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Research Article

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Efficiency Assessment for a Line Location Problem with positive/negative weights with Interval Coordination

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Abstract
A line location problem as a subdivision of facility location problem deals with finding a straight line in the plane, so that sum of weighted distances or maximum weighted distances from the line to the demand points is minimized. This paper provides a novel approach to study the semi-obnoxious median line location problem with Euclidean norm by using data envelopment analysis method. Since the presented procedure would consider lines as decision making units, so efficiency of lines is taken into account instead of their optimality. Moreover, due to the inherent uncertainty of the parameters of the line location problem in the real world such as weights or/and coordinates of demand points, the problem under interval data is studied in viewpoint of data envelopment analysis as well. Furthermore, some propositions and a numerical example are provided to investigate the problem. Finally, conclusion is given.

Keywords: Line location problem, Semi-obnoxious, DEA, Interval data, efficiency, median line

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1. Introduction

Line location problem as a branch of facility location problems deals with finding a straight line in the plane which minimizes the sum of weighted distances or maximum weighted distances from the line to the demand points. In these cases, the optimal line is named 1-median and 1-center line, respectively. Designing railways or highways, piping, and path location are some of real-world application of the line location problem [1]. In addition, semi-obnoxious line location problem considers the case that some of the demand points have positive weights and the rest have negative weights. The positive weight means that the interaction between the line and demand point is desirable, so the line is aimed to be as close to the demand point as possible. Vice versa, negative weight means that the interaction between the line and demand point is obnoxious. Therefore, it is preferred the demand point is located far from the line in order to reduce the mentioned adverse effects. For instance, markets, malls, and universities are preferred to be close to highways, whereas hospitals are tried to be located as much as possible far away roads and main streets because of pollution, noise and other undesirable environmental impacts. The semi-obnoxious case of the problem with Euclidean and rectilinear norms has been solved by using metaheuristic algorithms ([2], [3]). Moreover, in a basic line location problem, it is assumed that all parameters are fixed numbers. For example, weight and coordinate of demand points are considered to be exact numbers. But it is obvious that in many cases, the exact number cannot be determined for the parameters. Namely, consider a city that its number of residents changes significantly during the day. So, if the number of populations is given as weight of each point, considering this number inside an interval seems more realistic.

One of the efficiency evaluation methods is data envelopment analysis (DEA). DEA is a linear programming method used for assessment the relative efficiency of decision-making units (DMUs) with multiple inputs and multiple outputs. DEA is a nonparametric method that utilizes production frontier for measuring relative efficiency, which is initially considered as ratio of weighted outputs to weighted inputs. This method has attracted attention over three past decades and has many applications such as measurement efficiency of firms, banks and hospitals. DEA was first introduced by Charnes, Cooper and Rhodes [4], and then extended by Banker et al. [5] for measuring variable return to scale.

Due to importance of efficiency concept in optimization and decision-making problems and forasmuch as the semi-obnoxious line location problem has not ever been studied in view point of DEA so far, a novel approach is applied to consider the problem by using DEA. Therefore, lines are considered as DMUs and then efficiency of the DMUs is investigated instead of optimality of the line. Moreover, if several DMUs are obtained as efficient, a ranking method would be applied to determine the super-efficient DMU. Furthermore, due to the inherent uncertainty of the parameters of the line location problem in the real world such as weights or/and coordinates of demand points, the problem under interval data is studied in viewpoint of data envelopment analysis as well. Also, some properties and a numerical example are provided to investigate the problem. The main contributions of this paper are as follows:

1. For the first time, the semi-obnoxious line location problem is studied in view point of DEA.
2. Efficiency of the lines is considered instead of optimality of lines for the first time.
3. For the first time, the line location problem with interval data is studied.
4. Some properties of the problem in DEA are proved.

Also, for the case where the data such as weight of demand points and coordination of points are definitive, this technique can be applied for evaluating the efficiency of candidate lines obtained from meta-heuristic algorithms studied in the literature for finding location of semi-obnoxious lines in the plane, hence the optimal and efficient lines satisfying demand of existing points can be found.
This paper consists of following sections: The next section presents literature review. Section 3 deals with proposed method and some properties of the problem. section 4 presents a numerical example. Finally, conclusion is given in section 5.

2. Literature review

2.1.Line location problem

Wesolowsky [6] was the first who dealt with the line location problems and its applications. He suggested the first exact algorithm for tackling the median type of line location location problems. Morris and Norback [1] developed the problem by other metrics. They studied minisum and minimax types of the liner facility location problem with rectilinear norm and presented some properties of the problem. Furthermore, Lee and Cheng [7] applied Euclidean norm for solving the problem and proposed a proper algorithm with time complexity of $O(n^2)$. Drezner and Wesolowsky [8] were the first who discussed the obnoxious line location problem. They provided two algorithms: A non-linear path is found by iteratively solving network minimal-cut problems in the first one and the second algorithm considers the case that the route is assumed to be linear. In addition, their presented algorithm for obnoxious center line location problem took an $O(n^3)$ time. Furthermore, Chen and Wang [9] applied a geometric approach and solved obnoxious line location problem in $O(n^2 \log n)$ time. The first location model with positive and negative weights in networks was discussed by Burkard and Krarup [10]. They investigated 2-median semi-obnoxious facility location in trees. Since then, Fathali et al. [11] discussed the location problems with positive/negative weights on a graph where two objective functions are considered for studying the problem, namely the weighted sum of the minimum distances and the sum of the minimum weighted distances and used Ant Colony algorithm with a tabu restriction for tackling both objective functions. The semi-obnoxious location problem in the plane was presented in [12] [13].
Golpayegani et al. [2] developed the line location problem with the consideration of desirable and undesirable environmental effects of the line by using various norms. They first presented the problem with Euclidean norm and solved it by using particle swarm optimization (PSO). Then, they considered the problem in the plane with rectilinear norm and used genetic algorithm (GA) for tackling the problem [3]. As expected from metaheuristic algorithms, these methods do not provide exact solutions and the solutions are tried to be close to optimality as much as possible.

2.2 Facility location and DEA

Facility location problems with DEA approach has been studied widely in recent decade. Thomas et al. [14] investigated obnoxious facility location problem with DEA approach. To do so, they first determined optimal location of facilities and considered them as inputs of the DMUs. Then, a binary integer program was applied to analyze the location of sites, while assessment was implemented by a hierarchical process of DEA. Cook and Green [15] introduced a DEA model for finding the best sites for new facilities i.e., retail outlets in order to maximize ratio of benefits to costs considering existing budget constraints. Azadeh et al. [16] applied an integrated hierarchical DEA for optimal location of solar energy sites. Then, they evaluated the results obtained from presented DEA model by using principal component analysis (PCA) and numerical taxonomy (NT) methods. Klimberg and Ratick [17] used DEA method to determine optimal and efficient location/allocation patterns for both capacitated and un-capacitated cases in which minimizing total cost and obtaining better efficiency were considered simultaneously. Moheb-Alizadeh et al. [18] incorporated DEA method into location/allocation models in a fuzzy environment and presented a fuzzy multi-objective nonlinear programing model in order to study the pattern of location of facilities and the assignment of demands. Karbasian and Dashti [19] extended dispersion facilities location problem by using DEA and applied a fuzzy goal programing method to find optimal and efficient location patterns along with maximizing services to the demand points. Mohaghar et al. [20] investigated the problem of supplier selection and applied fuzzy VIKOR and assurance region DEA method for selection and ranking suppliers. Mitropoulos et al. [21] addressed the efficient allocation of resources in a healthcare system in order to determine location of health providers and assessment of service providers by using DEA and integer programing location allocation models. Bozorgi et al. [22] applied DEA for dynamic facility layout problem with equal departments. Also, an integrated computer simulation-stochastic DEA for solving job shop facility layout design problem was discussed by Azadeh et al. [23]. Darestani and Mohammadreza [24] presented a fuzzy DEA model for evaluating efficiency of the selected locations in a multi-objective covering facility location problem. Hong and Jeong [25] applied DEA technique for evaluating various alternative facility location/allocation schemes obtained from a multi-objective programing model in order to designing efficient supply chain network. Alhassam et al. [26] applied DEA for assessing efficiency of private and public primary health facilities in Ghana. Segall et al. [27] used DEA approach to the healthcare facility location problem in which demographic factors considered in the selection of candidate location of healthcare facility were classified into inputs and outputs. Moreover, DEA has attracted interests in locating wind plants. For example, Azadeh et al. [28] presented a fuzzy-DEA model for decision making on wind plant locations. Then, the results obtained from presented DEA model were examined by using principal component analysis and numerical taxonomy methods. Khanjarpanah and Jabbarzadeh [29] presented an optimization approach for locating power plants. Then, they defined sustainable criteria to assess the efficiency of candidate locations by using a novel DEA model under uncertainty. Furthermore, a novel multi-period network DEA was introduced by Khanjarpanah et al. [30] for locating wind-photovoltaic power plants. Ketabi et al. [31] applied DEA and super efficiency DEA models along with Analytic hierarchy process and computer simulation for optimizing facility layout problem.

2.3 DEA with interval data

Data envelopment analysis with interval data (IDEA) has been discussed by manyresearches. Despotis and Yianni [32] provided an alternative way for considering this type of data in DEA. They transformed a nonlinear IDEA model into a linear programming tantamount model, on the basis of the original data set, by using transformations only on the variables. Upper and lower bounds for the performance scores of the units were then described as natural results of their formulizations. Zhu [33] investigated and also compared two methods for solving nonlinear IDEA: the one is based on scale transformation and the other one by using standard CCR model. He showed that when there is a weight limitation, scale-based and variable-based approaches do not convert linear programming into a non-linear model of IDEA. Also, since strong introductory relations in a change-based and variable-based approach are not used correctly, he developed an improved and correct approach for strong introductory relationships. Jahanshaloo et al. [34] discussed sensitivity and stability for all DMUs, with interval data. Their method of classification remains unaltered under perturbations of the interval data. In 2009, Jahanshahloo et al. [35] proposed a model called interval generalized (IGDEA) model that enables the evaluation of the efficiency of several IDEA models in an integrated manner by combining the various preferences of decision makers. In addition, Hatami Marbini, Emrouznejad and Agrell [36] gave a comprehensive evaluation process to measure the relative efficiency of a group of DMUs in DEA with interval and negative data. In this method, DMUs are classified into three categories such as completely efficient, weak efficient, and inefficient. Bagheri et al. [37] introduced a DEA model with fuzzy data for evaluating the knowledge levels in a knowledge-based organization in various time intervals.

3. Background

3.1. Semi-obnoxious line location problem

The mathematical definition of the line location problem is as follows:
Consider a distance function d, given indices $M=\{1,2,...,N\}$ and a group of demand points $\varepsilon_x=\{E_{x1},E_{x2},...,E_{xN}\}$ in the plane where $E_{xi}=(a_{i1},a_{i2})$ and each demand point has weight w_i. The goal is finding a straight line $L_{p,s}=\{x: x=p+\lambda s,\ \lambda\in\mathbb{R}\}$ minimizing the following function:

$$f(L)=\sum_{i\in M} w_i\, d\left(E_{x_i},L\right) \tag{1}$$

Where $d\left(E_{x_i},L\right)=Min_{\ p\in L}\, d\left(E_{x_i},p\right)$ represents the distance between the line and the existing facilities.

3.2. Data Envelopment Analysis

Consider a group of decision-making units $\{DMU_k \mid k=1,2,...,n\}$ in which DMU_k produces
multiple outputs $y_{rk},\ (r=1,2,...,s)$ by utilizing multiple inputs $x_{jk},\ (j=1,2,...,m)$. In fact, the producer uses input vector $x\in\mathbb{R}_+^m$ to generate output vector $y\in\mathbb{R}_-^s$. Also, all data are supposed to be nonnegative, whereas at least one component of every input and output vector is positive. The production possibility set (PPS) is considered as follows [4]:

$$T_c=\left\{(X,Y): \sum_{k=1}^{n}\lambda_k x_k\le X,\ \sum_{k=1}^{n}\lambda_k y_k\ge Y,\lambda_k\ge 0,\ k=1,2,...,n\right\}$$

Definition 1: $DMU_\alpha=(x_\alpha,y_\alpha)$ is called Pareto efficient if there is not any $(x,y)\in T_c$ such that we have $(-x,y)\ge(-x_\alpha,y_\alpha)$ and $(-x,y)\ne(-x_\alpha,y_\alpha)$.

For evaluating the relative efficiency of DMU_α, $(\alpha=1,2,...,n)$ under assumption of constant returns to scale, two basic models provided by Charnes and Cooper [5] are presented. They will not be discussed in details.

$$
\begin{aligned}
&\theta^* = Min\ \theta - \varepsilon(1s^+ + 1s^-) && (2)\\
&s.t.\\
&\sum_{k=1}^{n}\lambda_k x_{ik} = \theta x_{i\alpha} - s_i^-, && i = 1,2,...,m\\
&\sum_{k=1}^{n}\lambda_k y_{rk} = y_{r\alpha} + s_r^+, && r = 1,2,...,S\\
&\lambda \geq 0,\ s^+ \geq 0,\ s^- \geq 0
\end{aligned}
$$

and,

$$
\begin{aligned}
&\varphi^* = Max\ \varphi + \varepsilon(1s^+ + 1s^-) && (3)\\
&s.t.\quad \sum_{k=1}^{n}\lambda_k x_{jk} = x_{j\alpha} - s_j^-, && j = 1,2,...,m\\
&\sum_{k=1}^{n}\lambda_k y_{rk} = \varphi y_{r\alpha} + s_r^+, && r = 1,2,...,S\\
&\lambda \geq 0,\ s^+ \geq 0,\ s^- \geq 0
\end{aligned}
$$

The models (2) and (3) are named input and output oriented, respectively. Also, DMU_α $(\alpha = 1,2,...,n)$ is Pareto efficient if and only if either 1 or 2 happen:

1. $\theta^* = 1$ and $s^+ = 0$ and $s^- = 0$ in model (2).
2. $\varphi^* = 1$ and $s^+ = 0$ and $s^- = 0$ in model (3).

3.3 A super-efficiency model: LJK– CCR model

It is clear that efficient DMUs obtained in most DEA models are not comparable because they get the same efficiency score. In order to provide a useful performance evaluation of all DMUs, ranking DMUs in DEA has become an attractive topic in recent decades. For a review of ranking methods, see Adler et al. [38]. Monfared et al. [39] proposed a method for ranking of Iranian universities. Rezai Balf et al. [40] provided a ranking method using Tchebycheff norm. Jahanshahloo et al. [41] presented a method using ideal points. They studied the problem to obtain an efficiency interval involving assessments from both the optimistic and the pessimistic viewpoints. In addition, Sexton et al. [42] based on a cross-efficiency ratio matrix proposed a method of ranking of DMUs. Also, Jahanshahloo et.al [43] presented the symmetric weight assignment technique in the cross-efficiency evaluation method. Furthermore, some models like Andersen and Petersen (AP) [44] model has some difficulties. Banker and Gifford [45][30] were the first to recognize the possibility of infeasibility of the AP model. They proved that the infeasibility could not happen for positive inputs. Also, Thrall [46] stated that the model presented by Anderson and Petersen may result in infeasibility and instability if some inputs are close to zero. Mehrabian et al. [47]expanded a super-efficiency model (called MAJ model) that resolves the disadvantages of the AP model such as infeasibility and instability. It can be seen, when the constant-return-to-scale DEA models are applied, the infeasibility could arise in the super-efficiency assessment if and only if there is zero in input [48].

Because of existence of some zero inputs in the DMU under evaluation in our problem, which will be noticed in section 3, the LJK-CCR model proposed by Shanling et al. [49] is applied. This model is as below:

$$Min(1+\frac{1}{m}\sum_{j=1}^{m}\frac{s_{j2}^{+}}{R_j^{-}}) \qquad (4)$$

$$\begin{aligned} s.t.\quad & \sum_{\substack{k=1\\k\neq o}}^{n}\lambda_k x_{ik}+s_{j1}^{-}-s_{j2}^{+}=x_{j\alpha}, && j=1,2,\ldots,m \\ & \sum_{\substack{k=1\\k\neq o}}^{n}\lambda_k y_{rk}+s_r^{+}=y_{r\alpha}, && r=1,2,\ldots,s \\ & \lambda_k\geq 0\ , && k=1,2,\ldots,n \\ & s_{j1}^{-}\geq 0,\ s_{j2}^{+}\geq 0\ , && j=1,2,\ldots,m \\ & s_r^{+}\geq 0\ , && r=1,2,\ldots,s \end{aligned}$$

in which X_k and Y_k are input and output vectors, respectively. Also, R_j^{-} is maximum of all *j*th inputs including *j*th input of DMU_k, i.e., $R_j^{-}=Max_j(x_{jk})$.

3.4. Interval DEA

Now, suppose that there are *n* DMUs which use *m* inputs to generate *s* outputs in which unlike the original DEA model, the input and/or output amounts are not exactly determined and are considered within bounded intervals, i.e. $x_{jk}\in\left[x_{jk}^{l},x_{jk}^{u}\right]$ and $y_{rk}\in\left[y_{rk}^{l},y_{rk}^{u}\right]$, where upper and lower bounds of the intervals are assumed to be constant and $x_{jk}^{l}>0$ and $y_{rk}^{l}>0$ [42]. If $y_{rk}^{l}=y_{rk}^{u}$ then the *r*th output of the *k*th *DMU* has a definite value. The following models provide the lower limit and upper limit of interval assessment, respectively:

$$\theta_{\alpha}^{*L}=Max\sum_{r=1}^{s} y_{ro}^{L}u_r \qquad (5)$$

$$\begin{aligned} s.t.\quad & \sum_{r=1}^{s} y_{rk}^{U}u_r-\sum_{i=1}^{m} x_{jk}^{L}v_j\leq 0, && k=1,2,\ldots,n;\ k\neq\alpha \\ & \sum_{r=1}^{s} y_{r\alpha}^{L}u_r-\sum_{i=1}^{m} x_{j\alpha}^{U}v_j\leq 0 \\ & \sum_{j=1}^{m} x_{j\alpha}^{U}v_j=1 \\ & u_r\geq\varepsilon,\ \ v_j\geq\varepsilon, && r=1,\ldots,s;\ \ j=1,\ldots,m \end{aligned}$$

and,

$$
\begin{aligned}
\theta_\alpha^{*U} = Max \sum_{r=1}^{s} y_{r\alpha}^{U} u_r \qquad\qquad & (6) \\
s.t. \quad \sum_{r=1}^{s} y_{rk}^{L} u_r - \sum_{j=1}^{m} x_{jk}^{U} v_j \le 0 \qquad & k = 1, 2, \ldots, n \; ; k \ne \alpha \\
\sum_{r=1}^{s} y_{r\alpha}^{U} u_r - \sum_{j=1}^{m} x_{j\alpha}^{L} v_j \le 0 \qquad & \\
\sum_{j=1}^{m} x_{j\alpha}^{L} v_j = 1 \qquad & \\
u_r \ge \varepsilon, \quad v_j \ge \varepsilon \qquad & r = 1, \ldots, s \;\; ; \;\; j = 1, \ldots, m
\end{aligned}
$$

The two above models provide the efficiency of each *DMU* that belongs to the bounded interval $\left[\theta_\alpha^{*L}, \theta_\alpha^{*U}\right]$. Therefore, the following relation is satisfied for the evaluated efficiency of each selected input and output:

$\theta_\alpha^{*L} \le \theta_\alpha^{*} \le \theta_\alpha^{*U}$

Relying on the idea of the interval efficiencies, all *DMU*s can be considered in one of the three following classes [35, 38]:

1. $E^{-} = \left\{ DMU_k, \;\; k = 1, \ldots, n \,|\, \theta_k^{*U} < 1 \right\}$

If $\theta_k^{*U} < 1$, it concludes that $\theta_k^{*L} < 1$. Then this class consists of all DMUs which are inefficient in their best and worst status.

2. $E^{+} = \left\{ DMU_k, \;\; k = 1, \ldots, n \,|\, \theta_k^{*L} < 1 \,,\; \theta_k^{*U} = 1 \right\}$

This class consists of all DMUs which are inefficient in their worst situation, whereas are efficient in their best status.

3. $E^{++} = \left\{ DMU_k, \;\; k = 1, \ldots, n \,|\, \theta_k^{*L} = 1 \right\}$

Then. If $\theta_k^{*L} = 1$, it concludes that $\theta_k^{*U} = 1$ then this class consists of all DMUs which are efficient both in their best and worst status.

In the next section, the SOLLP in DEA is introduced and some properties are proved.

4. Proposed Method

As mentioned, in the semi-obnoxious median line location problem, the aim is finding a straight line in the plane which minimizes sum of weighted distances from the demand points to the line. It implies that the line is located at maximum possible weighted distances from the points with negative weight and be close to the positive points as much as possible at the same time so that the objective function 1 be minimized.

Furthermore, the main goal in basic DEA approach is improving efficiency of a DMU under evaluation and obviously the common way to reach this goal is decreasing the inputs and increasing the outputs of the DMU.

Hence, if lines are considered as DMUs, then, finding an optimal semi-obnoxious median line in the plane can be transformed into the problem of finding the corresponding efficient DMU in DEA area.

However, prior to dealing with the new approach, a property of the semi-obnoxious line location proved by Golpayegani et.al., ([2], [3]) is presented in the following:

Proposition 1: In the line location problem with positive and negative weighted points, at least one of the demand points with positive weight lies on the optimal line. Now, the following definition provides the new method for investigating semi-obnoxious median line location by using DEA method:

Definition 2: Let *n* lines are given. Hence, these lines can be considered as *DMU*s as follows:

$$DMU_k(L_k) = (x_{1k}, x_{2k}, ..., x_{mk}, y_{1k}, y_{2k}, ..., y_{sk}) \tag{7}$$

in which $DMU_k; k = 1,...,n$ are defined corresponding to $L_k; k = 1,...,n$ in which, $x_{jk};\ j = 1,...,m$ & $k = 1,...,n$ is the distance from a point with positive weight $v_j \geq 0$ to the line *k* and $y_{rk};\ r = 1,...,s$ & $k = 1,...,n$ is considered to be a distance from a point with negative weight $u_r \leq 0$ to the line *j*. In fact, the demand points are considered as set of points with positive weight $v_j \geq 0$ and demand points with negative weight $u_r \leq 0$ where *n*=*m*+*s*. Moreover, since in semi-obnoxious median line location weighted distances of the demand points to the line are considered, so the relation (7) is taken into account as follows:

$$DMU_k(L_k) = (v_1 x_{1k}, v_2 x_{2k}, ..., v_m x_{mk}, |u_1| y_{1k}, |u_2| y_{2k}, ..., |u_s| y_{sk}) \tag{8}$$

In order to increase the efficiency score, classic DEA models rely on the assumption that inputs have to be minimized and outputs have to be maximized, then according to the relation (8) an efficient *DMU* actually is a line locating at maximum possible weighted distance from the points with negative weight and is close to the points with positive weight as much as possible, simultaneously.

Note: When lines are transformed to DMUs, two cases may occur:

1) all inputs and outputs of a DMU correspond to a line is equal to zero. In this case, we consider the corresponding DMU as efficient.
2) $\exists i, j;\ v_i x_{ik} \neq 0,\ |u_r| y_{rk} \neq 0\quad i = 1,...,m,\ j = 1,...,s,$, i.e. at least one point with positive weight and negative weight do not lie on the given line.

In this paper we deal with the second case and obtain some properties of the problem. The following theorem illustrates the meaningful relationship between optimal median line in the plane and the corresponding *DMU* in DEA.

Theorem 1: In a semi-obnoxious median line location problem if line L_t is optimal, so its corresponding *DMU* i.e. DMU_t has at least one input equal to zero.

Proof: First, consider the following model:

$$\theta^* = Min\,\theta - \varepsilon(1s^+ + 1s^-) \qquad (9)$$

$$s.t.\ \sum_{k=1}^{n}\lambda_k x_{1k} = \theta x_{1p} - s_1^-$$

$$\sum_{k=1}^{n}\lambda_k x_{jk} = \theta x_{jp} - s_j^-, \qquad j = 1,2,...,m$$

$$\sum_{k=1}^{n}\lambda_k y_{rk} = y_{rp} + s_r^+, \qquad r = 1,2,...,s$$

$$\lambda_k \geq 0, s^+ \geq 0, s^- \geq 0 \qquad k = 1,2,...,n$$

We prove the theorem by using four lemmas as follows:

Lemma 1: If $(\lambda^*, \theta^*, s^{-*}, s^{+*})$ is the optimal solution of the model (9), then $s^{-*} = 0$.

Proof:

$$\forall k \quad \lambda_k^* \geq 0,\ x_{1k} \geq 0 \Rightarrow \sum_{k=1}^{n}\lambda_k^* x_{1k} \geq 0 \Rightarrow 0\theta - s_1^{-*} \geq 0$$

Then, $s_1^{-*} \leq 0 \xrightarrow{s_1^{-*} \geq 0} s_1^{-*} = 0.$

Also. the above lemma shows that in optimality $\sum_{k=1}^{n}\lambda_k^* x_{1k} = 0$

Lemma 2: In the model (9), if $\lambda_t^* > 0,$ then $x_{1t} = 0,$.

Proof: By using ad absurdum, suppose that $\exists t;\ \lambda_t^* > 0 \ \ \& x_{1t} > 0,$. Therefore;

$$\forall k \quad \lambda_k^* \geq 0,\ x_{1k} \geq 0 \Rightarrow \sum_{k=1}^{n}\lambda_k x_{1k} \geq 0 \Rightarrow 0\theta - s_1^{-*} \geq 0$$

$$\sum_{k=1}^{n}\lambda_k^* x_{1k} = \lambda_k^* x_{tk} + \sum_{\substack{k=1\\k\neq t}}^{n}\lambda_k^* x_{1k} > 0 \Rightarrow 0 > 0$$

which is a contradiction. So, ad absurdum statement is rejected and the lemma is satisfied.

Lemma 3: In the model (9), we have $\lambda^* \neq 0$.

Proof: Using ad absurdum let $\lambda^* = 0$. According to output constraints;

$$\sum_{k=1}^{n}\lambda_k^* Y_k = Y_p + s^{+*} \Rightarrow Y_p + s^{+*} = 0 \Rightarrow Y_p = 0 \ \& \ s^{+*} = 0$$

This is a contradiction because Y_p cannot be equal to zero. So, $\lambda^* \neq 0$.

Lemma 4: let $(\lambda^*, \theta^*, s^{-*}, s^{+*})$ is the optimal solution of model (9), if $\lambda_t^* > 0,$ then DMU_t is strongly efficient.

Proof: consider the following model as dual of model (9):

$$\begin{aligned} & Max\, UY_p && (10) \\ & s.t. \quad VX_p = 1 \\ & \qquad UY_k - VX_k \le 0 && \forall k \\ & \qquad U \ge 1\varepsilon,\ V \ge 1\varepsilon \end{aligned}$$

Suppose that $\lambda_t^* > 0,$. Then, if (U^*, V^*) is the optimal solution for (10), according to the complementary slackness theorem we have:

$$U^* Y_k - V^* X_k = 0$$
$$U^* > 0,\ V^* > 0$$

Also, since $V^* X_t > 0$, therefore $V^* X_t = \beta > 0$.
Now, we use model (10) for evaluating DMU_t:

$$\begin{aligned} & Max\, UY_t && (11) \\ & s.t. \quad VX_t = 1 \\ & \qquad UY_j - VX_j \le 0 \quad , && \forall j \\ & \qquad U \ge 1\varepsilon,\ V \ge 1\varepsilon \end{aligned}$$

By putting $\overline{U} = \dfrac{U^*}{\beta}, \quad \overline{V} = \dfrac{V^*}{\beta},$, we have:

$$\begin{aligned} & U^* Y_k - V^* X_k = 0 \\ & \overline{U} Y_j - \overline{V} X_j = \frac{1}{\beta}(U^* Y_j - V^* X_j) \le \frac{1}{\beta}(0) = 0, \\ & \overline{V} X_t = \frac{1}{\beta}(V^* X_t) = \frac{1}{\beta}.\beta = 1 \qquad (12) \\ & \overline{V} = \frac{1}{\beta} V^* > 0 \Rightarrow \overline{V} \ge 1\varepsilon \\ & \overline{U} = \frac{1}{\beta} U^* > 0 \Rightarrow \overline{U} \ge 1\varepsilon \end{aligned}$$

and the objective function is:

$$\overline{U} Y_t = \frac{1}{\beta} U^* Y$$

Now, from (12) we conclude that, $\dfrac{1}{\beta}(U^* Y_t - V^* X_t) = 0 \Rightarrow \overline{U} Y_t - \overline{V} X_t = 0$,Using $\overline{V} X_t = 1$ we have $\overline{U} Y_t = 1$.

Therefore $(\overline{U}, \overline{V})$ is optimal for (11). Hence, DMU_t is strongly efficient.

The presented lemmas indicate there exist $t \in \{1,...,n\}$ so that DMU_t is strongly efficient in T_c and $x_{1t} = 0$. It means that in optimality, at least one input is equal to zero and it fulfills the theorem 1.

In the following, a numerical example is provided to illustrate the problem with interval data in the plane.

5. Numerical Example

Suppose that 5 weighted demand points in R^2 with interval coordination are given in Table 1:

Table1. Five demand points in the plane

Point	Coordination (x, y)	weight
A_1	(0.9 , 1.1)	1
A_2	(1.1, 0.9)	1
B_1	(-1.1, -0.9)	1
B_2	(-0.9, -1.1)	1
C_1	(0.9, -0.1)	1
C_2	(1.1, 0.1)	1
D_1	(-1.1, 0.9)	-1
D_2	(-0.9, 1.1)	-1
E_1	(-0.1, 2.1)	-1
E_2	(0.1, 2.1)	-1

Now, for finding the optimal line we consider the following 14 lines as candidates:
$L_1: y - x = 0.2$, $L_2: y - x = -0.2$, $L_3: x = 0.9$, $L_4: x = 1.1$, $L_5: x = -1.1$, $L_6: x = -0.9$, $L_7: y = 1.1$, $L_8: y = 0.9$, $L_9: y = -0.1$, $L_{10}: y = 0.1$, $L_{11}: x = 0.1$, $L_{12}: x = -0.1$, $L_{13}: x + 2y = 0.7$, $L_{14}: x + 2y = 1.3$

Due to the importance of the interval data, the input and output of the DMUs correspond to the 14 lines is aimed to be considered as intervals. As mentioned before, weights of demand points or distances from the demand points to the line, or both, may be fluctuated. So, if distances of points to the line or weights of points or both are given as intervals, inputs and outputs of the DMUs would be interval data. Therefore, suppose that inputs and outputs of the 14 DMUs have tolerance of 0.05. Hence, we form 14 DMUs as Table 2:

Table2. 14 DMU_s with 6 inputs and 2 outputs

DMU	Input	Output	DMU	Input	Output
1	$\begin{pmatrix} (0.0\,,0.05) \\ (0.23\,,0.33) \\ (0.0\,,0.05) \\ (0.24\,,0.34) \\ (0.80\,,0.90) \\ (0.80\,,0.90) \end{pmatrix}$	$\begin{pmatrix} (1.22\,,1.32) \\ (1{,}22\,,1.32) \end{pmatrix}$	8	$\begin{pmatrix} (0.15\,,0.25) \\ (0.0\,,0.05) \\ (1.75\,,1.85) \\ (1.95\,,2.05) \\ (0.95\,,1.05) \\ (0.75\,,0.85) \end{pmatrix}$	$\begin{pmatrix} (0.0\,,0.05) \\ (0.15\,,0.25) \end{pmatrix}$
2	$\begin{pmatrix} (0.23\,,0.33) \\ (0.0\,,0.05) \\ (0.23\,,0.33) \\ (0.0\,,0.05) \\ (0.52\,,0.62) \\ (0.52\,,0.62) \end{pmatrix}$	$\begin{pmatrix} (1.51\,,\ 1.61) \\ (1.51\,,\ 1.61) \end{pmatrix}$	9	$\begin{pmatrix} (1.15\,,1.25) \\ (0.95\,,1.05) \\ (0.75\,,0.85) \\ (0.95\,,1.05) \\ (0.0\,,0.05) \\ (0.15\,,0.25) \end{pmatrix}$	$\begin{pmatrix} (0.95\,,1.05) \\ (1.15\,,1.25) \end{pmatrix}$
3	$\begin{pmatrix} (0.0\,,0.05) \\ (0.15\,,0.25) \\ (1.95\,,2.05) \\ (1.75\,,1.85) \\ (0.0\,,0.05) \\ (1.95\,,2.05) \end{pmatrix}$	$\begin{pmatrix} (1.95\,,2.05) \\ (1.75\,,1.85) \end{pmatrix}$	10	$\begin{pmatrix} (0.95\,,1.05) \\ (0.75\,,0.85) \\ (0.95\,,1.05) \\ (1.15\,,1.25) \\ (0.15\,,0.25) \\ (0.0\,,0.05) \end{pmatrix}$	$\begin{pmatrix} (0.75\,,0.85) \\ (0.95\,,1.05) \end{pmatrix}$
4	$\begin{pmatrix} (0.15\,,0.25) \\ (0.0\,,0.05) \\ (2.15\,,2.25) \\ (1.95\,,2.05) \\ (0.15\,,0.25) \\ (0.0\,,0.05) \end{pmatrix}$	$\begin{pmatrix} (2.15\,,2.25) \\ (1.95\,,2.05) \end{pmatrix}$	11	$\begin{pmatrix} (0.75\,,0.85) \\ (0.95\,,1.05) \\ (1.15\,,1.25) \\ (0.95\,,1.05) \\ (0.75\,,0.85) \\ (0.95\,,1.05) \end{pmatrix}$	$\begin{pmatrix} (1.15\,,1.25) \\ (0.95\,,1.05) \end{pmatrix}$
5	$\begin{pmatrix} (1.95\,,2.05) \\ (2.15\,,2.25) \\ (0.0\,,0.05) \\ (0.15\,,0.25) \\ (1.95\,,2.05) \\ (2.15\,,2.25) \end{pmatrix}$	$\begin{pmatrix} (0.0\,,0.05) \\ (0.15\,,0.25) \end{pmatrix}$	12	$\begin{pmatrix} (0.95\,,1.05) \\ (1.15\,,1.25) \\ (0.95\,,1.05) \\ (0.75\,,0.85) \\ (0.95\,,1.05) \\ (1.15\,,1.25) \end{pmatrix}$	$\begin{pmatrix} (0.95\,,1.05) \\ (0.75\,,0.85) \end{pmatrix}$
6	$\begin{pmatrix} (1.75\,,1.85) \\ (1.95\,,2.05) \\ (0.15\,,0.25) \\ (0.0\,,0.05) \\ (1.75\,,1.85) \\ (1.95\,,2.05) \end{pmatrix}$	$\begin{pmatrix} (0.15\,,0.25) \\ (0.0\,,0.05) \end{pmatrix}$	13	$\begin{pmatrix} (1.02\,,1.13) \\ (0.98\,,1.08) \\ (1.56\,,1.66) \\ (1.65\,,1.75) \\ (0.0\,,0.05) \\ (0.22\,,0.32) \end{pmatrix}$	$\begin{pmatrix} (0.0\,,0.05) \\ (0.22\,,0.32) \end{pmatrix}$
7	$\begin{pmatrix} (0.0\,,0.05) \\ (0.15\,,0.25) \\ (1.95\,,2.05) \\ (2.15\,,2.25) \\ (1.15\,,1.25) \\ (0.95\,,1.05) \end{pmatrix}$	$\begin{pmatrix} (0.15\,,0.25) \\ (0.0\,,0.05) \end{pmatrix}$	14	$\begin{pmatrix} (0.75\,,0.85) \\ (0.71\,,0.81) \\ (1.83\,,1.93) \\ (1.92\,,2.02) \\ (0.22\,,0{,}32) \\ (0.0\,,0.05) \end{pmatrix}$	$\begin{pmatrix} (0.0\,,0.05) \\ (0.22\,,0.32) \end{pmatrix}$

Now, we use models (5) and (6). As can be seen, $DMU_l \in E^+\,(l=1,3,\ 4,\ 5,\ 6,7,8,9,12,13,14)$, $DMU_2 \in E^{++}$, and $DMU_{10,11} \in E^-$. According to the section 3.4 and what introduced for E^-, E^+ and E^{++}, it can be said that

DMU_{10} and DMU_{11} are inefficient, while $DMU_l\,(l = 1, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14)$ are inefficient in their worst situation and are efficient in their best situation. Furthermore, DMU_2 is efficient both in its best and worst situation. Also, as stated before, due to the presented producer in this paper, considering this rule for table 1, DMU_2 corresponds to the line L_2 and has at least one input equal to zero that confirms theorem 1. This example shows that in viewpoint of DEA concept, an efficient line (*DMU*) has at least one input equal to zero, i.e. its corresponding line passes through at least one point with positive weight in the plane in the semi-obnoxious facility location problem. Also, for the case where the data such as weight of demand points and coordination of points are definitive, we can use this technique for evaluating the efficiency of candidate lines obtained from meta-heuristic algorithms studied in the literature for finding location of semi-obnoxious lines in the plane. In fact, candidate lines are obtained from meta-heuristic algorithms in the first step, then, by using the presented producer in this paper, the efficiency of corresponding *DMU*s is evaluated in the next step. So, the optimal and efficient lines satisfying demand of existing points can be found.

6. Conclusion

This paper studied the semi-obnoxious line location problems in DEA area for the first time. To do so, a novel approach for defining lines as DMUs was presented and then relative efficiency of DMUs was assessed. In addition, in the case that there are several DMUs as efficient, a ranking method called LJK-CCR model was used for ranking the efficient DMUs. Furthermore, since in the line location problems in many real situations, the amount of parameters such as weight of demand points or coordinate of points may be imprecise numbers, so considering the problem in the presence of these data seemed crucial. This issue was studied by data envelopment analysis as well. In addition, it was proved that DMUs which are efficient have at least one input equal to zero. Finally, an illustrative example was given to investigate the proposed method. Also, for the case where the data such as weight of demand points and coordination of points are definitive, this technique can be applied for evaluating the efficiency of candidate lines obtained from meta-heuristic algorithms studied in the literature for finding location of semi-obnoxious lines in the plane, hence the optimal and efficient lines satisfying demand of existing points can be found.

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