$$W_C^J(\mathfrak{A}) = \{m\} \Leftrightarrow A = lI, lTr(C) = m.$$

b) For every $A \in \mathfrak{A}$ and $l \in \mathbb{C}$, let $A = lI$ and $lTr(C) \in L$. Then obviously the result follows from the definition of $W_C^J(A)$. Conversely, let the set $W_C^J(\mathfrak{A})$ is a subset of a straight line L. If $\mathfrak{A} \subseteq \{lI \mid l \in \mathbb{C}\}$, then clearly $\mathfrak{A} \subseteq \{lI \mid l \in \mathbb{C}, lTr(C) \in L\}$ and (i) is proved.

Now, let $\mathfrak{A}$ contains a nonscalar matrix A. Then (ii) follows from [4, Theorem 5.3].□

We denote by $\sigma_J^{\pm}(A)$ the sets of the eigenvalues of A with eigenvectors v such that $v^* Jv = \pm 1$. We note that a J-Hermitian matrix A is J-unitarily diagonalizable if and only if every eigenvalue of A belongs either to $\sigma_J^+(A)$ or to $\sigma_J^-(A)$. In other word, $\sigma_J^+(A)$ (respectively, $\sigma_J^-(A)$) consists of r(respectively, n-r) eigenvalues. Let A be a J-Hermitian matrix and let

$$\begin{aligned}
&a_1, \ldots, a_r \in \sigma_J^+(A), && a_1 \geq \cdots \geq a_r, \\
&a_{r+1}, \ldots, a_n \in \sigma_J^-(A), && a_{r+1} \geq \cdots \geq a_n, \\
&c_1, \ldots, c_r \in \sigma_J^+(C), && c_1 \geq \cdots \geq c_r, \\
&c_{r+1}, \ldots, c_n \in \sigma_J^-(C), && c_{r+1} \geq \cdots \geq c_n.
\end{aligned}$$

The eigenvalues of A are called to not interlace if either $a_r > a_{r+1}$ or $a_n > a_1$. If this condition does not hold, then we say that the eigenvalues of A are interlace.

Bebiano et al. [6] showed that if either the eigenvalues of A or C interlace and

$$a_1 \neq a_n, \quad a_r \neq a_{r+1}, \quad c_1 \neq c_n, \quad c_r \neq c_{r+1},$$

then $W_C^J(A)$ is the whole real line.

Now, due to this notation, we have the following proposition to identify $\mathfrak{A}$ and $W_C^J(\mathfrak{A})$.

Proposition 4. Let $\mathfrak{A} \subseteq M_n$. Then $W_C^J(\mathfrak{A}) \subseteq \mathbb{R}$ if and only if

a) $C = diag(c_1, \ldots, c_n) \in \mathbb{R}^n$ with the $c_i J_i$ pairwise distinct for $i = 1, \ldots, n$ and $\mathfrak{A}$ is a set of Hermitian matrices.

b) $\mathfrak{A}$ is a set of nonscalar *J*-Hermitian and *J*-unitarily diagonalizable matrices of A_i 's and $C \in \mathfrak{A}$. Also, for $k = 1, \ldots, n,$ let a_{i_k} and c_k be the eigenvalues of A_i 's and *C*, respectively, such that

$$\begin{aligned}
&a_{i_1}, \ldots, a_{i_r} \in \sigma_J^+(A_i), && a_{i_1} \geq \cdots \geq a_{i_r},\\
&a_{i_{r+1}}, \ldots, a_{i_n} \in \sigma_J^-(A_i), && a_{i_{r+1}} \geq \cdots \geq a_{i_n},\\
&c_1, \ldots, c_r \in \sigma_J^+(C), && c_1 \geq \cdots \geq c_r,\\
&c_{r+1}, \ldots, c_n \in \sigma_J^-(C), && c_{r+1} \geq \cdots \geq c_n.
\end{aligned}$$

If the eigenvalues of A_i 's and the eigenvalues of *C* do not interlace, then one of the following conditions holds:

a) $(a_{l_1} - a_{m_1})(c_{l_2} - c_{m_2}) < 0, \quad for\ all\ 1 \leq l_1, l_2 \leq r, \quad r+1 \leq m_1, m_2 \leq n.$

b) $(a_{l_1} - a_{m_1})(c_{l_2} - c_{m_2}) > 0, \quad for\ all\ 1 \leq l_1, l_2 \leq r, \quad r+1 \leq m_1, m_2 \leq n.$

Proof. The results follow by [4, Theorem 5.2] and [6, Proposition 2.1], respectively.□

3. Geometric interpretation for star-shapeness of *J*-*C*-Numerical range

After studying the properties of each concept, researchers always describe it geometrically. In this section, we study the star-shapeness of a matrix and a set of matrices.

Lemma 1[7, Lemma 1]. Consider $A, B \in M_n$ be partitioned as

$$A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad and \qquad B = \begin{bmatrix} b_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $a_{11}, b_{11} \in \mathbb{C}.$ Then

$$Tr\left(A\begin{bmatrix} e^{-i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix} B\begin{bmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}\right)$$
$$= a_{11}b_{11} + Tr\left(A_{22}B_{22}\right) + e^{i\theta}\left(A_{12}B_{21}\right) + e^{-i\theta}\left(B_{12}A_{21}\right).$$

The locus of which, when θ runs from 0 to 2π, forms an ellipse centered at $a_{11}b_{11} + Tr\left(A_{22}B_{22}\right)$ with length of major axis equal to $2\left(\left|A_{12}B_{21}\right| + \left|B_{12}A_{21}\right|\right)$.

We consider the following set:

$$SW_C^J\left(A\right) = \left\{S \in M_n \mid W_C^J\left(S\right) \subseteq W_C^J\left(A\right), \textit{for all } C \in M_n\right\}.$$

Then for any unitary *U*, we have $SW_C^J\left(A\right) = SW_C^J\left(U^{-1}AU\right)$. If $S \in SW_C^J\left(A\right)$, then $U^{-1}SU \in SW_C^J\left(A\right)$.

Now, using this definition, we prove that *J*-*C*-numerical range is star-shaped, but before expressing it, we need some lemmas, which we present below.

Lemma 2. Let $S = \left(s_{ik}\right) \in SW_C^J\left(A\right)$, let $1 \le l \le n$, let $m \in [0,1]$, and let $T = \left(t_{ik}\right)$ be defined by

$$t_{ik} = \begin{cases} ms_{ik} & \textit{if exactly one of } i \textit{ and } k \textit{ equals } l, \\ s_{ik} & \textit{otherwise.} \end{cases}$$

That is, *T* is obtained from *S* by multiplying *m* to the entries on the *l*th row and on the *l*th column, except for the $\left(l,l\right)$th entry of *S*. Then $T \in SW_C^J\left(A\right)$.

Proof. We assume, without loss of generality, that $l = 1$. For every *J*-unitary U_1 and U_2 and for every $\theta \in \mathbb{R}$, we set

$$w\left(U_1,U_2,\theta\right) := Tr\left(\left(U_1^{-1}CU_1\right)\begin{bmatrix} e^{-i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}\left(U_2^{-1}SU_2\right)\begin{bmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}\right).$$

Clearly, $w\left(U_1,U_2,\theta\right) \in \mathbb{C}$ and $w\left(U_1,U_2,\theta\right) \in W_C^J\left(S\right) \subseteq W_C^J\left(A\right)$. Since $\mathfrak{U}_{r,n-r}$ is path connected, we can choose two continuous functions

$$f_{U_1}, g_{U_2} : [0,1] \to \mathfrak{U}_{r,n-r}$$

such that $f_{U_1}(0) = U_1$, $g_{U_2}(0) = I$ and that both $f_{U_1}^{-1}(1)Cf_{U_1}(1)$ and $g_{U_2}^{-1}(1)Sg_{U_2}(1)$ are upper triangular. By using Lemma 1, for every $q \in [0,1]$, the points $w\left(f_{U_1}(q), g_{U_2}(q), \theta\right)$ form an ellipse $E(q)$ when θ runs through 0 to 2π. Because both $f_{U_1}(q)$ and $g_{U_2}(q)$ are continuous, $E(0)$ deforms continuously to become $E(1)$ when q runs from 0 to 1. Since both $f_{U_1}^{-1}(1)Cf_{U_1}(1)$ and $g_{U_2}^{-1}(1)Sg_{U_2}(1)$ are upper triangular, it follows from Lemma 1 that the length of the major axis of the ellipse $E(1)$ is zero; that is, $E(1)$ degenerates into a single point. Let $p \in \mathbb{C}$ be any point in the interior of $E(0)$. If $p = E(1)$, then $p \in W_C^J(S) \subseteq W_C^J(A)$. If $p \neq E(1)$, then p must be swept across by some ellipse $E(q)$ as $E(0)$ is deformed to become the degenerating ellipse $E(1)$ when q runs from 0 to 1. Thus, $p \in W_C^J(S) \subseteq W_C^J(A)$. The point

$$Tr\left(\left(U_1^{-1}CU_1\right)\begin{bmatrix} s_{11} & mS_{12} \\ mS_{21} & S_{22} \end{bmatrix}\right) = d_{11}s_{11} + Tr\left(D_{22}S_{22}\right) + m\left(D_{12}S_{21}\right) + m\left(S_{12}D_{12}\right)$$

where

$$U_1^{-1}CU_1 = \begin{bmatrix} d_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \quad S = \begin{bmatrix} s_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad d_{11}, s_{11} \in \mathbb{C}, \quad m \in [0,1]$$

is in the interior of the ellipse $E(0)$ and therefore is contained in $W_C^J(A)$. Because this is true for every J-unitary matrix U_1 and for every $C \in M_n$, so

$$\begin{bmatrix} s_{11} & mS_{12} \\ mS_{21} & S_{22} \end{bmatrix} \in SW_C^J(A). \square$$

Lemma 3. Let $S \in SW_C^J(S)$. Then for every $a \in [0,1]$, the following conditions hold:

a) $aS + (1-a)diag(S) \in SW_C^J(A)$.

b) $aS + (1-a)\dfrac{Tr(A)}{n} I_n \in SW_C^J(A)$.

The statement (b) means that the set $SW_C^J(A)$ is star-shaped with respect to star-center $\dfrac{Tr(A)}{n} I_n$.

Proof. Let $S=(s_{ik})\in SW_C^J(A)$ and $m\in[0,1]$ be such that $m^2=a$. By repeatedly applying the result of Lemma 2 on *S* and considering $1\le l\le n$, we obtain

$$aS+(1-a)diag(S)=(t_{ik}),$$

Where

$$t_{ik}=\begin{cases} s_{ik}, & i=k, \\ m^2 s_{ik}, & otherwise \end{cases}$$

is contained in $SW_C^J(A)$ and this prove (a).

It follows from [8, p. 77, Problem 3] that there exists *J*-unitary *U* such that

$$diag\left(U^{-1}\mathrm{S}\mathrm{U}\right)=\frac{Tr(S)}{n}I_n.$$

Now,

$$S\in SW_C^J(A)\Rightarrow\{Tr(S)\}=W_I^J(S)\subseteq W_I^J(A)=\{Tr(A)\}.$$
$$\Rightarrow Tr(S)=Tr(A).$$

Because $U^{-1}\mathrm{S}\mathrm{U}\in SW_C^J(A)$, so (a) implies that

$$T=a\left(U^{-1}\mathrm{S}\mathrm{U}\right)+(1-a)\frac{Tr(A)}{n}I_n=a\left(U^{-1}\mathrm{S}\mathrm{U}\right)+(1-a)diag\left(U^{-1}\mathrm{S}\mathrm{U}\right).$$

Thus

$$aS+(1-a)\frac{Tr(A)}{n}I_n=UTU^{-1}\in SW_C^J(A).\square$$

Now, we provide our result about star-shapeness of *J*-*C*-numerical range.

Theorem 5. Let $A,C\in M_n(\mathbb{C})$. Then $W_C^J(A)$ is star-shaped with respect to star-center $\frac{Tr(A)Tr(C)}{n}$.

Proof. Let $w\in W_C^J(A)$, let $a\in[0,1]$ and let *U* be a *J*-unitary matrix such that $w=Tr\left(CU^{-1}A\mathrm{U}\right)$. Because $A\in SW_C^J(A)$, so by Lemma 3(b), we have

$$S:=aA+(1-a)\frac{Tr(A)}{n}I_n\in SW_C^J(A).$$

Therefore,

$$aw+(1-a)\frac{Tr(A)Tr(C)}{n}=Tr\left(CU^{-1}SU\right)\in W_C^J(S)\subseteq W_C^J(A).\square$$

In Theorem 2, we gave some elementary properties of $W_C^J(\mathfrak{A})$. Now, by the star-shapeness of $W_C^J(\mathfrak{A})$ and connectivity of $\mathfrak{A}$, we have the following theorem.

Theorem 6. If $\mathfrak{A}$ is connected, then so is $W_C^J(\mathfrak{A})$.

Proof. By previous theorem, for every $A,C\in M_n(\mathbb{C}), W_C^J(A)$ is star-shaped with $\frac{Tr(A)Tr(C)}{n}$ as a star center, that is, for every $w\in W_C^J(A)$ and $a\in[0,1]$,

$$aw+(1-a)\frac{Tr(A)Tr(C)}{n}\in W_C^J(A).$$

Let $w_1=Tr\left(CU_1^{-1}A_1U_1\right)$ and $w_2=Tr\left(CU_2^{-1}A_2U_2\right)$, where $A_1,A_2\in\mathfrak{A}$ and U_1 and U_2 are *J*-unitary matrices. Then there are two line segment, one with end points w_1 and $\frac{Tr(A_1)Tr(C)}{n}$, and the other with end points w_2 and $\frac{Tr(A_2)Tr(C)}{n}$. Because $\mathfrak{A}$ is connected, so are the sets $\{Tr(A)\,|\,A\in\mathfrak{A}\}$ and $\left\{\frac{Tr(A)Tr(C)}{n}\,|\,A\in\mathfrak{A}\right\}$. Therefore, there is a path joining w_1 to $\frac{Tr(A_1)Tr(C)}{n}$, then to $\frac{Tr(A_2)Tr(C)}{n}$ and finally to w_2.$\square$

Now, if $\mathfrak{A}$ is not connected, then $W_C^J(\mathfrak{A})$ may also not be connected. See the two examples below.

Example 2. a) Let $J=I_n$, let $C=E_{11}$, let $A=diag(1+i,1-i)$, let $S_1=conv\{A,-A\}$, let $S_2=conv\{A,-A+4I_n\}$, and let $\mathfrak{A}=S_1\cup S_2$. Then $\mathfrak{A}$ is star-shaped with star-center *A*. Now,

$$W_C^J(S_1)=W(S_1)=\bigcup_{a\in[0,1]}W\left(aA+(1-a)(-A)\right)=\bigcup_{b\in[-1,1]}bW(A).$$

Also, because

$$W(A)=W(-A+2I_n)=conv\{1-i,1+i\},$$

we have

$$W_C^J(S_2) = W(S_2) = \bigcup_{a\in[0,1]} W\left(aA + (1-a)(-A+4I_n)\right)$$
$$= \bigcup_{a\in[0,1]} W\left((1-2a)(-A+2I_n)+2I_n\right)$$
$$= \bigcup_{b\in[-1,1]} bW\left(-A+2I_n\right)+2$$
$$= \bigcup_{b\in[-1,1]} bW(A)+2$$
$$= W_C^J(S_1)+2.$$

Thus,

$$W_C^J(\mathfrak{A}) = W(S_1 \cup S_2)$$
$$= W(S_1) \cup W(S_2)$$
$$= conv\{0,-1-i,-1+i\} \cup conv\{0,1-i,1+i,2\} \cup conv\{2,3-i,3+i\},$$

and $W_C^J(\mathfrak{A})$ is not star-shaped.

b) Let $J = I_n$, let $C = E_{11}$, let $A = diag(1+i,1-i)$, and let $\mathfrak{A} = conv\{A,-A\}$. Then

$$W_C^J(\mathfrak{A}) = W(\mathfrak{A})$$
$$= \bigcup_{a\in[0,1]} W\left(aA + (1-a)(-A)\right)$$
$$= \bigcup_{b\in[-1,1]} bW(A)$$
$$= conv\{0,-1-i,-1+i\} \cup conv\{0,1-i,1+i\}.$$

In the following, we check whether for a convex set $\mathfrak{A}, W_C^J(\mathfrak{A})$ is always star-shaped or not. Nevertheless, before that we need a lemma, which expresses the star-shapeness of $W_C^J(\mathfrak{A})$ according to certain states of $\mathfrak{A}$ or *C*. From now on, we denote by $SC_C^J(A)$ the set of all-star-centers of $W_C^J(A)$.

Lemma 4. Let $C \in M_n$ and $\mathfrak{A}$ be a convex matrix set.

a) If $\mathfrak{A}$ contains a scalar matrix mI, then $W_C^J(\mathfrak{A})$ is star-shaped with $mTr(C)$ as a star-center.

b) Let

i) $\bigcap_{A_i \in \mathfrak{A}} SC_C^J(A_i) \neq \varnothing$,

ii) $\bigcap_{i=1}^{3} SC_C^J(A_i) \neq \varnothing.$

In both cases, for every $m \in \bigcap\{SC_C^J(A) \mid A \in \mathfrak{A}\}, W_C^J(\mathfrak{A})$ is star-shaped with *m* as a star-center.

c) If $Tr(C)=0,$ then $W_C^J(\mathfrak{A})$ is star-shaped with 0 as a star-center.

d) If for every $A \in \mathcal{A}, Tr(A)=t,$ then $W_C^J(\mathfrak{A})$ is star-shaped with $tTr(C)$ as a star-center.

e) Let $\mathfrak{A} = conv\{A_1, A_2\}$ and let $m \in SC_C^J(A_1) \cap SC_C^J(A_2).$ Then $W_C^J(\mathfrak{A})$ is star-shaped with *m* as a star-center.

Proof. a) Let $mI = A_1$ and let $A_2 \in \mathfrak{A}.$ Then

$$conv\{mTr(C), W_C^J(A_2)\} \subseteq W_C^J(conv\{A_1, A_2\}) \subseteq W_C^J(\mathfrak{A}).$$

b) For every $w \in W_C^J(\mathfrak{A}),$ there is $B \in \mathfrak{A}$ such that $w \in W_C^J(B).$ Because $m \in SC_C^J(B),$ so the line segment joining *m* and *w* will lie in $W_C^J(B) \subseteq W_C^J(\mathfrak{A}),$ and part (i) follows. Part (ii) can be obtained from Helly's Theorem and part (i).

c) The result follows from Theorem 5 and (b).

d) Because for every $A \in \mathfrak{A}, Tr(A)=t,$ so $\bigcap\{SC_C^J(A) \mid A \in \mathfrak{A}\} = tTr(C).$ Now, the result follows from (b).

e) Assume that $w \in W_C^J(\mathfrak{A}).$ Then there are $U_0 \in \mathfrak{U}_{r,n-r}$ and $a \in [0,1]$ such that $w = Tr\left(CU_0^{-1}(aA_1 + (1-a)A_2)U_0\right).$ It suffices to prove that

$$conv\{m, Tr(CU_0^{-1}A_1U_0), Tr(CU_0^{-1}A_2U_0)\} \subseteq W_C^J(\mathfrak{A}). \tag{1}$$

Let $U_1 \in \mathfrak{U}_{r,n-r}$ such that $Tr(CU_1^{-1}A_1U_1) = m.$ Since $m \in SC_C^J(A_1) \cap SC_C^J(A_2),$ we have

$$conv\{Tr(CU_0^{-1}A_1U_0), m\} \cup conv\{Tr(CU_0^{-1}A_2U_0), m\} \subseteq W_C^J(\mathfrak{A}).$$

Furthermore,

$$\begin{aligned} &conv\{Tr(CU_0^{-1}A_1U_0), Tr(CU_0^{-1}A_2U_0)\} \\ &= \{Tr\left(CU_0^{-1}(aA_1 + (1-a)A_2)U_0\right) \mid a \in [0,1]\} \subseteq W_C^J(\mathfrak{A}). \end{aligned}$$

Thus

$$\begin{aligned} d &= conv\left\{Tr\left(CU_0^{-1}A_1U_0\right), Tr\left(CU_0^{-1}A_2U_0\right)\right\} \\ &\cup conv\left\{Tr\left(CU_0^{-1}A_1U_0\right), m\right\} \\ &\cup conv\left\{Tr\left(CU_0^{-1}A_2U_0\right), m\right\} \subseteq W_C^J\left(\mathfrak{A}\right). \end{aligned}$$

We need to prove equation (1).

If d is a line segment or a point, then equation (1) holds obviously. Suppose that d is nondegenerate. Since $\mathfrak{U}_{r,n-r}$ is path-connected, we define a continuous function

$$\begin{aligned} f &: [0,1] \rightarrow \mathfrak{U}_{r,n-r} \\ f&(0) = U_0, \\ f&(1) = U_1. \end{aligned}$$

For $a \in [0,1]$, we set

$$\begin{aligned} g(a) &:= conv\left\{Tr\left(Cf(a)^{-1}A_1f(a)\right), Tr\left(Cf(a)^{-1}A_2f(a)\right)\right\} \\ &\cup conv\left\{Tr\left(Cf(a)^{-1}A_1f(a)\right), m\right\} \\ &\cup conv\left\{Tr\left(Cf(a)^{-1}A_2f(a)\right), m\right\} \subseteq W_C^J(\mathfrak{A}). \end{aligned}$$

Also, we set

$$M := max\left\{a \mid w \in conv\left(g(k)\right),\ for\ all\ \ 0 \le k \le a\right\}.$$

For every $w \in conv\left(g(0)\right)$, because

$$g(1) = conv\left\{Tr\left(Cf(a)^{-1}A_2f(a)\right), m\right\}$$

and $g(1)$ degenerates, by the continuity of *f*, we have

$$w \in g(M) \subseteq W_C^J\left(\mathfrak{A}\right)$$

and the result follows. □

Now, we are ready to present our theorem, which actually generalizes part (e) of the above Lemma.

Theorem 7. Suppose that $C \in M_n$, that *S* be a (finite or infinite) family of matrices in M_n and that $\mathfrak{A} = conv\left(S\right)$. If $m \in \bigcap_{A \in S} SC_C^J\left(A\right)$, then $W_C^J\left(\mathfrak{A}\right)$ is star-shaped with star-center *m*.

Proof. If S has two elements, then the result holds from Lemma 4. Assume that $|S|\geq 3$ and that $w \in W_C^J(\mathfrak{A})$. Then there exist $S_1,\ldots,S_l \in S$ and $a_1,\ldots,a_l > 0$ with $a_1+\cdots+a_l=1$ and $U \in \mathfrak{U}_{r,n-r}$ such that

$$w_i = Tr\left(CU^{-1}S_iU\right), \qquad i=1,\ldots,l$$
$$w = Tr\left(CU^{-1}(a_1S_1+\cdots+a_lS_l)U\right).$$

Therefore, $w \in conv\{w_1,\ldots,w_l\}$. The half line through m and w intersects a line segment joining some w_i and w_k with $1\leq i \leq k \leq l$ such that $w \in conv\{m,w_i,w_k\}$. Now, again Theorem 5 yields

$$conv\{m,w_i,w_k\} \subseteq W_C^J\left(conv\{S_i,S_k\}\right) \subseteq W_C^J(\mathfrak{A}).\square$$

4. The joint *J*-*C*-numerical range

Many researchers have investigated the joint numerical range (see [9, 10, 11, 12]) and the joint *C*-numerical range (see [2, 13, 14, 15]). Following them in the introduction and in definition 2 for $(A_1,\ldots,A_k) \in M_n^k$, we introduce the joint *J*-*C*-numerical range as follows:

$$W_C^J(A_1,\ldots,A_k) = \left\{\left(Tr(CU^{-1}A_1U),\ldots,Tr(CU^{-1}A_kU)\right) \mid U \in M_n, U^*JU=J\right\}$$
$$\subseteq \mathbb{C}^k.$$

In this section, after stating a definition, we generalize this concept and study it.

Definition 3. Let $C, A_1,\ldots,A_k \in M_n$, and consider the *k*-tuple $K=(A_1,\ldots,A_k)$. Also, let S be a nonempty subset of M_n^k. We define *J*-*C*-numerical range of S as follows:

$$W_C^J(S) = \bigcup\left\{W_C^J(K) \mid K \in S\right\},$$

and we call it the generalized joint *J*-*C*-numerical range.

Obviously, if $S=\{K\}$, then $W_C^J(S)=W_C^J(K)$.

Now, we investigate the preliminary properties of the generalized joint *J*-*C*-numerical range.

Theorem 8. Let $C \in M_n$ be a nonscalar matrix and let $\varnothing \neq S \subseteq M_n^k$.

a) For every $U \in \mathfrak{U}_{r,n-r}, W_C^J(S) = W_C^J\left(U^{-1}SU\right)$.

b) Consider $a,b \in \mathbb{C}$ with $a \neq 0$ and $K=(A_1,\ldots,A_k) \in M_n^k$.

i) For $i=1,\ldots,k$, we set $B_i = aA_i + bI$. Then for every $L=(B_1,\ldots,B_k)\in M_n^k$,

$$W_C^J(L) = aW_C^J(K) + bTr(C).$$

ii) We set $B = aC + b$, then

$$W_B^J(K) = \left\{a(w_1,\ldots,w_k) + b\left(Tr(A_1),\ldots,Tr(A_k)\right) \mid (w_1,\ldots,w_k)\in W_C^J(K)\right\}.$$

c) If C and $A_1,\ldots,A_k \in K$ are J-Hermitian, then $W_C^J(K), W_C^J(S)\subseteq \mathbb{R}^k$.

d) If S is bounded, then so is $W_C^J(S)$.

e) If S is compact, then so is $W_C^J(S)$.

f) If S is connected, then so is $W_C^J(S)$.

Proof. Due to the generalized joint *J*-*C*-numerical range definition, Proposition 1 and Theorem 2, parts (a)-(e) are proved.

f) For every $K,L\in S$ with $K=(A_1,\ldots,A_k)$ and $L=(B_1,\ldots,B_k)$ and for *J*-unitary matrices $V_0,V_1\in\mathfrak{U}_{r,n-r}$, there is a path joining U_a with $a\in[0,1]$ joining V_0 and U_a. Therefore, there is a path joining

$$\left(Tr\left(CV_0^{-1}A_1V_0\right),\ldots,Tr\left(CV_0^{-1}A_kV_0\right)\right)$$

to

$$\left(Tr\left(CV_1^{-1}A_1V_1\right),\ldots,Tr\left(CV_1^{-1}A_kV_1\right)\right),$$

which is connected to $\left(Tr\left(CV_1^{-1}B_1V_1\right),\ldots,Tr\left(CV_1^{-1}B_kV_1\right)\right)$.□

Also, we have the following theorem to simultaneously describe the geometric form of $W_C^J(S)$ and to identify *S*.

Theorem 9. Let $C\in M_n$ is a nonscalar matrix and let $\varnothing\neq S\subseteq M_n^k$. Then the following conditions hold:

a) The set $W_C^J(S)$ is a polygon $\left\{(w_1,\ldots,w_k)\right\}\in\mathbb{C}^k$ if and only if

$$S=\left\{(m_1I_n,\ldots,m_kI_n) \mid Tr(C)(m_1,\ldots,m_k)=(w_1,\ldots,w_k)\right\}.$$

b) $W_C^J(S) \subseteq \mathbb{R}^k$ if and only if $C = diag(c_1, \dots, c_n) \in \mathbb{R}^n$ with the $c_l J_l$ pairwise distinct, for $l = 1, \dots, n$, and for every $K = (A_1, \dots, A_k) \in M_n^k$, A_i 's($i = 1, \dots, k$) are Hermitian matrices.

Proof. Part (a) is obvious and part (b) follows from [4, Theorem 5.2]. □

Also, if C is a *J*-Hermitian and *J*-unitarily diagonalizable matrix, then one can write $W_C^J(A_1, \dots, A_k) \subseteq \mathbb{C}^k$ in the form $W_C^J(A_{11}, A_{12}, \dots, A_{k1}, A_{k2}) \subseteq \mathbb{R}^{2k}$, where

$$A_{l1} = Re^J(A_l) = \frac{1}{2}\left(A_l + A_l^{\#}\right), \quad A_{l2} = Im^J(A_l) = \frac{1}{2i}\left(A_l - A_l^{\#}\right), \quad l = 1, \dots, k.$$

The problem of determining conditions on S such that $W(S)$ is star-shaped still remains an open and challenging problem. For example, Lau et al.[2] Showed that for $J = I_n, C = E_{11}, S_1 = (A_1, B_1, I_3)$ in which $A_1 = diag(0,1,0)$ and $B_1 = diag(1,0,-1)$, and $S_2 = (A_2, B_2, O_3)$ in which $A_2 = diag(1,0,0)$ and $B_2 = diag(0,-1,1)$, if $S = conv\{S_1, S_2\}$, then $W_C^J(S) = W(S)$ is the union of the triangular disk with vertices

$$(1-a,a,a), \quad (a,a-1,a), \quad (0,1-2a,a)$$

and is not star-shaped. Hence, $W(S)$ may not be star-shaped in general.

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