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Characterization A_9 and $L_4(3)$ by Order and Irreducible Character Degree

M. Mokhtari [1], A. Iranmanesh [*2], Abolfazl Tehranian [1]
[1] Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran,
[2] Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares.

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Abstract
Characterization of finite groups is one of the central themes of research in group theory. In this connection, classifying finite groups by the structure of their character degrees is an interesting problem in group theory. At first, it was shown that the alternating groups, the sporadic simple groups, the simple classic groups of Lie type, the simple exceptional groups of Lie type and symmetric groups are uniquely determined by their character degrees. In the following, characterization of finite groups by less information than their character degrees, namely by their orders and one or both of their largest and second largest irreducible character degrees, was investigated. In this regard, simple K_3-groups, Mathieu groups, $L(2, p)$ for all odd prime numbers p, the alternating group A_8 and some of simple K_4-groups were uniquely determined.
In this paper, our main goal is to characterize the alternating group A_9 and $L_4(3)$ from simple K_4-groups, with this method. As a result, we show that the above groups are uniquely determined by the structure of complex group algebra.

Keywords: Almost simple group, Character degree, Order, Complex group algebra, Characterization

* Corresponding author: Email: Iranmanesh@modares.ac.ir

1. Introduction

Throughout this paper, let G be a finite group,number and the set of be a prime p irreducible characters of G is denotedof irreducible for the set $cd(G)$ We write $Irr(G)$. character degrees of G and the first column of the ordinary character table of is G denoted by $X_1(G)$.

Characterization of finite groups is one of the central themes of research in group theory. There are various characterizations of finite groups by given properties, such as their prime graph or order of their elements, etc. (see [1,2]). In this connection, classifying finite groups by the structure of their character degrees is an interesting problem in group theory. It is known that non-abelian simple groups are uniquely determined by their character tables.

It was shown in [3] that the symmetric groups are uniquely determined by their character degrees. It was verified in [4,5,6], that the alternating groups, the sporadic simple groups, the simple classic groups of Lie type and simple exceptional groups of Lie type are uniquely determined by their character degrees. In [7,8], the authors proved that $PGU_3(q^2)$ and $PGL(n,q)$ are uniquely determined by the structure of their complex group algebras.

A finite grouphas exactly three or four distinct $|G|$ group, if- K_4 group or - K_3 is called a G prime divisors, respectively. It is well-known that there exist eight simplesgroups (cf. - K_3 [9]). In [10,11,12], authors characterized simplegroups, all Mathieu groups and - K_3 character degrees, namely by their orders and one by less information than their $PSL(2,p)$ or both of their largest and second largest irreducible character degrees.

In [13], it is proved that if 2^a+1 oris determined $PSL(2,2^a)$ is prime, then 2^a-1 uniquely by its order and its largest irreducible character degree. Afterward, in [14,15], the authors proved some finite simple K_4-groups and alternating group A_8 can be determined uniquely by their orders and at most two irreducible character degrees of their character table. In [16], the authors determined the finite groups with the same order and the same largest and second largest irreducible character degrees as $PGL(2,p^2)$ for all odd prime numbers p .

In this paper, we continue this investigation. The goal of this paper is to characterize two simples K_4-groups, the alternating group A_9 and $L_4(3)$, by their orders and at most two irreducible character degrees of their character table.

In the whole of this paper, we use the following notation: Let n be an integer, then we denote by $\pi(n)$ the set of all prime divisors ofis $\pi(|G|)$ is a finite group, then G . If n denoted by $\pi(G)$. For two natural numbers a andwe write p, and a prime number n . $p^{n+1} \nmid a$ but $p^n \mid a$ i.e., $p^n \| a$, when $|a|_p = p^n$ For a prime r , the set of Sylow r -subgroups of G is denoted by $Syl_r(G)$ and $|Syl_r(G)| = n_r(G)$. If H is a characteristic subgroup of G, we write $HchG$. Also, for the subgroup H of G, we set $H_G = \bigcap_{g \in G} H^g$.

All other notations are standard and we refer to [17].

For the proof of our mail result, we need some lemmas as follows:

Lemma 1.1. (Ito's Theorem) [18, theorem 6.15]. Let A be a normal subgroup of G which is abelian. Then $\chi(1)$ divides $|G:A|$ for all $\chi \in Irr(G)$.

Lemma 1.2. [18, Theorem 6.2, 6.8, 11.29]. Let N be a normal subgroup of G and $\chi \in Irr(G)$. Let θ be an irreducible constituent of χ_N and suppose N are $\theta_1 = \theta, ..., \theta_t$ θ distinct conjugates of in. $e = [\chi_N, \theta]$ where $\chi_N = e\sum_{i=1}^{t} \theta_i$, . Then G Also $\frac{\chi(1)}{\theta(1)} \mid |G:N|$.

Lemma 1.3. [11]. Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$.

Lemma 1.4. [14]. Let G be a finite group and forelementary -$t$be a N, let $t \in \pi(G)$ abelian subgroup of.is a C, where $C_G(N) = N \times C$, then $|C_G(N)|_t = |N|$. If G .$C_G(N)$ Hall subgroup of - $(\pi(C_G(N)) - \{t\})$

2. Theorems and Result

First, we express main theorems:

Theorem 1. Let G be a finite group such that $|G| = |A_9|$ and $189, 162 \in cd(G)$. Then $G \cong A_9$.

Proof. By hypotheses, we have $|G| = 2^6.3^4.5.7$. Let $\chi, \psi \in Irr(G)$ such that $\chi(1) = 189$ and $\psi(1) = 162$. First, we prove that G is not a solvable group.

Let be G solvable group. Let H be a Hall subgroup of G of order $2^6.3^4.7$, and so $|G:H| = 5$. Then $G/H_G \cong S_5$. Since orders of solvable subgroups of S_5 divided by 5 are $5, 10$ and 20, it follows that $|H_G| = 2^6.3^4.7, 2^5.3^4.7$ or $2^4.3^4.7$.

Let , and $\frac{\chi(1)}{\theta(1)} \mid |G:H_G| = 5$ Then $\left[\theta, \chi_{H_G}\right] \neq 0$. such that $\theta \in Irr(H_G)$ Let $|H_G| = 2^6.3^4.7$.

so, $\left[\varphi, \psi_{H_G}\right] \neq 0$. such that $\varphi \in Irr(H_G)$ let Also, $\theta(1) = 189$. Then, $\frac{\psi(1)}{\varphi(1)} \mid |G:H_G|$

which is a contradiction., $\theta(1)^2 + \varphi(1)^2 \succ |H_G|$ But $\varphi(1) = 162$. and so, $= 5$, Let $\left|H_G\right| = 2^5.3^4.7$ or $2^4.3^4.7$. Let $\theta \in Irr(H_G)$ such that $\left[\theta, \chi_{H_G}\right] \neq 0$. Then $\frac{\chi(1)}{\theta(1)} |$ $\left|G : H_G\right|$, and so, $\theta(1) = 189$. Butis G , which is a contradiction. Therefore $\theta(1)^2 \succ \left|H_G\right|$ not a solvable group.

So, by using Lemma 1.3, we get that G has a normal seriessuch that $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ abelian simple groups and-is a direct product of isomorphic non K/H . $\left|G/K\right| | \left|Out(K/H)\right|$

Then, by classification of finite simple groups, since $\left|K/H\right| | \left|G\right|$, we have the following cases:

Case 1. $K/H \cong A_5$. Then $\theta \in Irr(H)$ Let $2^3.3^3.7.$ or $\left|H\right| = 2^4.3^3.7$, and so $\left|G/K\right| | 2$ such that $\left[\theta, \chi_H\right] \neq 0$. Then $\frac{\chi(1)}{\theta(1)} | \left|G : H\right|$, and so, $\theta(1) = 189$ or $\theta(1)^2 \succ \left|H\right|$,. But 63 which is a contradiction.

Case 2. $K/H \cong L_2(7)$. Then $\left|H\right| = 2^3.3^3.5$,and so , $\left|G/K\right| | 2$ or $2^2.3^3.5$. Let $\varphi \in Irr(H)$ such that $\left[\varphi, \psi_H\right] \neq 0$. Then $\frac{\psi(1)}{\varphi(1)} | \left|G : H\right|$, and so $\varphi(1) = 54$ orand $\left|H\right| = 2^3.3^3.5$. If 27 . $\varphi(1) = 27$ which is a contradiction. So $\varphi(1)^2 \succ \left|H\right|$, then, $\varphi(1) = 54$

Letand so $2^3.3^3$ order of H be a Hall subgroup of F be solvable group. Let H Let $1 \le i \le 3$. for $\left|F_H\right| = 2^i.3^3$ In a similar way was as above, we get $\left|H : F\right| = 5$. ushT. $\sigma(1) = 27$ and so, $\frac{\varphi(1)}{\sigma(1)} | \left|H : F_H\right|$, Then $\left[\sigma, \varphi_{F_H}\right] \neq 0$. such that $\sigma \in Irr(F_H)$ which is a contradiction. $\sigma(1)^2 \succ \left|F_H\right|$,

Therefore H is not a solvable group. So by using Lemma 1.3, we get that H has a normal seriesabelian -is a direct product of isomorphic non M/N such that $1 \trianglelefteq M \trianglelefteq N \trianglelefteq H$ simple groups and $\left|H/M\right| | \left|Out(M/N)\right|$. Then $M/N \cong A_5$ or A_6.

(i) If $M/N \cong A_5$, then $\left|N\right| = 18$ or 9 . Let $\sigma \in Irr(N)$ such that $\left[\sigma, \varphi_N\right] \neq 0$. Then $\frac{\varphi(1)}{\sigma(1)} | \left|H : N\right|$, and so $\sigma(1) = 9$. Thus $\sigma(1)^2 \succ \left|N\right|$, which is a contradiction.

(ii) If $M/N \cong A_6$. Then $\left|N\right| = 9$ and so by Lemma 1.1, we get, which $\varphi(1) | \left|H : N\right|$ is impossible.

If $\left|H\right| = 2^2.3^3.5$, hence $\varphi(1)^2 \succ \left|H\right|$, which is a contradiction.

Case 3. $K/H \cong A_6$. Then $\left|G/K\right| | 4$, and so $\left|H\right| = 2^i.3^2.7$ for $1 \le i \le 3$. Similarly, we get a contradiction.

Case 4. $K/H \cong L_2(8)$. Then $|G/K| \mid 3$, and so $|H| = 2^3.3^2.5$ or $2^3.3.5$.. If $|H| = 2^3.3^2.5$, since $Mult(L_2(8)) = 1$, thenBut, by checking the character table of $G \cong L_2(8) \times H$. , which is contradiction. 189 has no irreducible character of degree G we see that $L_2(8)$,
If $|H| = 2^3.3.5$, since $Mult(L_2(8)) = 1$, thensuch that $\theta \in Irr(K)$ Let $K \cong L_2(8) \times H$.
But, by checking the .63 or $\theta(1) = 189$ and so, $\frac{\chi(1)}{\theta(1)} \mid |G:K| = 3$, Then $[\theta, \chi_K] \neq 0$.
character table of $L_2(8)$, we see that K has no irreducible character of degree 189 or 63, which is a contradiction.

Case 5. $K/H \cong A_7$. Then $|G/K| \mid 2$, and so $|H| = 72$ or 36. Let $\varphi \in Irr(H)$ such that $[\varphi, \psi_H] \neq 0$. Then $\frac{\psi(1)}{\varphi(1)} \mid |G:H|$, and so, $\varphi(1) = 18$ or 9. Hence $\varphi(1)^2 \succ |H|$, which is a contradiction.

Case 6. $K/H \cong U_3(3)$. Then $|G/K| \mid 2$, and so, $|H| = 30$ or 15.
If $|H| = 30$, sinceBut, by checking the character $G \cong U_3(3) \times H$., then $Mult(U_3(3)) = 1$ table of $U_3(3)$, we see that G has no irreducible character of degree 189, which is contradiction.
If $|H| = 15$, then, which is a $\psi(1) \mid |G:H|$ is cycle group and so by Lemma 1.1, H contradiction.

Case 7. $K/H \cong A_8$. Then $|G/K| \mid 2$, and so, $|H| = 9$. So by Lemma 1.1, we get $\psi(1) \mid |G:H|$, which is a contradiction.

Case 8. $K/H \cong L_3(4)$. Then $|G/K| \mid 12$, and so, $|H| = 9$ or 3. So similarly case 7, we get a contradiction.

Case 9. $K/H \cong U_4(2)$. Then $|G/K| \mid 2$, and so, $|H| = 7$. So by Lemma 1.1, we get $\chi(1) \mid |G:H|$, which is a contradiction.

Case 10. $K/H \cong A_9$. Then $|G/K| \mid 2$, and so, $|H| = 1$. Hence $G \cong A_9$.

Theorem 2. Let G be a finite group such that $|G| = |L_4(3)|$ and $729, 640 \in cd(G)$ .Then $G \cong L_4(3)$.

Proof. By hypotheses, we have $|G| = 2^7.3^6.5.13$. Let $\chi, \psi \in Irr(G)$ such that $\chi(1) = 729$ and $\psi(1) = 640$. First we prove that G is not a solvable group.
Let G be a solvable group and M be a minimal normal subgroup of. Then for some G $t \in \pi(G)$,, M , χ, ψ Thus by Lemma 1.1 to G. elementary abelian subgroup of -t is M shows that $t = 13 = |M|$. Then, since $G/C_G(M) < M$, we deduce that $Aut(M) \cong \mathbb{Z}_{12} \hookrightarrow$ $(\pi(C_G(M))$ is C where $C_G(M) = M \times C$, and so by Lemma 1.4, $13 \mid\mid |G|$ But $C_G(M)$.

$C_G(M)$.Hall subgroup of $- -\{13\})$ Also,Hence, $G.$ $\trianglelefteq C$ implies that $G \trianglelefteq CchC_G(M)$ there exists a minimal normal subgroup N ofwhich is a $N \leq C$, such that G contradiction.

So, by using Lemma 1.3, we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$.

Then, by the classification of finite simple groups, we have the following cases:

Case 1. $K/H \cong A_5$. Then $|G/K| \mid 2$, and so, $|H| = 2^i.3^5.13$ for $i = 4, 5$.

If $|H| = 2^5.3^5.13$, Let $\theta, \varphi \in Irr(H)$ such that $[\theta, \chi_H] \neq 0$, $[\varphi, \psi_H] \neq 0$. Then $\frac{\chi(1)}{\theta(1)} \mid |G : H|$ and $\frac{\psi(1)}{\varphi(1)} \mid |G : H|$. Hence $\theta(1) = 243, \varphi(1) = 32$.

First, let H be a solvable group and M be a minimal normal subgroup ofThen, H. similarly is C where $C_H(M) = M \times C$, ,and so H elementary abelian subgroup of -13 M is a $(\pi(C_G(M)) - \{13\})$-Hall subgroup of $C_H(M)$, which is a normal subgroup of H, Hence, there exists a minimal normal subgroup N of H such that $N \leq C$, which is a contradiction.

Therefore H is not a solvable group. So by using Lemma 1.3, we get that H has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq H$ such that M/N is a direct product of isomorphic non-abelian simple groups and $|H/M| \mid |Out(M/N)|$. Then $M/N \cong L_3(3)$ and so $|N| = 18$ or. 9

Let , $\theta(1) \mid |H : N_3|$,Then by using Lemma 1.1 $N_3 \trianglelefteq H.$, and so $N_3 chN$ Hence $|N| = 18.$ which is a contradiction.

Let $|N| = 9.$ Then by using Lemma 1.1, $\theta(1) \mid |H : N|$, which is a contradiction.

Now, if $|H| = 2^4.3^5.13$. Let $\theta \in Irr(H)$ such that $[\theta, \chi_H] \neq 0$. Then $\frac{\chi(1)}{\theta(1)} \mid |G : H|$. Hence $\theta(1) = 243$ and so, $\theta(1)^2 \succ |H|$, which is a contradiction.

Case 2. $K/H \cong A_6$. Then $|G/K| \mid 4$, and so, $|H| = 2^i.3^4.13$ for $2 \leq i \leq 4$. Similar case 1, we get a contradiction.

Case 3. $K/H \cong L_3(3)$. Then $|G/K| \mid 2$, and so, $|H| = 2^i.3^3.5$ for $i = 2, 3$.

If $|H| = 2^3.3^3.5$, Let $\theta, \varphi \in Irr(H)$ such that $[\theta, \chi_H] \neq 0$, $[\varphi, \psi_H] \neq 0$. Then $\frac{\chi(1)}{\theta(1)} \mid |G:H|$ and $\frac{\psi(1)}{\varphi(1)} \mid |G:H|$. Hence $\theta(1) = 27$, $\varphi(1) = 8$.

First, let H be a solvable group andis M Then H.be a minimal normal subgroup of M 5-elementary abelian subgroup of H and so similarly, we get a contradiction.

Therefore H is a non-solvable group. Then by using Lemma 1.3, we get that H has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq H$ such that M/N is a direct product of isomorphic non-abelian simple groups and $|H/M| \mid |Out(M/N)|$. Then $M/N \cong A_5$ or A_6.

(i) Let $M/N \cong A_5$. Then $|N| = 18$ or 9. Similar case 1, which implies a contradiction.

(ii) Let $M/N \cong A_6$. Then $|N| = 3$ and so by Lemma 1.1, $\theta(1) \mid |H:N|$, which is a contradiction.

If $|H| = 2^2.3^3.5$, Let $\theta \in Irr(H)$ such that $[\theta, \chi_H] \neq 0$. Then $\frac{\chi(1)}{\theta(1)} \mid |G:H|$. Hence $\theta(1) = 27$, and so, $\theta(1)^2 \succ |H|$, which is a contradiction.

Case 4. $K/H \cong U_4(2)$. Then $|H| = 234$ or 117.

If $|H| = 234$, Let $\theta, \varphi \in Irr(H)$ such that $[\theta, \chi_H] \neq 0$, $[\varphi, \psi_H] \neq 0$. Then $\frac{\chi(1)}{\theta(1)} \mid |G:H|$ and $\frac{\psi(1)}{\varphi(1)} \mid |G:H|$. Hence $\theta(1) = 9$, $\varphi(1) = 2$. Similarly, H is a non-solvable group. So, by using Lemma 1.3, we get thatis M/N such that $1 \trianglelefteq M \trianglelefteq N \trianglelefteq H$ has a normal series H a direct product of isomorphic non-abelian simple groups and $|H/M| \mid |Out(M/N)|$. Since abelian simple group, which is impossible.-there exists no non, $|M/N| \mid |H|$

If $|H| = 117$, let $\theta \in Irr(H)$ such that $[\theta, \chi_H] \neq 0$. Then $\frac{\chi(1)}{\theta(1)} \mid |G:H|$. Hence $\theta(1) = 9$. It is obvious thatSo, $13, 39$.is a solvable group and has no irreducible character of degree H by checking, we haveand $H' \cong \mathbb{Z}_{13}$

$$X_1(H) = \{1,1,1,1,1,1,1,1,1,3,3,3,9\}$$

Let M be a minimal normal subgroup of H. Then, similarlyelementary abelian -13is M subgroup of H and so $C_H(M) = M \times C$, where C is a $(\pi(C_G(M)) - \{13\})$-Hall subgroup of $C_H(M)$, which is a normal subgroup of H. Hence, there exists a minimal normal subgroup N of H such that $N \leq C$, which is a contradiction.

Case 5. $K/H \cong L_4(3)$. Then $|G/K| \mid 4$, and so, $|H| = 1$. Hence $G \cong L_4(3)$.

Now, we get the following results from the theorems:

Corollary 2.1. Let $X_1(G) = X_1(H)$. such that $L_4(3)$, or $H = A_9$ be a finite group and G Then $G \cong H$.

Corollary 2.2. Let $H = A_9$ or $L_4(3)$. If G is a group such that $\mathbb{C}G \cong \mathbb{C}H$, then $G \cong H$. . Thusis uniquely determined by the structure of its complex group algebra. H

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