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Research Article

A New Approach on Scrambling of the Halton Sequence with Sampling

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Abstract
Although many scramble methods have been introduced for the Halton sequence, not all of them have been widely acclaimed due to their computational complexity and difficult implementation. Sampling the Halton sequence is an easy and fast way to achieve a more uniform and, of course, more efficient sequence.

Keywords: Halton sequence, scrambling, discrepancy, Weyl sequence, continued fraction, sampling.

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1. Introduction

Quasi-Monte Carlo methods are a variant of ordinary Monte Carlo methods that employ highly uniform quasirandom numbers in place of Monte Carlo's pseudorandom numbers. Clearly, the generation of appropriate high-quality quasirandom sequences is crucial to the success of using quasi-Monte Carlo methods. The Halton sequence, which is one of the standard low-discrepancy sequences, and one of its important advantages is that the Halton sequence is easy to implement due to its definition via the radical inverse function. However,the original Halton sequence suffers from correlations between radical inverse functions with different bases used for different dimensions. These correlations result in poorly distributed two-dimensional projections. A standard solution to this problem is to use a randomized (scrambled) version of the Halton sequence which have both good two-dimensional projections and a smaller discrepancy, which is a measure of deviation from uniformity [1,5].

2. The Halton sequences

Let $p \geq 0$ be an integer, then any integer $n \geq 0$ can be written in the form:

$$n = a_0 + a_1 p + \cdots + a_m p^m \qquad (1)$$

where, $0 \leq a_j \leq p$ and $m = \lceil \log_p n \rceil$ is the maximum number of digits needed to represent all *n*-values.

The radical inverse function φ for base p is defined by

$$\varphi_p(n) = \frac{a_0}{p} + \frac{a_1}{p^2} + \cdots + \frac{a_m}{p^{m+1}} \qquad (2)$$

The Van der Corput sequence in base p is defined as the one-dimensional point set $\left\{\varphi_p(n)\right\}_{n=0}^{\infty}$ Halton (1960) extended this definition to the s-dimensional sequence as

$$X_n = \left(\varphi_{p_1}(n), \varphi_{p_2}(n) \cdots, \varphi_{p_s}(n)\right) \qquad (3)$$

where $n = 0,1,2,\ldots$ and the dimensional bases $p_1, p_2, \ldots, p_s$ are pairwise coprime. In practice, we always use the first *s* primes as the bases.

As mentioned, the Halton sequence has poor two-dimensional projections (for dimensions greater than 10) because of the correlation between the radical inverse functions used for different dimensions [1]. Examples of these poor projections can be seen in Figure.1 (*b*, *c* and *d*).

In Figure.1 it can be seen that for small bases, the dispersion of the sequence points is acceptable. As the dimensions increase, the points fall into clusters of parallel lines with the $y = x$ line [1], and as these dimensions become larger, the number of clusters decreases, but are closer together, and the points accumulate in them, and the space becomes wider. All of the scrambles introduced are intended to break these lines and thus fill the space more uniformly.

The correlation between points of the Halton sequence can be broken by scrambling the digits of the sequence in a way that preserves the low-discrepancy properties. This was first formally described by Braaten and Weller[3], who defined the scrambled radical inverse function $S_p(n)$ as

$$S_p(n) = \frac{\pi_p(a_0)}{p} + \frac{\pi_p(a_1)}{p^2} + \cdots + \frac{\pi_p(a_m)}{p^{m+1}} \qquad (4)$$

Here, $\pi_p(a_j)$ is a permutation on the digits $\{0, 1, \ldots, p-1\}$ which holds the digit 0 fixed. There are $(p-1)!$ such permutations. The scrambled Halton sequence is then given by

$$X_n = \left(S_{p_1}(n), S_{p_2}(n) \cdots, S_{p_s}(n)\right) \qquad (5)$$

Note that since each set of permutations (one permutation for each dimension) always leads to the same scrambled version of the Halton sequence, the scrambling defined by 4 and 5 is a deterministic scrambling. If one is searching for the best possible deterministic scrambling under some criterium, then the search space should only contain permutations $\pi_p(a_j)$ [3].

3. The Halton sequence Scrambles

There are many scrambles for the Halton sequence that the interested reader can see them in [3,5]. In this article, we will review only the Warnock's PhiCF scramble and Chi's optimal scramble, and then compare the results obtained with our methods with the results of these two scrambles.

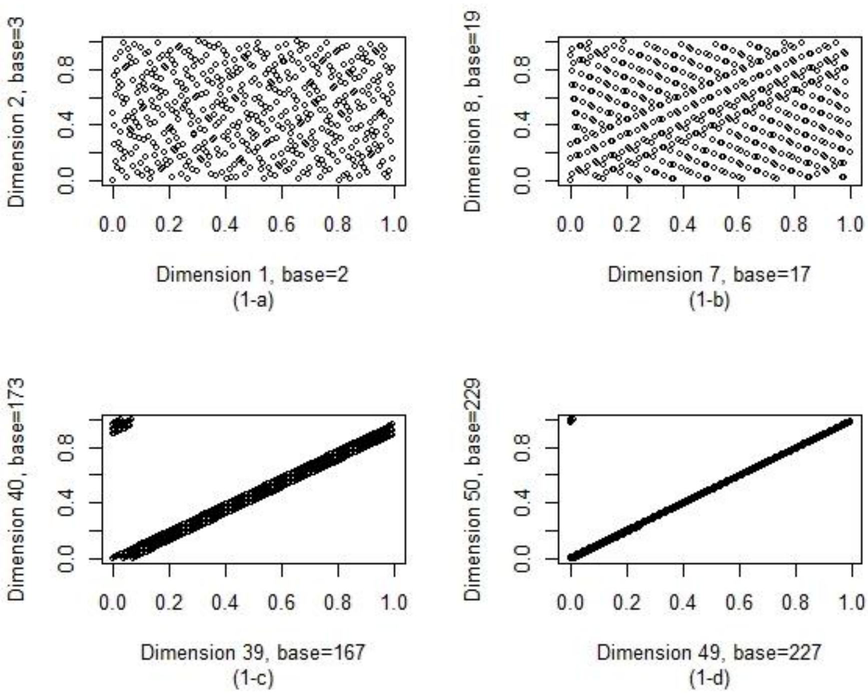

Figure 1. Different two-dimensional projections of the Halton sequence

3.1. Warnock's PhiCF sequence

Warnock combined the initial behavior of the Weyl sequence with the asymptotic behavior of the Halton sequence to construct what he called the PhiCF sequence. The name of this sequence is due to the use of continued fraction expansion in its structure. He replaces each a_j in 1 with $S(p).a_j \ mod \ p$ to obtain a new kind of radical inverse function

$$\omega_p(n) = \frac{S(p).a_0 \ mod \ p}{p} + \frac{S(p).a_1 \ mod \ p}{p^2} + \cdots + \frac{S(p).a_m \ mod \ p}{p^{m+1}} \qquad (6)$$

where $S(p)$ is defined to be a number such that $S(p)/p$ is close to the fractional part of $\sqrt{p}$.

Since p_i is different for each $S(p_i)$, then the scrambled version of a sequence generated by a different $S(p_i)$ is expected to be independent. This is due to the fact that the Weyl sequences, consisting of multiples of square roots of primes have good discrepancy in low dimension. The square roots of primes are independent [1].

3.2. Mascagni and Chi's optimal scrambling

Mascagni and Chi [1], considered the linear scrambling $\pi_p(a_j) = w_i .a_j \ mod \ p$ with w_i an integer. In [6] is an example that says for a prime modulus p , and a primitive root W modulo p as multiplier, we have that the discrepancy, $D_N^{(2)}$, satisfies

$$(p-1)D_{p-1}^{(2)} \leq 2 + \sum_{j=1}^{q} a_j \qquad (7)$$

where a_j is the *j*th digit in the continued fraction expansion of $\frac{W}{p}$ with $a_q = 1$. Using this criterion and inspired by Warnock's PhiCF Scramble, Mascagni and Chi tabulated the w_i coefficients up to the 50th dimension for their optimal Scramble.

4. Sampling from Halton sequence

This seems to be the simplest and best way to break the correlation between dimensions. If one can reorder or shuffle the digits of each point in the Halton sequence for different dimensions, the correlations between different dimensions can be made very small. This is due to the fact that there are gaps between the most significant digits of $\varphi_p(n),\ p < 10$, which have good two-dimensional projections. However, when, there are no gaps for the most significant digits of $\varphi_p(n)$ and the most significant digits cycle from 1 to 9 without jumps [1]. Figure 2. shows the improvement of the two-dimensional projections mentioned in the previous sections using this method.

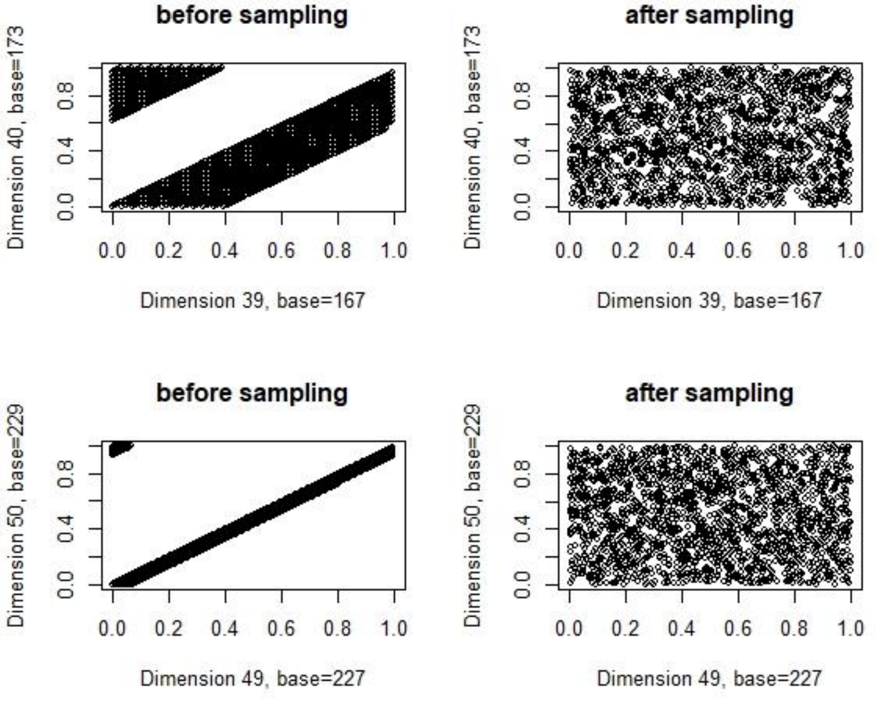

Figure 2. 2000 points of Halton sequence before(left) and after(right) sampling

Here, HSW is the Halton sequence after sampling with replacement, HSWO is the Halton sequence after sampling without replacement, CHi is Chi's optimal sequence and PhiCF is the Warnock's PhiCF sequence.

Another criterion by which we can compare sequences is discrepancy. It can be seen from Figure 3. that in 8 dimensions, for less than 1000 points, the discrepancy plot of the CHi and PhiCF sequences are below the other sequences. But as the number of points increases, this superiority belongs to the HSWO. Almost the entire path of the plot the discrepancy of HSW on top of the other three plot. As the dimension increases to 16, the situation changes dramatically. The discrepancy plot for CHi and PhiCF are almost identical, but much higher than the our sequences (Figure 4. up). For better view, we have drawn another plot by changing the scale of the vertical axis (Figure 4. down). Although HSW looks better at low points (less than 400), the discrepancy for HSWO is lower than the HSW sequence as the number of points increases. Therefore, in general, HSWO has a higher performance based on discrepancy.

5. Numerical results

In the following, we use the studied sequences in estimating test integrals. The integrals used here are:

$$I_1(f) = \int_0^1 \cdots \int_0^1 \prod_{j=1}^{s} \frac{\pi}{2} \sin(\pi x_j) dx_1 \cdots dx_s = 1 \qquad (8)$$

$$I_2(f)=\int_0^1\cdots\int_0^1\prod_{j=1}^{s}\frac{|4x_j-2|+a_j}{1+a_j}dx_1\cdots dx_s=1 \qquad (9)$$

where the a_j are parameters. The integrand in 9 allows a tuning of the relative importance of the variables, as well as of their interactions, by appropriate choices of the parameters. The key-concept used for this is effective dimension. The effective dimension of the function in 9 is tabulated [7]. There are four choices of parameters as follows:

(1) $a_j = 0 \quad for \ \ 1 \le j \le s$

(2) $a_j = 1 \quad for \ \ 1 \le j \le s$

(3) $a_j = j \quad for \ \ 1 \le j \le s$

(4) $a_j = j^2 \quad for \ \ 1 \le j \le s$

For the first choice of parameters, all variables are equally important and the truncation dimension is approximately the same as the nominal dimension. This is the most difficult case for numerical integration. For another choices of a_j, the importance of the successive variables is decreasing. In general, when a_j becomes bigger, the variables are decreasing quickly in importance and the effective dimension becomes smaller [3].

Figures 5 to 9 show the estimated values of the test integrals in dimensions 10, 20, 30 and 40. In all Figures, in 10 dimensions, all sequences converge to the exact value and have little fluctuation. As the dimensions increase, although the sequences deviate slightly from the exact value of the integrals, this deviation is very small for sequence HSWO and also it fluctuates less than other sequences. Therefore, according to these figures, it can be said that sequence HSWO is more stable in estimating test integrals than other sequences. The numerical results of the estimates are in Tables 1 to 5.

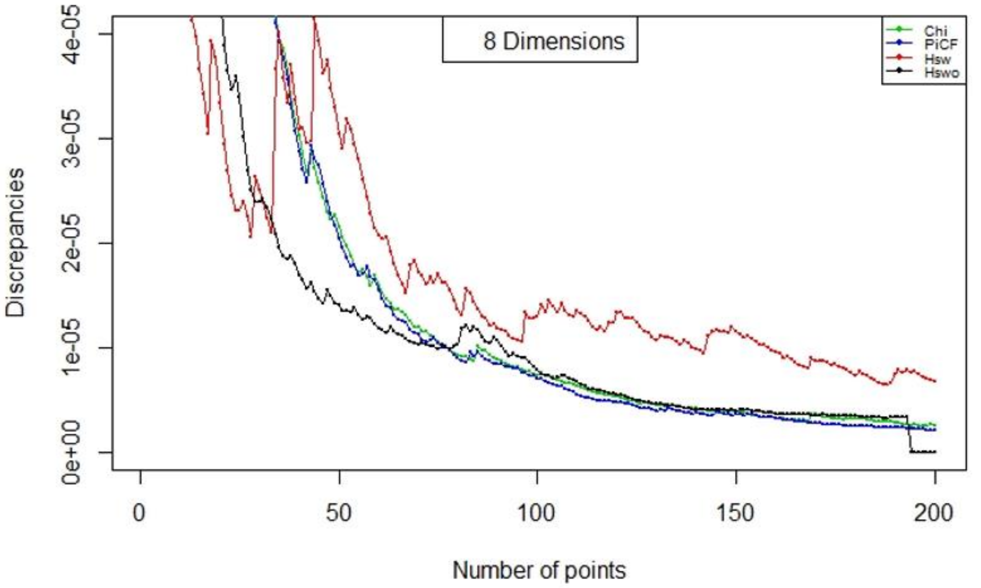

Figure 3. Discrepancies for several Halton sequences in 8 dimensions

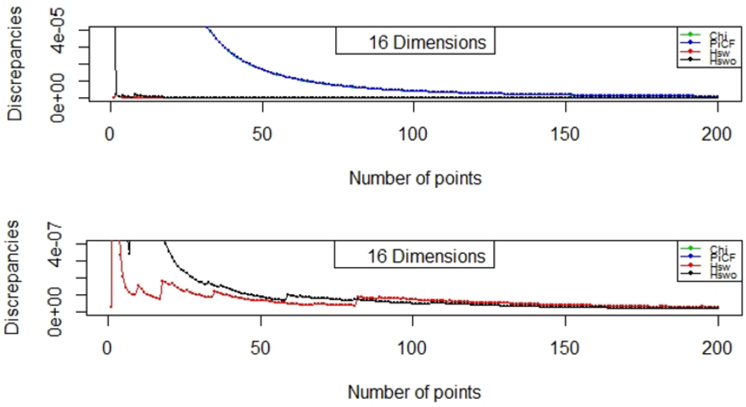

Figure 4. Discrepancies for several Halton sequences in 16 dimensions

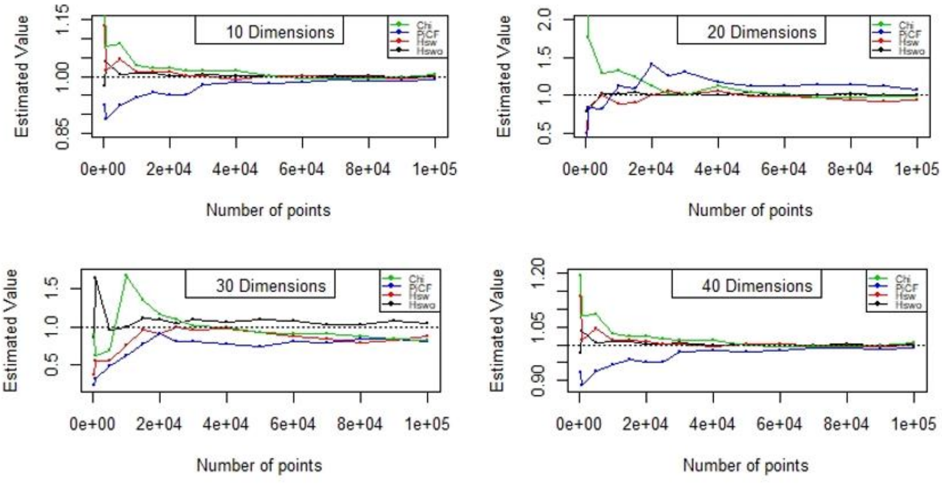

Figure 5. Estimates of the integral $\boldsymbol{I_1(f)}$ using various Halton sequences

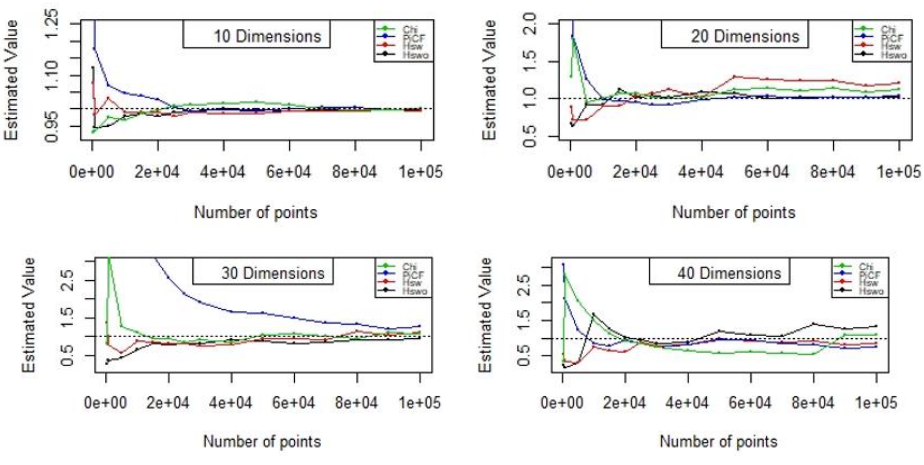

Figure 6. Estimates of the integral $\boldsymbol{I_2(f)}$ with $\boldsymbol{a_j = 0}$ using various Halton sequences

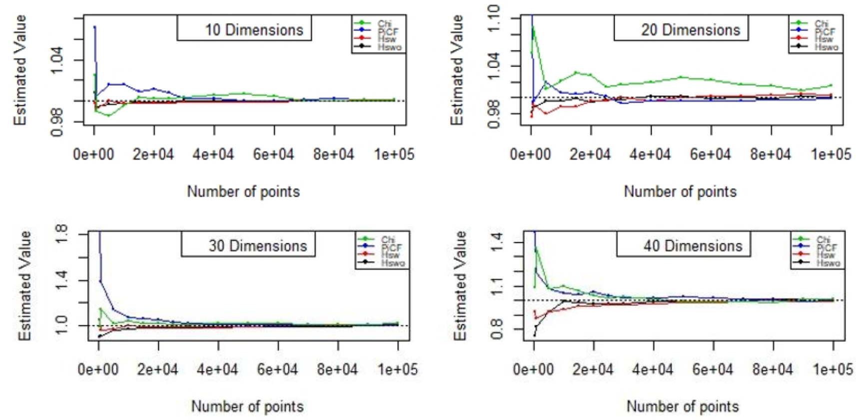

Figure 7. Estimates of the integral $I_2(f)$ with $a_j = 1$ using various Halton sequences

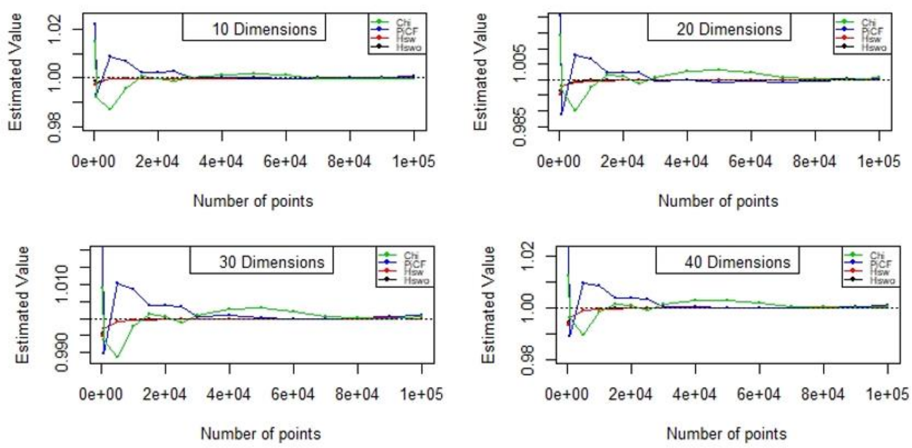

Figure 8. Estimates of the integral $I_2(f)$ with $a_j = j$ using various Halton sequences

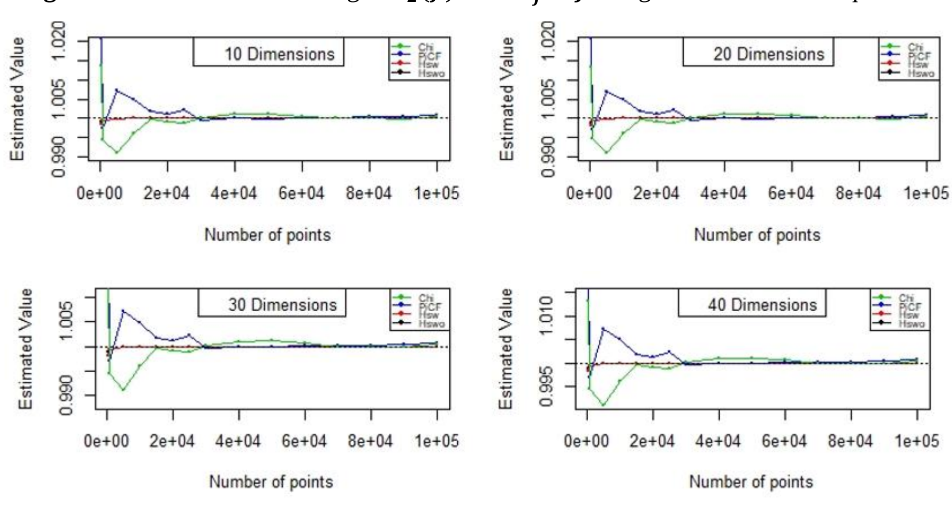

Figure 9. Estimates of the integral $I_2(f)$ with $a_j = j^2$ using various Halton sequences

6. Conclusion

In this article, we are introduced to the new Halton sequel scramble. We have seen that the non-replacement sampling method has both simplicity in execution and very high efficiency in more uniform distribution of points. Graphs of the estimated values of the test integrals show that the $HSWO$ is not only closer to the actual value but also has less fluctuations than other scrambles. Combining sampling with digits permutation or with reverse radical inverse function will be an idea for our future research.

Table 1. Estimates of the integral $\boldsymbol{I_1(f)}$ using various Halton sequences

generator	N	s=5	s=10	s=20	s=30	s=40
Chi	500	1.0614	1.1944	2.9632	0.8895	0.8895
PhiCF	500	0.8162	0.9223	0.2297	0.2260	0.2260
HSW	500	0.8922	1.1356	0.4991	0.3616	0.3616
HSWO	500	1.0016	0.9759	0.7917	0.8566	1.0767
Chi	1000	0.9844	1.0810	1.7651	0.6137	0.6137
PhiCF	1000	0.8801	0.8874	0.8410	0.3118	0.3118
HSW	1000	0.9377	1.0145	0.8264	0.5440	0.5440
HSWO	1000	0.9970	1.0386	0.7971	1.6366	1.2874
Chi	5000	1.0503	1.0869	1.2982	0.6835	0.6835
PhiCF	5000	1.0171	1.0470	1.0100	0.5510	0.5510
HSW	5000	0.9569	0.9253	0.8208	0.4750	0.4750
HSWO	5000	1.0015	1.0054	1.0295	0.9594	0.7626
Chi	10000	1.0013	1.0092	1.0217	0.9919	1.7507
PhiCF	10000	1.0062	1.0132	0.8859	0.7635	0.7635
HSW	10000	0.9777	0.9428	1.1243	0.6185	0.6185
HSWO	10000	1.0350	1.0350	1.3198	1.6734	1.6734
Chi	20000	1.0002	1.0034	1.0005	1.0955	1.3144
PhiCF	20000	1.0088	1.0108	0.9992	0.9071	0.9071
HSW	20000	0.9814	0.9522	1.4187	0.9056	0.9056
HSWO	20000	1.0198	1.0235	1.1168	1.1593	1.1593
Chi	50000	1.0004	1.0003	1.0322	1.0949	0.9828
PhiCF	50000	1.0019	1.0033	0.9927	0.9169	0.9169
HSW	50000	1.0020	0.9816	1.1283	0.7463	0.7463
HSWO	50000	0.9950	1.0002	1.0428	0.9313	0.9313
Chi	70000	1.0000	1.0002	1.0125	1.0245	0.9834
PhiCF	70000	1.0002	0.9989	0.9663	0.8354	0.8354
HSW	70000	0.9994	0.9911	1.1442	0.7973	0.7973
HSWO	70000	0.9966	0.9965	0.9717	0.9056	0.9056
Chi	100000	1.0002	1.0017	0.9963	1.0484	0.9120
PhiCF	100000	1.0005	0.9982	0.9330	0.8755	0.8755
HSW	100000	0.9991	0.9912	1.0794	0.8074	0.8074
HSWO	100000	1.0065	1.0056	0.9860	0.8269	0.8269

Table 2. Estimates of the integral $\boldsymbol{I_2(f)}$ with $\boldsymbol{a_j = 0}$ using various Halton sequences

generator	N	s=5	s=10	s=20	s=30	s=40
Chi	500	1.0024	1.1193	0.6592	0.2812	0.2264
PhiCF	500	1.0094	1.0765	0.881	1.3761	0.5476
HSW	500	0.9415	1.3837	2.6997	60.6243	3.0763
HSWO	500	0.9287	1.4227	0.8019	0.3357	0.3357

Chi	1000	0.9821	0.9436	0.6299	0.362	0.1798
PhiCF	1000	1.0001	0.9826	0.7139	0.7749	0.3402
HSW	1000	0.9193	1.1756	1.8391	30.8132	2.1033
HSWO	1000	0.9316	1.2205	3.1918	2.8029	2.8029
Chi	5000	1.0011	0.9499	0.9126	0.4356	0.2973
PhiCF	5000	0.9967	1.0306	0.7159	0.5707	0.3199
HSW	5000	1.0128	1.067	1.2569	7.5243	1.2339
HSWO	5000	0.9765	0.8818	1.2812	2.0284	2.0284
Chi	10000	0.9991	0.9781	0.9235	0.664	1.6802
PhiCF	10000	0.9984	0.9906	0.9057	0.8752	0.7388
HSW	10000	1.0209	1.0448	0.9893	4.0488	0.8383
HSWO	10000	0.9688	0.9186	1.1214	1.4821	1.4821
Chi	20000	1.0001	0.9777	1.0233	0.7919	1.0352
PhiCF	20000	0.9999	0.9897	1.0433	0.8272	0.6252
HSW	20000	1.0034	1.0271	0.9487	2.5576	0.9713
HSWO	20000	0.9971	1.0332	0.9316	0.9248	0.9248
Chi	50000	0.9998	0.9961	1.065	0.8885	1.1911
PhiCF	50000	0.9995	0.9873	1.301	0.9456	0.9905
HSW	50000	0.9969	0.9942	1.0171	1.6338	0.9654
HSWO	50000	1.0189	1.033	1.0316	0.5701	0.5701
Chi	70000	0.9999	0.9962	1.0204	0.8618	1.0573
PhiCF	70000	0.9996	0.9941	1.2423	0.9106	0.8928
HSW	70000	0.9942	1.0034	0.9974	1.3644	0.8502
HSWO	70000	1.0012	1.0443	0.9977	0.5744	0.5744
Chi	100000	1.0001	0.9985	1.0272	0.9289	1.2608
PhiCF	100000	0.9998	0.9978	1.176	1.0508	0.8357
HSW	100000	0.9981	0.996	1.0272	1.195	0.7236
HSWO	100000	0.9982	1.0331	1.1035	1.1058	1.1058

Table 3. Estimates of the integral $I_2(f)$ with $a_j = 1$ using various Halton sequences

generator	N	s=5	s=10	s=20	s=30	s=40
Chi	500	1.0000	1.0079	0.9815	0.8962	0.7541
PhiCF	500	0.9986	0.9978	0.9756	1.0516	0.9248
HSW	500	0.9791	1.0708	1.1042	1.8841	1.4721
HSWO	500	1.0256	1.1040	0.9910	1.0900	1.0900
Chi	1000	0.9969	0.9940	0.9868	0.9095	0.8206
PhiCF	1000	0.9976	0.9919	0.9912	0.9626	0.8760
HSW	1000	0.9599	1.0041	0.9939	1.3896	1.1939
HSWO	1000	0.9893	1.0309	1.1423	1.3591	1.3591
Chi	5000	0.9998	0.9965	0.9962	0.9539	0.9255
PhiCF	5000	0.9995	0.9996	0.9791	0.9719	0.9201
HSW	5000	1.0051	1.0158	1.0198	1.1364	1.0779
HSWO	5000	0.9847	0.9876	1.0148	1.0776	1.0776
Chi	10000	0.9997	0.9982	0.9964	0.9633	0.9917
PhiCF	10000	0.9996	0.9976	0.9886	0.9948	0.9400
HSW	10000	1.0076	1.0161	1.0070	1.0681	1.0498
HSWO	10000	0.9954	1.0044	1.0390	1.0961	1.0961
Chi	20000	0.9999	0.9985	0.9944	0.9761	0.9801
PhiCF	20000	0.9999	0.9919	0.9958	0.9910	0.9587
HSW	20000	1.0022	1.0116	1.0070	1.0529	1.0558
HSWO	20000	1.0022	1.0118	1.0192	1.0325	1.0325

Chi	50000	0.9999	0.9996	1.0012	0.9887	0.9941
PhiCF	50000	0.9999	0.9987	1.0008	0.9899	0.9856
HSW	50000	0.9989	0.9998	0.9962	1.0116	1.0238
HSWO	50000	1.0063	1.0144	1.0158	0.9919	0.9919
Chi	70000	0.9999	0.9997	0.9998	0.9885	0.9966
PhiCF	70000	0.9999	0.9995	1.0021	0.9952	0.9933
HSW	70000	0.9985	1.0010	0.9955	1.0022	1.0087
HSWO	70000	1.0005	1.0078	1.0128	0.9926	0.9926
Chi	100000	1.0000	1.0001	1.0021	0.9964	1.0078
PhiCF	100000	1.0000	1.0001	1.0045	0.9985	0.9943
HSW	100000	0.9996	1.0007	0.9969	1.0022	0.9970
HSWO	100000	1.0016	1.0051	1.0085	1.0058	1.0058

Table 4. Estimates of the integral $\boldsymbol{I_2(f)}$ with $\boldsymbol{a_j = j}$ using various Halton sequences

generator	N	s=5	s=10	s=20	s=30	s=40
Chi	500	0.9990	0.9988	0.9964	0.9954	0.9940
PhiCF	500	0.9978	0.9969	0.9953	0.9946	0.9934
HSW	500	1.0077	1.0219	1.0206	1.0266	1.0296
HSWO	500	1.0029	1.0036	1.0029	1.0033	1.0033
Chi	1000	0.9987	0.9984	0.9978	0.9968	0.9959
PhiCF	1000	0.9988	0.9979	0.9978	0.9968	0.9960
HSW	1000	0.9859	0.9931	0.9888	0.9898	0.9890
HSWO	1000	0.9988	1.0019	1.0047	1.0064	1.0064
Chi	5000	0.9998	0.9997	0.9995	0.9993	0.9991
PhiCF	5000	0.9997	0.9967	0.9993	0.9992	0.9990
HSW	5000	1.0069	1.0088	1.0079	1.0102	1.0093
HSWO	5000	0.9875	0.9890	0.9903	0.9910	0.9910
Chi	10000	0.9999	0.9999	0.9998	0.9997	0.9996
PhiCF	10000	0.9999	0.9998	0.9998	0.9995	0.9994
HSW	10000	1.0053	1.0066	1.0067	1.0084	1.0083
HSWO	10000	0.9974	0.9992	1.0004	1.0013	1.0013
Chi	20000	1.0000	0.9999	0.9998	0.9998	0.9997
PhiCF	20000	1.0000	0.9999	0.9998	0.9997	0.9997
HSW	20000	1.0015	1.0024	1.0022	1.0037	1.0034
HSWO	20000	1.0004	1.0016	1.0021	1.0021	1.0021
Chi	50000	1.0000	1.0000	1.0000	0.9999	0.9999
PhiCF	50000	1.0000	0.9999	0.9999	0.9999	0.9998
HSW	50000	0.9993	0.9995	0.9992	1.0001	0.9997
HSWO	50000	1.0051	1.0062	1.0064	1.0062	1.0062
Chi	70000	1.0000	1.0000	1.0000	0.9999	0.9999
PhiCF	70000	1.0000	1.0000	0.9999	0.9999	0.9999
HSW	70000	0.9995	0.9997	0.9993	0.9999	0.9998
HSWO	70000	1.0003	1.0011	1.0011	1.0011	1.0011
Chi	100000	1.0000	1.0000	1.0001	1.0001	1.0001
PhiCF	100000	1.0000	1.0000	1.0001	1.0001	1.0001
HSW	100000	1.0002	1.0004	1.0000	1.0006	1.0005
HSWO	100000	0.9998	1.0001	1.0002	1.0001	1.0001

Table 5. Estimates of the integral $\boldsymbol{I_2(f)}$ with $\boldsymbol{a_j = j^2}$ using various Halton sequences

generator	N	s=5	s=10	s=20	s=30	s=40
Chi	500	0.9991	0.9990	0.9989	0.9989	0.9988
PhiCF	500	0.9984	0.9983	0.9982	0.9982	0.9982
HSW	500	1.0192	1.0210	1.0207	1.0209	1.0210
HSWO	500	1.0027	1.0028	1.0028	1.0028	1.0028
Chi	1000	0.9994	0.9994	0.9994	0.9993	0.9993
PhiCF	1000	0.9995	0.9994	0.9994	0.9994	0.9994
HSW	1000	0.9964	0.9974	0.9970	0.9970	0.9970
HSWO	1000	0.9996	0.9998	1.0000	1.0001	1.0001
Chi	5000	0.9876	0.9877	0.9977	0.9978	0.9978
PhiCF	5000	1.0068	1.0071	1.0070	1.0071	1.0070
HSW	5000	0.9999	0.9999	0.9999	0.9999	0.9999
HSWO	5000	0.9999	0.9999	0.9999	0.9999	0.9999
Chi	10000	0.9976	0.9977	0.9999	0.9999	0.9999
PhiCF	10000	1.0046	1.0048	1.0048	1.0049	1.0049
HSW	10000	0.9999	0.9999	0.9999	0.9999	0.9999
HSWO	10000	1.0000	1.0000	1.0000	1.0000	1.0000
Chi	20000	1.0000	0.9999	0.9999	0.9999	0.9999
PhiCF	20000	1.0000	1.0000	1.0000	1.0000	1.0000
HSW	20000	1.0011	1.0012	1.0012	1.0013	1.0013
HSWO	20000	0.9998	0.9999	0.9999	0.9999	0.9999
Chi	50000	1.0049	1.0049	1.0050	1.0050	1.0050
PhiCF	50000	0.9997	0.9998	0.9998	0.9998	0.9998
HSW	50000	1.0001	1.0001	1.0002	1.0002	1.0003
HSWO	50000	0.9999	0.9999	0.9999	0.9999	0.9999
Chi	70000	0.9999	0.9999	0.9999	0.9999	0.9999
PhiCF	70000	1.0000	1.0000	1.0000	1.0000	1.0000
HSW	70000	1.0000	1.0001	1.0000	1.0001	1.0001
HSWO	70000	1.0002	1.0002	1.0002	1.0002	1.0002
Chi	100000	0.9994	0.9994	0.9994	0.9994	0.9994
PhiCF	100000	1.0004	1.0005	1.0004	1.0005	1.0005
HSW	100000	1.0000	1.0000	1.0000	1.0001	1.0000
HSWO	100000	1.0000	1.0000	1.0001	1.0001	1.0001

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