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Research Article

Observers and Relative Entropy Functional

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Abstract

In this paper, we will use the mathematical modeling of one-dimensional observers to present the notion of the *relative entropy functional* for relative dynamical systems. Also, the invariance of the entropy of a system under topological conjugacy is generalized to the relative entropy functional. Moreover, from observer viewpoint, a new version of the Jacobs Theorem is obtained. It has been proved that relative entropy functional is equivalent to the Kolmogorov entropy for dynamical systems, from the viewpoint of observer χ_X, where χ_X is the characteristic function on compact metric space X.

Keywords : Kolmogorov entropy; Relative dynamical system; Invariant; Relative entropy functional

1 Introduction

The concept of entropy is originated from the physical and engineering sciences, but now it plays a ubiquitous role in all areas of science. The term entropy was first used by the German physicist Rudolf Clausius in 1865 to denote a thermodynamic function. The term entropy was first used by the German physicist Rudolf Clausius in 1865. Since then, it has been continually extended and applied by researchers in numerous areas of science, such as physics, information theory, chaos theory, ergodic theory, data mining, and dynamical systems. Measure-theoretic entropy of dynamical systems was first appeared in the paper [1] by Kolmogorov in 1958. Kolmogorovs entropy was improved by Sinai in 1959 [10].

The importance of entropy as a persistent object under the conjugacy of dynamic systems has been studied by several researchers [2, 9]. Therefore, systems with different entropies cannot be conjugate. Moreover, any physical variation on a dynamical system should be identified by an "observer". Also, a method is required to compare the perspective of different observers.

This paper is an attempt to present a new approach to the entropy of a relative dynamical system [5], using the concept of an observer [2, 4]. So first, we should mathematically identify the observer. A modeling for an observer of a set X is defined as a fuzzy set $\Theta : X \to [0, 1]$ [2, 6, 7, 12]. These kinds of fuzzy sets are called

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"one-dimensional observers". After this identification, a method is required to compare different observers and evaluate their perspectives. So, the notion of the one-dimensional observer is used to define *relative entropy functional* for topological dynamical systems. This definition will be expected to have the fundamental properties of the entropy and also, as a special case, coincides with the Kolmogorov entropy from the viewpoint of the observer χ_X (characteristic function on X, where X denotes the base space of the system).

In this article, the set of all probability measures on X preserving T is denoted by $M(X,T)$. We also write $E(X,T)$ for the set of all ergodic measures of T.

2 Preliminaries

In what follows, we provide the preliminaries that are necessary for the rest of this paper.

Definition 2.1 ([2, 6, 7, 12]). *Let X be a compact metric space and Θ be a one-dimensional observer of X, i.e., $\Theta : X \to [0,1]$ is a fuzzy set. Moreover, let $T : X \to X$ be a continuous map. In this case, (X,T,Θ) is referred as a relative dynamical system.*

Definition 2.2 ([4]). *Let X be a compact metric space and $E \subseteq X$. Then the relative probability measure of E with respect to the one-dimensional observer Θ, is the fuzzy set $m^T_\Theta(E) : X \to [0,1]$, which is defined by*

$$m^T_\Theta(E)(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x))\Theta(T^i(x)),$$

where χ_E is the characteristic function of E.

According to this definition with a fixed observer Θ and a dynamical system T, we can associate to each subset E, a mapping $m^T_\Theta(E)$.

Theorem 2.1 ([4]). *Let (X,β,m) be a probability space, and $\Theta : X \to [0,1]$ be the characteristic function χ_X. Moreover, let $T : X \to X$ be an ergodic map. Then for each $x \in X$, $m^T_\Theta(E)(x)$ is almost everywhere equal to $m(E)$, where $E \in \beta$.*

Therefore, relative probability measures are extensions of probability measures. Note that, in physical systems, the role of Θ is critical. In fact Θ determines our looking to the state space.

In the rest of this paper, the relative measure with respect to an observer Θ at $x \in X$ is denoted by m_x, i.e.

$$m_x(E) = m^T_\Theta(E)(x), \qquad \text{for any } E \subseteq X.$$

In what follows some classical results are presented, that are needed in the continuation.

Theorem 2.2 (Choquet [8]). *Suppose Y is a compact convex metrizable subset of a locally convex space E, and $x \in Y$. Then there exists a probability measure τ on Y which represents x and is supported by the extreme points of Y, i.e., $\Phi(x) = \int_Y \Phi d\tau$ for every continuous linear functional Φ on E, and $\tau(\text{ext}(Y)) = 1$.*

Let $f : X \to \mathbb{R}$ be a bounded measurable function and $\mu \in M(X,T)$. It is known that $E(X,T)$ equals the extreme points of $M(X,T)$. By applying the Choquet's Theorem for $E = M(X)$, the space of finite regular Borel measures on X, and $Y = M(X,T)$, and using the linear functional $\Phi : M(X) \to \mathbb{R}$ given by $\Phi(\mu) = \int_X f d\mu$, we have the following result [9].

Corollary 2.1 ([9]). *Suppose that $T : X \to X$ is a continuous map on the compact metric space X. Then, for each $\mu \in M(X,T)$, there is a unique measure τ on the Borel subsets of the compact metrizable space $M(X,T)$, such that $\tau(E(X,T)) = 1$ and*

$$\int_X f(x) d\mu(x) = \int_{E(X,T)} \left(\int_X f(x) dm(x) \right) d\tau(m),$$

for every bounded measurable function $f : X \to \mathbb{R}$.

Under the hypothesis of 2.1,

$$\mu = \int_{E(X,T)} m d\tau(m),$$

which is called the *ergodic decomposition* of μ .

Theorem 2.3 (Jacobs [11]). *Let $T : X \to X$ be a continuous map on a compact metrizable space. If $\mu \in M(X,T)$ and $\mu = \int_{E(X,T)} m\mathrm{d}\tau(m)$ is the ergodic decomposition of μ, then we have:*

(i) If ξ is a finite Borel partition of X, then

$$h_\mu(T,\xi) = \int_{E(X,T)} h_m(T,\xi)\mathrm{d}\tau(m).$$

(ii) $h_\mu(T) = \int_{E(X,T)} h_m(T)\mathrm{d}\tau(m)$ (both sides could be ∞).

3 Main results

In this section we introduce the notion of entropy from the viewpoint of different observers. This notion describes a relative perspective of complexity and uncertainty in dynamical systems.

Definition 3.1 ([4]). *Let $T : X \to X$ be a continuous map on the topological space X, $x \in X$ and A be a Borel subset of X. Then*

$$m_x(A) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_A(T^i(x))\Theta(T^i(x)).$$

Also, let $x \in X$, $\xi = \{A_1, A_2, \dots, A_n\}$, and $\eta = \{B_1, B_2, \dots, B_m\}$ be finite Borel partitions of X. We define

$$\Psi_\Theta(x,T,\xi) := -\sum_{i=1}^{n} m_x(A_i)\log m_x(A_i),$$

and

$$\Psi_\Theta(x,T,\xi|\eta) := -\sum_{i,j} m_x(A_i\cap B_j)\log\frac{m_x(A_i\cap B_j)}{m_x(B_j)}.$$

(Assume that $\log 0 = -\infty$ and $0 \times \infty = 0$).

Note that $\Psi_\Theta(x,T,\xi|\eta)$ is the conditional version of $\Psi_\Theta(x,T,\xi)$. Also, it is clear that $\Psi_\Theta(x,T,\xi) \geq 0$.

Definition 3.2 ([11]). *Let η and ξ be two given partitions. ξ is a refinement of η, and is denoted by $\eta \prec \xi$, if every element of η is a union of elements of ξ.*

Definition 3.3 ([11]). *Given two partitions ξ and η, their common refinement is defined as follows*

$$\xi \vee \eta = \{A_i \cap B_j \mid A_i \in \xi, B_j \in \eta\}.$$

Theorem 3.1. *Let $T : X \to X$ be a continuous map on the topological space X and $x \in X$. If ξ and η are finite Borel partitions of X, then we have*

$$\Psi_\Theta(x,T,\xi\vee\eta) = \Psi_\Theta(x,T,\xi) + \Psi_\Theta(x,T,\eta \mid \xi).$$

Proof. Suppose $\xi = \{A_1, A_2, \dots, A_n\}$ and $\eta = \{B_1, B_2, \dots, B_m\}$ are finite Borel partitions of X. We can write

$$m_x(A_i\cap B_j) = \frac{m_x(A_i\cap B_j)}{m_x(A_i)}\cdot m_x(A_i).$$

So we have

$$\begin{aligned}
&\Psi_\Theta(x,T,\xi\vee\eta)\\
&= -\sum_{i,j} m_x(A_i\cap B_j)\log\frac{m_x(A_i\cap B_j)}{m_x(A_i)}\\
&\quad -\sum_{i,j} m_x(A_i\cap B_j)\log m_x(A_i)\\
&= -\sum_{i,j} m_x(A_i\cap B_j)\log m_x(A_i)\\
&\quad +\Psi_\Theta(x,T,\eta\mid\xi)\\
&= \Psi_\Theta(x,T,\xi) + \Psi_\Theta(x,T,\eta\mid\xi).
\end{aligned}$$

Theorem 3.2. *Let $T : X \to X$ be a continuous map on the topological space X and $x \in X$. If ξ and η are finite Borel partitions of X, then*

$$\Psi_\Theta(x,T,\xi|\eta) \leq \Psi_\Theta(x,T,\xi).$$

Proof. Suppose $\xi = \{A_1, A_2, \dots, A_n\}$ and $\eta = \{B_1, B_2, \dots, B_m\}$ are finite Borel partitions of X. Let $1 \leq i \leq n$ be fixed, $m_x(X) = t$ and

$$\alpha_k = \frac{m_x(B_k)}{t}, \qquad x_k = \frac{m_x(A_i\cap B_k)}{m_x(B_k)}.$$

So, by using the convexity of the function $\varphi(x) = x\log x$, we have

$$\varphi\left(\sum_k \frac{m_x(B_k)}{t}\cdot\frac{m_x(A_i\cap B_k)}{m_x(B_k)}\right) \leq \sum_k \frac{m_x(B_k)}{t}\cdot\varphi\left(\frac{m_x(A_i\cap B_k)}{m_x(B_k)}\right).$$

Obviously $\sum_k m_x(A_i \cap B_k) = m_x(A_i)$. So the left hand side of above is equal to

$$\varphi\left(\frac{m_x(A_i)}{t}\right) = \frac{m_x(A_i)}{t} \cdot \log \frac{m_x(A_i)}{t}.$$

Now, by multiplying both sides by t and summation over i, we will obtain

$$\begin{aligned} &\sum_i m_x(A_i) \cdot \log \frac{m_x(A_i)}{t} \\ &\leq \sum_{i,k} m_x(B_k) \cdot \frac{m_x(A_i \cap B_k)}{m_x(B_k)} \log \frac{m_x(A_i \cap B_k)}{m_x(B_k)} \\ &= \sum_{i,k} m_x(A_i \cap B_k) \log \frac{m_x(A_i \cap B_k}{m_x(B_k)}. \end{aligned}$$

Therefore,

$$-\Psi_\Theta(x, T, \xi) - t\log t \leq -\Psi_\Theta(x, T, \xi|\eta).$$

Since, $t\log t \leq 0$, one deduces the result.

Theorem 3.3. *Let $T : X \to X$ be a continuous map on the topological space X and $x \in X$. If ξ and η are finite Borel partitions of X, then*

$$\Psi_\Theta(x, T, \xi \vee \eta) \leq \Psi_\Theta(x, T, \xi) + \Psi_\Theta(x, T, \xi).$$

Proof. The proof is obtained using Theorems 3.1 and 3.2.

Theorem 3.4. *Let $r \geq 1$ be a fixed integer and $x \in X$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\xi = \{A_1, \dots, A_r\}$ and $\eta = \{C_1, \dots, C_r\}$ are two partitions of X, with $\sum_{i=1}^r m_x(A_i \Delta C_i) < \delta$, then*

$$\Psi_\Theta(x, T, \xi|\eta) + \Psi_\Theta(x, T, \eta \mid \xi) < \varepsilon.$$

Proof. Suppose $\epsilon > 0$ is given and $m_x(X) = t$. Choose $\delta > 0$ such that $\delta < \frac{t}{4}$, and

$$-r(r-1)\delta \log \delta - (t-\delta)\log(t-\delta) < \frac{\epsilon}{2}.$$

Let $\zeta = \{A_i \cap C_j : i \neq j\} \cup (\cup_{i=1}^r (A_i \cap C_i))$. Then $\xi \vee \eta = \eta \vee \zeta$. Since $A_i \cap C_j \subset \cup_{n=1}^r (A_n \Delta C_n)$, we have

$$m_x(A_i \cap C_j) < \delta \; (i \neq j),$$

and

$$m_x(\cup_{i=1}^r (A_i \cap C_i)) > t - \delta.$$

Hence

$$\begin{aligned} &\Psi_\Theta(x, T, \zeta) < \\ &-r(r-1)\delta \log \delta - (t-\delta)\log(t-\delta) < \frac{\epsilon}{2}. \end{aligned}$$

Therefore, by applying Theorems 3.1 and 3.3 we have

$$\begin{aligned} &\Psi_\Theta(x, T, \eta) + \Psi_\Theta(x, T, \xi|\eta) \\ &\quad = \Psi_\Theta(x, T, \xi \vee \eta) \\ &\quad = \Psi_\Theta(x, T, \eta \vee \zeta) \\ &\quad \leq \Psi_\Theta(x, T, \eta) + \Psi_\Theta(x, T, \zeta) \\ &\quad < \Psi_\Theta(x, T, \eta) + \frac{\epsilon}{2}, \end{aligned}$$

and so $\Psi_\Theta(x, T, \xi|\eta) < \frac{\epsilon}{2}$. Since $\xi \vee \eta = \xi \vee \zeta$, we easily obtain that $\Psi_\Theta(x, T, \eta|\xi) < \frac{\epsilon}{2}$.

Definition 3.4. *Suppose $T : X \to X$ is a continuous map on the topological space X, $x \in X$, and ξ is a finite Borel partition of X. We define the map $h_\Theta(\cdot, T, \xi) : X \to [0, \infty]$ as follows*

$$h_\Theta(x, T, \xi) = \limsup_{l \to \infty} \frac{1}{l} \Psi_\Theta\left(x, T, \vee_{i=0}^{l-1} T^{-i}\xi\right).$$

Definition 3.5. *Let $T : X \to X$ be a continuous map on the topological space X, $x \in X$, and A be a Borel subset of X. Also, let $\Xi = \{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X, such that $\operatorname{diam}(\xi_n) \to 0$ as $n \to \infty$. We define the map $h_\Theta(\cdot, T, \Xi) : X \to [0, \infty]$ as follows*

$$h_\Theta(x, T, \Xi) = \lim_{n \to \infty} h_\Theta(x, T, \xi_n).$$

Remark 3.1. *In the Definition 3.5, without loss of generality, we may assume that $\xi_n \prec \xi_{n+1}$, as otherwise we can replace ξ_n with $\eta_n = \vee_{k=1}^n \xi_k$. Hence,*

$$a_n(x) = \limsup_{l \to \infty} \frac{1}{l} \Psi_\Theta\left(x, T, \vee_{i=0}^{l-1} T^{-i}\xi_n\right)$$

is an increasing sequence with respect to n. Therefore, $\lim_{n\to\infty} a_n(x)$ exists as a non-negative extended real number.

Definition 3.6. *Suppose $T : X \to X$ is a continuous map on the topological space X, $x \in X$ and ξ is a finite Borel partition of X. We define the relative entropy of T at x as follows*

$$h_\Theta(T, m_x) = \sup_\xi h_\Theta(x, T, \xi).$$

Definition 3.7. *Suppose X is a compact metric space, $T : X \to X$ is a continuous map on X, and $\mu \in M(X,T)$. Also suppose that $\Xi = \{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of finite Borel partitions of X, such that $\operatorname{diam}(\xi_n) \to 0$ as $n \to \infty$. The relative entropy functional of T, with respect to μ, is defined as $L^T_\Theta(\cdot, m, \Xi) : C(X) \to \mathbb{R}$, where*

$$L^T_\Theta(f, m, \Xi) = \int_X f(x)h_\Theta(x, T, \Xi)\mathrm{d}\mu(x),$$

for all $f \in C(X)$. As previous, we assume that $0 \times \infty := 0$.

Theorem 3.5. *Let X be a compact metric space, and $T : X \to X$ be a continuous map on X. Also, let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X, such that $\operatorname{diam}(\xi_n) \to 0$ as $n \to \infty$. Then, for all $x \in X$*

$$h_\Theta(T, m_x) = \lim_{n\to\infty} h_\Theta(x, T, \xi_n).$$

Proof. Let $\varepsilon > 0$. If $h_\Theta(x, T, \xi) < \infty$, then we choose a finite partition $\xi = \{A_1, \dots, A_r\}$ of X such that

$$h_\Theta(x, T, \xi) > h_\Theta(T, m_x) - \varepsilon.$$

Otherwise, if $h_\Theta(T, m_x) = \infty$, we choose the partition ξ such that $h_\Theta(T, m_x, \xi) > \frac{1}{\varepsilon}$. So, we can select δ correspond to ε and r as indicated in the Theorem 3.4.

Let $P_i \subset A_i$ be compact subsets with $m_x(A_i \setminus P_i) < \frac{\delta}{r+1}$, and $\delta' = \inf_{i\neq j} d(P_i, P_j)$. Let us pick n such that $\operatorname{diam}(\xi_n) < \frac{\delta'}{2}$. For $1 \le i < r$, let E^i_n be the union of all the elements of ξ_n that intersect P_i, and let E^r_n be the union of the remaining elements of ξ_n. Note that, each $C \in \xi_n$ can intersect at most one P_i, since $\operatorname{diam}(\xi_n) < \frac{\delta'}{2}$. Thus, $\xi'_n = \{E^1_n, \dots, E^r_n\}$ is so that $\xi'_n \le \xi_n$ and

$$\begin{aligned} m_x(E^i_n \Delta A_i) &= m_x(E^i_n \setminus A_i) \\ &\quad + m_x(A_i \setminus E^i_n) \\ &\le m_x(X \setminus \cup^r_{j=1} P_j) \\ &\quad + m_x(A_i \setminus P_i) < \delta. \end{aligned}$$

By using the Theorem 3.4, we obtain that $\Psi_\Theta(x, T, \xi|\xi'_n) < \varepsilon$. Therefore, if n is such that $\operatorname{diam}(\xi_n) < \frac{\delta'}{2}$, then

$$\begin{aligned} h_\Theta(x, T, \xi) &\le h_\Theta(x, T, \xi') + \varepsilon \\ &\le h_\Theta(x, T, \xi_n) + \varepsilon. \end{aligned}$$

So we conclude that, by assumption of $\operatorname{diam}(\xi_n) < \dfrac{\delta'}{2}$, we have

$$h_\Theta(x, T, \xi) > h_\Theta(T, m_x) - 2\varepsilon,$$

if $h_\Theta(T, m_x) < \infty$, and $h_\Theta(x, T, \xi) > \dfrac{1}{\varepsilon}$, if $h_\Theta(T, m_x) = \infty$. Therefore, $\lim_{n\to\infty} h_\Theta(x, T, \xi_n)$ exists and is equal to $h_\Theta(T, m_x)$.

Theorem 3.6. *Suppose $T : X \to X$ is a continuous map on the compact metric space X. Let $\Xi = \{\xi_n\}_{n\in\mathbb{N}}$ and $\Pi = \{\eta_n\}_{n\in\mathbb{N}}$ be two sequences of finite Borel partitions of X, such that both $\operatorname{diam}(\xi_n)$ and $\operatorname{diam}(\eta_n) \to 0$ as $n \to \infty$. Then*

$$L^T_\Theta(f, m, \Xi) = L^T_\Theta(f, m, \Pi).$$

Proof. Let $x \in X$ be arbitrary. We obtain

$$\limsup_{l\to\infty} \frac{1}{l}\Psi_\Theta\left(x, T, \vee^{l-1}_{i=0} T^{-i}\xi_n\right) = h_\Theta(x, T, \xi_n), \tag{3.1}$$

$$\limsup_{l\to\infty} \frac{1}{l}\Psi_\Theta\left(x, T, \vee^{l-1}_{i=0} T^{-i}\eta_n\right) = h_\Theta(x, T, \eta_n) \tag{3.2}$$

Applying Equations (3.1) and (3.2) and Theorem 3.5, we conclude that

$$\begin{aligned} h_\Theta(x, T, \Xi) &= \lim_{n\to\infty} h_\Theta(x, T, \xi_n) \\ &= h_\Theta(T, m_x) \\ &= \lim_{n\to\infty} h_\Theta(x, T, \eta_n) \\ &= h_\Theta(x, T, \Pi). \end{aligned}$$

So, if $f \in C(X)$, then

$$f(x)h_\Theta(x, T, \Xi) = f(x)h_\Theta(x, T, \Pi),$$

for all $x \in X$. Therefore,

$$L^T_\Theta(f, m, \Xi) = L^T_\Theta(f, m, \Pi).$$

Remark 3.2. *By using results of the Theorem 3.6, we conclude that the definition of relative entropy functional is independent of the selection of finite Borel partitions. Therefore, given any invariant measure μ and any sequence of finite Borel partitions $\Xi = \{\xi_n\}_{n\in\mathbb{N}}$, with $\operatorname{diam}(\xi_n) \to 0$, there exist unique relative entropy functional $L_\Theta^T(f,\mu,\Xi)$. So, we can write $L_\Theta^T(f,\mu)$ instead of $L_\Theta^T(f,\mu,\Xi)$, without confusion.*

Example 3.1. *Let $X = \frac{\mathbb{R}}{\mathbb{Z}}$, β denote the Borel sigma-algebra, $\Theta = \chi_X$, and $f(x) = 1$. Also, let $T : X \to X$ be the doubling map $T(x) = 2x \pmod 1$. We know that T preserves Lebesgue measure m, and so is ergodic. Hence, by Theorem 2.1, for each $x \in X$ and $A \subset X$, we have $m_x(A) = m(A)$. Let*

$$\xi_n = \left\{ \left[\frac{i}{2^n}, \frac{i+1}{2^n} \right) : i = 0, 1, ..., 2^n - 1 \right\},$$

then we see that $h_\Theta(x,T,\xi_n) = \log 2$ and thus letting $n \to \infty$, gives that $h_\Theta(x,T,\Xi) = \log 2$. So, for each $\mu \in M(X,T)$ we have $L_\Theta^T(f,\mu,\Xi) = \log 2$.

Theorem 3.7. *Suppose $T : X \to X$ is a continuous map on the compact metric space X. Then,*

(i) The relative entropy functional $f \mapsto L_\Theta^T(f,\mu)$ is linear, for any given $\mu \in M(X,T)$.

(ii) The map $\mu \mapsto L_\Theta^T(f,\mu)$ is affine, for any given $f \in C(X)$.

Proof. The proof is trivial.

Definition 3.8. *Two relative dynamical systems (X,T_1,Θ_1) and (Y,T_2,Θ_2) are said to be conjugate, if there exists a homeomorphism $\varphi : X \to Y$ such that*

$$\varphi \circ T_1 = T_2 \circ \varphi,$$

and

$$\Theta_2(T_2 \circ \varphi(x)) = \Theta_1(T_1(x)),$$

for all $x \in X$.

Theorem 3.8. *Let $T : X \to X$ be a continuous map on compact metric space X. Then, If two relative dynamical systems (X,T_1,Θ_1) and (Y,T_2,Θ_2) are conjugate, and $\mu \in M(X,T)$, then*

$$L_{\Theta_1}^{T_1}(f,\mu) = L_{\Theta_2}^{T_2}(f\varphi^{-1},\mu\varphi^{-1}),$$

for all $f \in C(X)$.

Proof. Note that

$$m_\Theta^{T_1}(A)(x) = m_\Theta^{T_2}(\varphi(A))(\varphi(x)),$$

for $x \in X$ and the Borel set $A \subset X$. So,

$$\Psi_\Theta(x,T_1,\xi) = \Psi_\Theta(\varphi(x),T_2,\varphi(\xi)),$$

for any finite Borel partition ξ. Now, by using definition of $h_\Theta(\cdot,T,\Xi)$, we conclude that

$$h_{\Theta_1}(\cdot,T_1,\Xi) = h_{\Theta_2}(\cdot,T_2,\varphi(\Xi)) \circ \varphi,$$

for any sequence $\Xi = \{\xi_n\}_{n\in\mathbb{N}}$ of finite Borel partitions of X, with $\operatorname{diam}(\xi_n) \to 0$.

Note that, $\varphi(\Xi) = \{\varphi(\xi_n)\}_{n\in\mathbb{N}}$ and $\operatorname{diam}(\varphi(\xi_n)) \to 0$. Therefore, for $\mu \in M(X,T_1)$, and $f \in C(X)$ we have

$$\begin{aligned} & L_{\Theta_1}^{T_1}(f,\mu) \\ &= \int_X f(x) h_{\Theta_1}(x,T_1,\Xi) \mathrm{d}\mu(x) \\ &= \int_X f(x) h_{\Theta_1}(\varphi(x),T_2,\varphi(\Xi)) \mathrm{d}\mu(x) \\ &= \int_Y f(\varphi^{-1}(x)) h_{\Theta_1}(x,T_2,\varphi(\Xi)) \mathrm{d}(\mu\varphi^{-1})(x) \\ &= L_{\Theta_2}^{T_2}(f\varphi^{-1},\mu\varphi^{-1}). \end{aligned}$$

From observer viewpoint, the following version of Jacobs theorem, can be obtained as follows.

Theorem 3.9. *Let $T : X \to X$ be a continuous map on compact metric space X. If $\mu = \int_{E(X,T)} m \mathrm{d}\tau(m) \in M(X,T)$ is the ergodic decomposition of μ, then*

$$L_\Theta^T(f,\mu) = \int_{E(X,T)} L_\Theta^T(f,m) \mathrm{d}\tau(m),$$

for all $f \in C(X)$.

Proof. Suppose $\Xi = \{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of finite Borel partitions of X, such that $\operatorname{diam}(\xi_n) \to 0$. Now, let $f \in C^+(X)$. Using the Corollary 2.1, we obtain that

$$\begin{aligned} & L_\Theta^T(f,\mu,\xi) \\ &= \int_X f(x) h_\Theta(x,T,\xi) \mathrm{d}\mu(x) \\ &= \int_{E(X,T)} \left(\int_X f(x) h_\Theta(x,T,\xi) \mathrm{d}m(x) \right) \mathrm{d}\tau(m) \\ &= \int_{E(X,T)} \int_X L_\Theta^T(f,m,\xi) \mathrm{d}\tau(m). \end{aligned}$$

For the rest of proof, write $f = f^+ - f^-$, for $f \in C(X)$, where $f^+, f^- \in C^+(X)$.

Theorem 3.10. *Let $T : X \to X$ be a continuous map on compact metric space X. Moreover, let $x \in X$ and $\mu \in M(X,T)$. Then, $L^T_\Theta(1,\mu) = h_\Theta(T, m_x)$.*

Proof. Let $\Xi = \{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X, such that $\operatorname{diam}(\xi_n) \to 0$. Let $\mu \in M(X,T)$. Similar to proof of the Theorem 3.6, we can obtain

$$h_\Theta(x,T,\Xi) = h_\Theta(T, m_x), \qquad \forall\, x \in X.$$

Therefore,

$$\begin{aligned} L^T_\Theta(1,\mu) &= \int_X h_\Theta(T, m_x) d\mu(x) \\ &= h_\Theta(T, m_x). \end{aligned}$$

Theorem 3.11. *Let $T : X \to X$ be a continuous map on compact metric space X. Moreover, let $x \in X$, and $\mu \in M(X,T)$. Then, the relative entropy functional $f \mapsto L^T_\Theta(f,\mu)$ is a continuous linear function on $C(X)$, and*

$$\| L^T_\Theta(\cdot,\mu) \| = h_\Theta(T, m_x).$$

Proof. Let $\Xi = \{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X, such that $\operatorname{diam}(\xi_n) \to 0$. Then, for $f \in C(X)$, we have

$$\begin{aligned} \left|L^T_\Theta(f,\mu)\right| &= \left|\int_X f(x) h_\Theta(x,T,\Theta) \mathrm{d}\mu(x)\right| \\ &\le \int_X |f(x)|\, h_\Theta(x,T,\mu)\mathrm{d}\mu(x) \\ &\le \|f\|_\infty \int_X h_\Theta(x,T,\mu)\mathrm{d}\mu(x) \\ &= \|f\|_\infty\, L^T_\Theta(1,\mu) \\ &= \|f\|_\infty h_\Theta(T, m_x). \end{aligned}$$

Finally we conclude that, the relative entropy functional is a continuous function and $\| L^T_\Theta(\cdot,\mu) \| \le h^S_\Theta(T, m_x)$. The equality holds using the Theorem 3.10.

In the following, we extract the Kolmogorov entropy from relative entropy functional, as a special case.

Theorem 3.12. *Let $T : X \to X$ be a continuous map on compact metric space X. If $\Theta : X \to [0,1]$ is the characteristic function χ_X, then $L^T_\Theta(1,\mu) = h_\mu(T)$.*

Proof. Let $\Xi = \{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X, such that $\operatorname{diam}(\xi_n) \to 0$. Using the Definition 3.6, we deduce that $h_\Theta(x,T,\xi) = h_\Theta(T, m_x)$. Let $m \in E(X,T)$. By applying Theorem 2.1, we have $m_x(A) = m(A)$, for each Borel set A and $x \in X$. So by replacing m_x with m, we have $h_\Theta(x,T,\xi) = h_m(T)$. Therefore,

$$L_\Theta(1,m) = \int_X h_\Theta(x,T,\xi)\mathrm{d}m(x) = h_m(T).$$

Now, let $\mu \in M(X,T)$, and $\mu = \int_{E(X,T)} m \mathrm{d}\tau(m)$ be the ergodic decomposition of μ. Using the Theorems 2.3 and 3.9, we have

$$\begin{aligned} L^T_\Theta(1,\mu) &= \int_{E(X,T)} L^T_\Theta(1,m)\mathrm{d}\tau(m) \\ &= \int_{E(X,T)} h_m(T)\mathrm{d}\tau(m) \\ &= h_\mu(T). \end{aligned}$$

4 Conclusions

In this paper, we introduced a new notion of *relative entropy functional* for relative dynamical systems from the viewpoint of observer Θ by using a sequence of Borel partitions. This notion is an extension of Kolmogorov entropy, as we proved that if $\Theta : X \to [0,1]$ is the characteristic function χ_X, then $L^T_\Theta(1,\mu)$ is the Kolmogorov entropy of T. It is important to highlight that, *relative entropy functional* is an invariant object under the relative conjugate relation, and so, it can be used to obtain a new method for comparing between the perspectives of observers. Moreover, it can be used to measure complexity and/or uncertainty of the system from the viewpoint of observers. This notion is a continuous linear functional on $C(X)$, such that its norm equals the relative entropy of T, at each $x \in X$.

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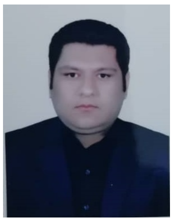

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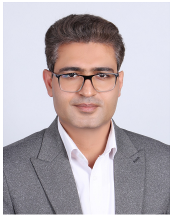

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