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Research Article

Optimal Control Strategy on Transmission Dynamics of HIV-HSV-II Coinfection Model

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Abstract
In this paper, optimal control problem is applied to Human immunodeficiency viruses (HIV) and Herpes simplex virus type 2 (HSV-2) coinfection model formulated by a system of ordinary differential equations. Optimal control strategy was employed to study the effect of combining different intervention strategy on the transmission dynamics of HIV-HSV-II coinfection diseases. The necessary conditions for the existence of the optimal controls were established using Pontryagin's Maximum Principle. Optimal control system was performed with help of Runge-Kutta forward-backward sweep numerical approximation method. Finally, numerical simulation illustrated that a combination of all controls is the most effective strategy to minimize the disease from the community.

Keywords: Coinfection, Stability, Optimal control, Numerical, Simulation.

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1. Introduction

Human immunodeficiency viruses (HIV) are an RNA retrovirus. HIV translates its RNA to DNA with a viral enzyme called reverse transcriptase [1]. The target cell of HIV is CD4 T cells. A healthy human body has about 1000/mm^3 of CD4 T cells. When the CD4 T cells of a patient decline to 200/mm^3 or below, then that person is classified as having AIDS [2]. In the world, new HIV infections among young women aged 15–24 years were reduced by 25% between 2010 and 2018. The annual number of deaths from AIDS-related illness among people living with HIV globally has fallen from a peak of 1.7 million in 2004 to 770 000 in 2018. The global decline in deaths has largely been driven by progress in eastern and southern Africa, which is home to 54% of the world's people living with HIV. AIDS-related mortality in the region declined by 44% from 2010 to 2018.The annual number of new infections since 2010 has declined from 2.1 million to 1.7 million in 2018 [3].

Herpes simplex virus type 2 (HSV-2) infections is widespread throughout the world and is almost exclusively sexually transmitted, causing genital herpes [4]. Genital herpes infections frequently have no symptoms, or mild symptoms that go unrecognized. When symptoms do occur, genital herpes is characterized by one or more genital or anal blisters or open sores called ulcers. In addition to genital ulcers, symptoms of new genital herpes infections often include fever, body aches, and swollen lymph nodes. HSV-2 is mainly transmitted during sex, through contact with genital surfaces, skin, sores or fluids of someone infected with the virus. HSV-2 can be transmitted from skin in the genital or anal area that looks normal and is often transmitted in the absence of symptoms [5]. An estimated 491 million (13%) people aged 15 to 49 years worldwide were living with the infection in 2016. More women are infected with HSV-2 than men in 2016 it was estimated that 313 million women and 178 million men were living with the infection [6, 7].

Epidemiological analysis has recognized a link between the prevalence of HSV-II and HIV. In fact, individuals infected with HSV-II are at greater risk of acquiring HIV after exposure, underscoring the fact that herpes infection is an important cofactor for HIV transmission. While the prevalence of HIV is much lower than that of HSV-II, the global burden of HIV is significant [8]. In many countries, the major public health significance of HSV-II relates to its potential role in facilitating HIV transmission. HSV-II is highly widespread in most regions experiencing severe HIV epidemics, with infection rates rising sharply with age to arrive at levels of 70% or more among adult women and men in some African countries [9].

Several mathematical models involving Ordinary Differential Equations (ODEs) have been developed to describe the transmission dynamics of HIV-HSV-II coinfection and control. These models have been used by several authors to increase the understanding of mechanisms involved in the transmission dynamics of HIV-HSV-II coinfection [10,11]. Moreover, many studies have used autonomous system of ODEs to assess the impact of using different control strategies, such as, vaccination, prevention, Screening and treatment on the transmission dynamics of coinfection disease [12,13].

The study of Pontryagin et al. [14] has laid the foundation for comprehending how to introduce control into compartmental models, as well as deriving Optimal Control (OC) strategy for containing the transmission dynamics of various infectious diseases. Particularly, formulation of Optimal Control Problem (OCP) depends on the desired aim to be achieved. The Cost Functional (CF) or objective functional is considered based on the successive time for implementing the control intervention. One may choose to implement the control strategy at the final time of the control intervention period or at each time the control intervention is

administered. Hence, it is necessary to decide whether there is the need to include terminal costs or not. The three well-known standard forms of OCP are Lagrange, Bolza, and Mayer formulations [15]. In mathematical epidemiology, the most often used technique for OCP formulation of the transmission dynamics of infectious diseases is Lagrange form. The theory of OC has been applied to compartmental models in order to derive the efficient strategies for control implementation that are helpful to decision makers in the controlling and decline of the spread of infectious diseases [16-18].

The goal of this work is to study the effect of incorporating various control strategies on mathematical model of HIV and HSV-II co-infection in [19].

2. Model Assumption

HIV-HSV-II coinfection model divided the total population at time t, denoted by $N(t)$ in Susceptible individuals $S(t)$, unawared HIV infected individuals $I_{uh}(t)$, unawared HSV-II infected individuals $I_{us}(t)$, unawared HIV-HSV-II coinfected individuals $I_{uhs}(t)$, screened HIV infected individuals $I_{sh}(t)$, screened HSV-II infected individuals $I_{ss}(t)$, screened HIV-HSV-II coinfected individuals $I_{shs}(t)$, individuals with AIDS $A(t)$, individuals with HSV-II $H(t)$, individuals with both AIDS and HSV-II $AH(t)$ and recovered individuals $R(t)$. It is assumed that susceptible individuals are recruited into the population at a constant rate Π. Susceptible individuals may acquire HIV infection with force of infection $\lambda_h = \frac{\beta_1(I_{uh}+q_1 I_{sh})}{N_h}$ when they come into effective contact with an infectious individual at the rate β_1 that may lead to infection. Also, susceptible individuals may acquire HSV-II infection with force of infection $\lambda_s = \frac{\beta_2(I_{us}+q_2 I_{ss})}{N_s}$ when they come into effective contact with an infectious individual at the rate β_1 that may lead to infection. The unawared HIV infected individuals are screened and join the screened HIV infected subclass at a rate α. However, some of the unawared HIV infected individuals progress to AIDS at a rate δ and others join the unawared HIV-HSV-II coinfection subclass at a rate ϕ. Furthermore, screened HIV infected individuals' progress to AIDS at a rate ω and also joined the screened HIV-HSV-II coinfection subclass at a rate φ. Also, the unawared HIV-HSV-II coinfection individuals are screened and join the screened HIV-HSV-II coinfected subclass at a rate θ. But, some of the unawared HIV-HSV-II coinfected individuals progress to AIDS and HSV-II coinfection subclass at rate ρ. The screened HIV-HSV-II coinfection is also progress to AIDS and HSV-II coinfection subclass at rate σ. The unawared HSV-II infected individuals are screened and joined the screened HSV-II infected subclass at a rate γ and others join the unawared HIV-HSV-II coinfection subclass at rate ψ. However, some of them are progress to HSV-II subclass with rate ε and recovered naturally by body immunity at rate κ. The screened HSV-II infected individuals are treated at rate ϵ and joined the recovered subclass with this rate. Some of them are progress to HSV-II subclass and screened coinfection of HIV-HSV-II subclass with rate η and τ respectively. AIDS individuals and HSV-II individuals are also progress to coinfection of AIDS and HSV-II subclass with rate ν and χ respectively. Finally, recovered individuals revert to susceptible subclass after losing their immunity at a rate ϑ. All individuals suffer natural mortality at a rate μ and sick individuals die of AIDS, HSV-II and AIDS-HSV-II coinfection at rate ξ [19].

The above assumptions can be written as linear system of differential equation as follows [19]:

$$\frac{dS}{dt} = \Pi + \vartheta R - (\lambda_h + \lambda_s + \mu)S$$
$$\frac{dI_{uh}}{dt} = \lambda_h S - (\phi\lambda_s + \alpha + \delta + \mu)I_{uh}$$
$$\frac{dI_{us}}{dt} = \lambda_s S - (\psi\lambda_h + \varepsilon + \gamma + \kappa + \mu)I_{us}$$
$$\frac{dI_{uhs}}{dt} = \phi\lambda_s I_{uh} + \psi\lambda_h I_{us} - (\rho + \theta + \mu)I_{uhs} \qquad (\mathbf{1})$$
$$\frac{dI_{sh}}{dt} = \alpha I_{uh} - (\varphi + \omega + \mu)I_{sh}$$
$$\frac{dI_{ss}}{dt} = \gamma I_{us} - (\tau + \eta + \epsilon + \mu)I_{ss}$$
$$\frac{dI_{shs}}{dt} = \theta I_{uhs} + \varphi I_{sh} + \tau I_{ss} - (\sigma + \mu)I_{shs}$$
$$\frac{dA}{dt} = \delta I_{uh} + \omega I_{sh} - (\nu + \mu + \xi)A$$
$$\frac{dH}{dt} = \varepsilon I_{us} + \eta I_{ss} - (\chi + \pi + \mu + \xi)H$$
$$\frac{dR}{dt} = \kappa I_{us} + \epsilon I_{ss} + \pi H - (\vartheta + \mu)R$$
$$\frac{dAH}{dt} = \rho I_{uhs} + \sigma I_{shs} + \nu A + \chi H - (\mu + \xi)AH$$

With initial condition $S(0) = S_0,\ I_{uh}(0) = I_{uh0},\ I_{us}(0) = I_{us0},\ I_{uhs}(0) = I_{uhs0},\ I_{sh}(0) = I_{sh0},\ I_{ss}(0) = I_{ss0},\ I_{shs}(0) = I_{shs0},\ A(0) = A_0,\ H(0) = H_0,\ AH(0) = AH_0,\ R(0) = R_0$.

3. Stability Analysis of HIV-HSV-II Co-infection Model

In this section, the analysis of HIV-HSV-II co-infection in model equation (1) was considered.

3.1 Invariant Region

Theorem 1: The total population size N of the system of model equation (1) is bounded in the invariant regionΩ. That is, size of N is bounded for all t.

Proof: In model equation (1) the total population of N is given a

$$N = S + I_{uh} + I_{us} + I_{uhs} + I_{sh} + I_{ss} + I_{shs} + A + H + AH + R$$

Differentiating N both sides with respect to t leads to

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI_{uh}}{dt} + \frac{dI_{us}}{dt} + \frac{dI_{uhs}}{dt} + \frac{dI_{sh}}{dt} + \frac{dI_{ss}}{dt} + \frac{dI_{shs}}{dt} + \frac{dA}{dt} + \frac{dH}{dt} + \frac{dAH}{dt} + \frac{dR}{dt} \qquad (2)$$

Substituting model equation (1) into equation (2), we can get

$$\frac{dN}{dt} = \Pi - \mu N - \xi(A + H + AH) \qquad (3)$$

In the absence of mortality due to disease ($\xi = 0$), then equation (3) become

$$\frac{dN}{dt} \leq \Pi - \mu N \qquad (4)$$

Rearranging and integrating both sides of (4), we get

$$\int \frac{dN}{\Pi - \mu N} \leq \int dt$$

$\Leftrightarrow \frac{-1}{\mu}\ln(\Pi - \mu N) \leq t + c_{12}$, where c_{12} is integration constant

$\Rightarrow \ln(\Pi - \mu N) \geq -\mu t + c_{13}$, where $c_{13} = -\mu c_{12}$

$\Rightarrow (\Pi - \mu N) \geq ce^{-\mu t}$, where $c = e^{-c_{13}}$

Then, applying initial condition$N(0) = N_0$, we obtain $\mathrm{c} = \Pi - \mu N_0$

$$\Rightarrow \Pi - \mu N \geq (\Pi - \mu N_0)e^{-\mu t}$$

$$\Rightarrow N \leq \frac{\Pi}{\mu} - [\frac{\Pi - \mu N}{\mu}]e^{-\mu t} \qquad (5)$$

As $t \to \infty$ in equation (5), the population size $N(t) \to \frac{\Pi}{\mu}$ which implies that $0 \leq N(t) \leq \left(\frac{\Pi}{\mu}\right)$. Thus, the feasible solution set of the model equation (1) enters and remains in the region:$\Omega = \{(S,\ I_{uh},\ I_{us},\ I_{uhs},\ I_{sh},\ I_{ss},\ I_{shs}\ ,\ A,\ H,\ AH,\ R\ ) \in \mathbb{R}_+^{11} : N_h \leq \Pi/\mu\}$.

Therefore, the model equation (1) is wellposed epidemiologically and mathematically. Hence, it is sufficient to study the dynamics of the model in the regionΩ.

3.2. Existence of solution

Lemma 1: Solutions of the model equations (1) together with the initial conditions$S(0) > 0$, $I_{uh}(0) > 0$, $I_{us}(0) > 0$, $I_{uhs}(0) > 0$, $I_{sh}(0) > 0$, $I_{ss}(0) > 0$, $I_{shs}(0) > 0$, $A(0) > 0$, $H(0) > 0, AH(0) > 0$, $R(0) > 0$ exist in $\mathbb{R}_+^{11}$i.e., the solution of the model variables $S(t)$, $I_{uh}(t)$, $I_{us}(t)$, $I_{uhs}(t)$, $I_{sh}(t)$, $I_{ss}(t)$, $I_{shs}(t)$, $A(t)$, $H(t)$, $AH(t)$and $R(t)$ exist for all t and will remain in$\mathbb{R}_+^{11}$.

Proof: Existence of solution for $(\ S,\ I_{uh},\ I_{us},\ I_{sh},\ I_{ss},\ A,\ H,\ R\ )$ [19]. Now, positivity for $(I_{uhs},\ I_{shs},\ AH)$ are shown below in table 1. Let

$f_{10}(\ S, I_{uh}, I_{us},\ I_{uhs},\ I_{sh},\ I_{ss}, I_{shs}\ , A,\ H, AH, R) = \phi I_{uh} + \psi I_{us} - (\rho + \theta + \mu)I_{us}$

$f_{11}(S, I_{uh}, I_{us},\ I_{uhs},\ I_{sh},\ I_{ss}, I_{shs}\ , A,\ H, AH, R) = \theta I_{uhs} + \varphi I_{sh} + \tau I_{ss} - (\sigma + \mu)I_{shs}$

$f_{12}(S, I_{uh}, I_{us},\ I_{uhs},\ I_{sh},\ I_{ss}, I_{shs}\ , A,\ H, AH, R) = \rho I_{uhs} + \sigma I_{ssh} + \nu A + \chi H - (\mu + \xi)AH$

According to Derrick and Groosman theorem as in [19], let Ω denote the region$\Omega = \{(S,\ I_{uh},\ I_{us},\ I_{uhs},\ I_{sh},\ I_{ss},\ I_{shs}\ , A,\ H,\ AH,\ R) \in \mathbb{R}_+^{11} :\ N \leq \Pi/\mu\}$. Then equations (1) have a unique solution if $(\partial f_i)/(\partial x_j)$, $i, j = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ are continuous and bounded inΩ. Here,$x_1 = S,\ x_2 = I_{uh},\ x_3 = I_{us}, x_4 = I_{uhs},\ x_5 = I_{sh},\ x_6 = I_{ss},\ x_7 = I_{shs}\ ,\ x_8 = A, x_9 = H,\ x_{10} = AHandx_{11} = R$. The continuity and the boundedness are verified as here under:

Thus, all the partial derivatives $(\partial f_i)/(\partial x_j), i, j = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ exist, continuous and bounded inΩ. Hence, by Derrick and Groosman theorem, a solution for the model (1) exists and is unique.

Table 1. Continuity and boundedness of the model solution

$\left\|\frac{(\partial f_1)}{(\partial S)}\right\| = \left\|\frac{(\partial f_1)}{(\partial I_{sh})}\right\| = \left\|\frac{(\partial f_1)}{(\partial I_{ss})}\right\| = \left\|\frac{(\partial f_1)}{(\partial I_{shs})}\right\| = \left\|\frac{(\partial f_1)}{(\partial A)}\right\| = \left\|\frac{(\partial f_1)}{(\partial H)}\right\| = \left\|\frac{(\partial f_1)}{(\partial AH)}\right\| = \left\|\frac{(\partial f_1)}{(\partial R)}\right\| = 0 < \infty$ $\left\|\frac{(\partial f_1)}{(\partial I_{uh})}\right\| = \|\phi\| < \infty, \quad \left\|\frac{(\partial f_1)}{(\partial I_{us})}\right\| = \|\psi\| < \infty, \quad \left\|\frac{(\partial f_1)}{(\partial I_{uhs})}\right\| = \|-(\rho+\theta+\mu)\| < \infty$
$\left\|\frac{(\partial f_2)}{(\partial S)}\right\| = \left\|\frac{(\partial f_2)}{(\partial I_{uh})}\right\| = \left\|\frac{(\partial f_2)}{(\partial I_{us})}\right\| = \left\|\frac{(\partial f_2)}{(\partial A)}\right\| = \left\|\frac{(\partial f_2)}{(\partial H)}\right\| = \left\|\frac{(\partial f_2)}{(\partial AH)}\right\| = \left\|\frac{(\partial f_2)}{(\partial R)}\right\| = 0 < \infty$ $\left\|\frac{(\partial f_2)}{(\partial I_{uhs})}\right\| = \|\theta\| < \infty, \quad \left\|\frac{(\partial f_2)}{(\partial I_{sh})}\right\| = \|\varphi\| < \infty, \quad \left\|\frac{(\partial f_2)}{(\partial I_{ss})}\right\| = \|\tau\| < \infty, \quad \left\|\frac{(\partial f_2)}{(\partial I_{shs})}\right\| = \|-(\sigma+\mu)\| < \infty$
$\left\|\frac{(\partial f_3)}{(\partial S)}\right\| = \left\|\frac{(\partial f_3)}{(\partial I_{uh})}\right\| = \left\|\frac{(\partial f_3)}{(\partial I_{us})}\right\| = \left\|\frac{(\partial f_3)}{(\partial I_{sh})}\right\| = \left\|\frac{(\partial f_3)}{(\partial I_{ss})}\right\| = \left\|\frac{(\partial f_3)}{(\partial R)}\right\| = 0 < \infty$ $\left\|\frac{(\partial f_3)}{(\partial I_{uhs})}\right\| = \|\rho\| < \infty, \left\|\frac{(\partial f_3)}{(\partial I_{shs})}\right\| = \|\sigma\| < \infty, \left\|\frac{(\partial f_3)}{(\partial A)}\right\| = \|\nu\| < \infty, \left\|\frac{(\partial f_3)}{(\partial H)}\right\| = \|\chi\| < \infty, \left\|\frac{(\partial f_3)}{(\partial AH)}\right\| = \|-(\mu+\xi)\| < \infty$

3.3.Positivity of Solution

In this section, we show all the solution of the model equation (1) remains positive for future time if their respective initial values are positive.

Theorem 2: Let$\Omega = \{(S, I_{uh}, I_{us}, I_{uhs}, I_{sh}, I_{ss}, I_{shs}, A, H, AH, R) \in \mathbb{R}_+^{11}; S_0(0) > 0, I_{uh0}(0) > 0, I_{us0}(0) > 0, I_{uhs0}(0) > 0, I_{sh0}(0) > 0, I_{ss0}(0) > 0, I_{shs0}(0) > 0, A_0(0) > 0, H_0(0) > 0, AH_0(0) > 0, , R_0(0) > 0\}$ then the solutions of $\{S, I_{uh}, I_{us}, I_{uhs}, I_{sh}, I_{ss}, I_{shs}, A, H, AH, R\}$ are positive for all $t \geq 0$.

Proof: Since positivity of $S(t)$, $I_{uh}(t)$, $I_{us}(t)$, $I_{sh}(t)$,$I_{ss}(t)$, $A(t)$, $H(t)$ and$R(t)$are shown in [19] separately. Now let us show $I_{uhs}(t)$, $I_{shs}(t)$ and $AH(t)$ are positive for future time.

From model equation (1) we have:

$\frac{dI_{uhs}}{dt} = \phi I_{uh} + \psi I_{us} - (\rho+\theta+\mu)I_{uhs}$, eliminating the positive terms $(\phi I_{uh} + \psi I_{us})$we get,

$\Leftrightarrow \frac{dI_{uhs}}{dt} \geq -(\rho+\theta+\mu)I_{uhs}$, using variables separable method we get,

$\Rightarrow \frac{dI_{uhs}}{I_{uhs}} \geq -(\rho+\theta+\mu)dt$, integrating both side we can get,

$$\Rightarrow \int \frac{dI_{uhs}}{I_{uhs}} \geq -\int (\rho+\theta+\mu)\, dt$$

$\Rightarrow \ln I_{uhs} \geq -(\rho+\theta+\mu)t + c_{13}$, where c_{13} is integration constant

$\Rightarrow I_{uhs}(t) \geq I_{uhs0}\mathrm{e}^{-(\rho+\theta+\mu)t}$,$I_{uhs0} = \mathrm{e}^{c_{13}}$and $\mathrm{e}^{-(\rho+\theta+\mu)t} \geq 0$,for all $t \geq 0$.

Hence, it can be concluded that$I_{uhs}(t) \geq 0$.

From model equation (1) we have:

$\frac{dI_{shs}}{dt} = \theta I_{uhs} + \varphi I_{sh} + \tau I_{ss} - (\sigma+\mu)I_{shs}$, eliminating the positive terms $(\theta I_{uhs} + \varphi I_{sh} + \tau I_{ss})$we get,

$\Leftrightarrow \frac{dI_{shs}}{dt} \geq -(\sigma+\mu)I_{shs}$, using variables separable method we get,

$\Rightarrow \frac{dI_{shs}}{I_{shs}} \geq -(\sigma+\mu)dt$, integrating both side we can get,

$$\Rightarrow \int \frac{dI_{shs}}{I_{shs}} \geq -\int (\sigma+\mu)\, dt$$

$\Rightarrow \ln I_{shs} \geq -(\sigma+\mu)t + c_{14}$, where c_{14} is integration constant

$\Rightarrow I_{shs}(t) \geq I_{shs0}\mathrm{e}^{-(\sigma+\mu)t}$,$I_{\mathrm{shs0}} = \mathrm{e}^{14}$and $\mathrm{e}^{-(\sigma+\mu)t} \geq 0$,for all $t \geq 0$.

Hence, it can be concluded that$I_{shs}(t) \geq 0$.

From model equation (1) we have:

$\frac{dAH}{dt} = \rho I_{uhs} + \sigma I_{shs} + \nu A + \chi H - (\mu+\xi)AH$, eliminating the positive terms $(\rho I_{uhs} + \sigma I_{shs} + \nu A + \chi H)$we get,

$\Leftrightarrow \frac{dAH}{dt} \geq -(\mu+\xi)AH$, using variables separable method we get,

$\Rightarrow \frac{dAH}{AH} \geq -(\mu+\xi)dt$, integrating both side we can get,

$$\Rightarrow \int \frac{dAH}{AH} \geq -\int (\mu+\xi)\, dt$$

$\Rightarrow \ln \mathrm{AH} \geq -(\mu+\xi)t + c_{15}$, where c_{15} is integration constant

$\Rightarrow AH(t) \geq AH_0\mathrm{e}^{-(\mu+\xi)t}$,$AH_0 = \mathrm{e}^{\mathrm{c}_{15}}$and $\mathrm{e}^{-(\mu+\xi)t} \geq 0$,for all $t \geq 0$.

Hence, it can be concluded that $AH(t) \geq 0$.

Therefore, the model variables $I_{uhs}(t)$, $I_{shs}(t)$ and $AH(t)$ representing population sizes of various types of cells are positive quantities and will remain in $\mathbb{R}_+^{11}$for all$t$.

3.4. Stability Analysis of the Disease-free Equilibrium (DFE)

The disease-free equilibrium of the HIV-HSV-II coinfection is obtained by equating the system of model equation (1) to zero. At disease free equilibrium, there are no infections and recovery. Then we can get;

$$E_1 = \left\{\left(\frac{\Pi}{\mu}\right), \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0\right\}$$

The local stability of the DFE,E_1, can be established using the next generation operator method in Van den Driessche and Watmouth [22] on the system (1). It follows that the basic reproduction number of the HIV-HSV-II model equation (1), denoted by $\Re_{hs}$ is given by $\Re_{hs} = \max\{\Re_h, \Re_s\}$ as obtained in [19].

$$\text{Where, } \Re_h = \left[\frac{\beta_1(\omega+\mu)+\beta_1 q_1 \alpha(\alpha+\delta+\mu)}{(\alpha+\delta+\mu)(\omega+\mu)}\right] \text{ \& } \Re_s = \left[\frac{\beta_2(\eta+\epsilon+\mu)+\beta_2 q_2 \gamma(\varepsilon+\gamma+\kappa+\mu)}{(\varepsilon+\gamma+\kappa+\mu)(\eta+\epsilon+\mu)}\right]$$

Theorem 3: The disease-free equilibrium points E_1 of the system (1) is locally asymptotically stable whenever the basic reproduction number is less than one ($\Re_{\mathrm{hs}} < 1$) and unstable if otherwise.

Proof: To proof this theorem first we obtained the Jacobian matrix of the model equation (1) at the disease-free equilibrium E_1 is given by:

$$J(E_1)=\begin{bmatrix} -\mu & -\beta_1 & -\beta_2 & 0 & -\beta_1 q_1 & -\beta_2 q_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1 - r_1 & 0 & 0 & \beta_1 q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_2 - r_2 & 0 & 0 & \beta_2 q_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi & \psi & -r_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & -r_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 & -r_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & \varphi & \tau & -r_6 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & \omega & 0 & 0 & -r_7 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 & \eta & 0 & 0 & -r_8 & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 & 0 & \sigma & \nu & \chi & -r_9 & 0 \\ 0 & 0 & \kappa & 0 & 0 & \epsilon & 0 & 0 & \pi & 0 & -r_{10} \end{bmatrix}$$

Now, the eigenvalues of $J(E_1)$ are required to be found. The characteristic equation $det[J(E_1)-\lambda I]=0$ is expanded and simplified as follows:

$$\begin{vmatrix} -\mu-\lambda & -\beta_1 & -\beta_2 & 0 & -\beta_1 q_1 & -\beta_2 q_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\beta_1 - r_1)-\lambda & 0 & 0 & \beta_1 q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\beta_2 - r_2)-\lambda & 0 & 0 & \beta_2 q_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi & \psi & -r_{3-\lambda} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & -r_4-\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 & -r_5-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & \varphi & \tau & -r_6-\lambda & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & \omega & 0 & 0 & -r_7-\lambda & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 & \eta & 0 & 0 & -r_8-\lambda & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 & 0 & \sigma & \nu & \chi & -r_9-\lambda & 0 \\ 0 & 0 & \kappa & 0 & 0 & \epsilon & 0 & 0 & \pi & 0 & -r_{10}-\lambda \end{vmatrix} = 0 \quad (6)$$

From the Jacobian matrix of (6), we obtained a characteristic polynomial:

$[-\mu-\lambda][-r_{10}-\lambda][-r_9-\lambda][-r_8-\lambda][-r_7-\lambda][-r_6-\lambda][4\lambda^4+L_1\lambda^3+L_2\lambda^2+L_3\lambda+L_4]=0$ (7)

Where $L_1=r_4-\beta_1+r_5-\beta_2$

$L_2=2[\alpha\beta_1 q_1-\gamma\beta_2 q_2-r_4(\beta_1-r_1)-r_5(\beta_2-r_2)]+(r_5-\beta_2)(r_4-\beta_1)$

$L_3=\alpha\beta_1 q_1(r_5-\beta_2)-\gamma\beta_2 q_2(r_4-\beta_1)-r_4(r_5-\beta_2)(\beta_1-r_1)-r_5(\beta_2-r_2)(r_4-\beta_1)$

$L_4=\gamma\beta_2 q_2 r_4(\beta_1-r_1)+\gamma\beta_2 q_2\alpha\beta_1 q_1+r_4 r_5(\beta_2-r_2)(\beta_1-r_1)-r_5\alpha\beta_1 q_1(\beta_2-r_2)$

Thus, from equation (7) clearly, we see that:

$\lambda_1=-\mu,\ \lambda_2=-r_{10},\ \lambda_3=-r_9,\ \lambda_4=-r_8,\ \lambda_5=-r_7,\ \lambda_6=-r_6, \lambda_7=-r_3$

It can be observed that the eigenvalues$\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6$and λ_7 are absolutely negative quantities.

For the last expression, that is,

$$4\lambda^4+L_1\lambda^3+L_2\lambda^2+L_3\lambda+L_4=0 \quad (8)$$

We applied Routh-Hurwitz criteria. By the principle of Routh-Hurwitz criteria, (8) has strictly negative real root if and only if $L_1>0, L_2>0, L_3>0, L_4>0$ and$L_1L_2L_3>L_3^2+L_1^2L_4$.

Therefore, it is concluded that the DFE E_1 of the system of differential equations (1) is locally asymptotically stable if $\Re_{hs} < 1$ and unstable if$\Re_{hs} > 1$. Here,

$r_1 = (\phi + \alpha + \delta + \mu), r_2 = (\psi + \varepsilon + \gamma + \kappa + \mu), r_3 = (\rho + \theta + \mu), r_4 = (\varphi + \omega + \mu),$

$r_5 = (\tau + \eta + \epsilon + \mu), r_6 = (\sigma + \mu), r_7 = (\nu + \mu + \xi), r_8 = (\chi + \pi + \mu + \xi),$

$r_9 = (\mu + \xi), r_{10} = (\vartheta + \mu).$

3.5. Global Stability of Disease-Free Equilibrium

The global stability of disease-free equilibrium is determined using Castillo-Chavez and Song [23] technique. The model equation (1) can be re-written as

$dX/dt = F(X, Y)$

$dY/dt = G(X, Y), \quad G(X, 0) = 0$

Where, X stands for the uninfected population, that is $X = (S,\ R)$ and Y also stands for the infected population, that is $Y = (I_{uh},\ I_{us},\ I_{uhs},\ I_{sh},\ I_{ss},\ I_{shs}, A, H, AH)$. The disease-free equilibrium point of the model is denoted by $U = (X^*, 0)$. The point $U = (X^*, 0)$ to be globally asymptotically stable equilibrium for the model provided that $\Re_{hs} < 1$ and the following conditions must be met:

(H_1). For$dX/dt = F(X, 0)$, X^* is globally asymptotically stable.

(H_2). $G(X, Y) = AY - \tilde{G}(X, Y), \quad \tilde{G}(X, Y) \geq 0$ for $(X, Y) \in \Omega$.

Where $A = D_Y G(U, 0)$ is a Metzler matrix (the off diagonal elements of A are non-negative) and G is the region where the model make biologically sense.

If the model equation(1) met the above two criteria, then the following theorem holds.

Theorem 4: The point $U = (X^*, 0)$ is globally asymptotically stable equilibrium provided that $\Re_{\mathrm{hs}} < 1$ and the condition (H_1) and $(H_2)$are satisfied.

Proof: From system (1) we can get $F(X, Y)$ and $G(X, Y)$;

$$dX/dt = F(X, Y) = \begin{bmatrix} \Pi + \vartheta R - (\lambda_h + \lambda_s + \mu)S \\ \kappa I_{us} + \epsilon I_{ss} + \pi H - (\vartheta + \mu)R \end{bmatrix} \text{ and}$$

$$dY/dt = G(X, Y) = \begin{bmatrix} \lambda_h S - r_1 I_{uh} \\ \lambda_s S - r_2 I_{us} \\ \phi I_{uh} + \psi I_{us} - r_3 I_{uhs} \\ \alpha I_{uh} - r_4 I_{sh} \\ \gamma I_{us} - r_5 I_{ss} \\ \theta I_{uhs} + \varphi I_{sh} + \tau I_{ss} - r_6 I_{shs} \\ \delta I_{uh} + \omega I_{sh} - r_7 A \\ \varepsilon I_{us} + \eta I_{ss} - r_8 H \\ \rho I_{uhs} + \sigma I_{shs} + \nu A + \chi H - r_9 AH \end{bmatrix}$$

Consider the reduced system

$$\frac{dX}{dt}_{|Y=0} = \begin{bmatrix} \Pi - \mu S \\ 0 \end{bmatrix} \qquad (9)$$

From (9), it is obvious that $X^* = [(\Pi/\mu), 0]$ is the global asymptotic point. This can be verified from the solution, namely $S = [\Pi/\mu] + [S(0) - (\Pi/\mu)]e^{-\mu t}$. As $t \to \infty$, the solution $(S) \to [\Pi/\mu]$, implying that the global convergence of (9) in Ω.

From the equation for infected compartments in the model we have:

$$A = \begin{bmatrix} \beta_1 - r_1 & 0 & 0 & \beta_1 q_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_2 - r_2 & 0 & 0 & \beta_2 q_2 & 0 & 0 & 0 & 0 \\ \phi & \psi & -r_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & -r_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & -r_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & \varphi & \tau & -r_6 & 0 & 0 & 0 \\ \delta & 0 & 0 & \omega & 0 & 0 & -r_7 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 & \eta & 0 & 0 & -r_8 & 0 \\ 0 & 0 & \rho & 0 & 0 & \sigma & \nu & \chi & -r_9 \end{bmatrix}$$

Since A is Metzler matrix, i.e., all off diagonal elements are nonnegative. Then, $G(X,Y)$ can be written as, $G(X,Y) = AY - \tilde{G}(X,Y)$, where

$$\tilde{G}(X,Z) = \begin{bmatrix} \beta_1(I_{uh} + q_1 I_{sh})\left[1 - \frac{S}{N}\right] \\ \beta_2(I_{us} + q_2 I_{ss})\left[1 - \frac{S}{N}\right] \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{G}_1(X,Z) \\ \tilde{G}_2(X,Z) \\ \tilde{G}_3(X,Z) \\ \tilde{G}_4(X,Z) \\ \tilde{G}_5(X,Z) \\ \tilde{G}_6(X,Z) \\ \tilde{G}_7(X,Z) \\ \tilde{G}_8(X,Z) \\ \tilde{G}_9(X,Z) \end{bmatrix} \quad (10)$$

It follows that, in equation (10)$\tilde{G}_1(X,Y) \geq 0, \tilde{G}_2(X,Y) \geq 0,$ and $\tilde{G}_3(X,Y) = \tilde{G}_4(X,Y) = \tilde{G}_5(X,Y) = \tilde{G}_6(X,Y) = \tilde{G}_7(X,Y) = \tilde{G}_8(X,Y) = \tilde{G}_9(X,Y) = 0.$

Hence,$\tilde{G}(X,Y) \geq 0$. Therefore, condition (H_1) and (H_2) are satisfied and we conclude that U is globally asymptotically stable for$\Re_{hs} < 1$.

3.6.Endemic Equilibrium Point

The endemic equilibrium denoted by $(S^*,\ I^*_{uh},\ I^*_{us},\ I^*_{uhs},\ I^*_{sh},\ I^*_{ss},\ I^*_{shs},\ A^*,\ H^*,\ AH^*,\ R^*)$ and it occur when the disease persist in the community. To obtain it we equate all the model (1) to zero. Then we obtained;

$S^* = \frac{\Pi+\vartheta R^*}{(\lambda^*_h+\lambda^*_s+\mu)}, I^*_{uh} = \frac{(\Pi+\vartheta R^*)\lambda^*_h}{(\lambda^*_h+\lambda^*_s+\mu)(\phi+\alpha+\delta+\mu)},\ I^*_{us} = \frac{(\Pi+\vartheta R^*)\lambda^*_s}{(\lambda^*_h+\lambda^*_s+\mu)(\psi+\varepsilon+\gamma+\kappa+\mu)},$

$I^*_{uhs} = \frac{\phi I^*_{uh}+\psi I^*_{us}}{(\rho+\theta+\mu)},\ I^*_{sh} = \frac{\alpha I^*_{uh}}{(\varphi+\omega+\mu)}, I^*_{ss} = \frac{\gamma I^*_{us}}{(\tau+\eta+\epsilon+\mu)},$

$I^*_{shs} = \frac{\theta I^*_{uhs}+\varphi I^*_{sh}+\tau I_{ss}}{(\sigma+\mu)}, A^* = \frac{\lambda^*_h(\pi+\vartheta R^*)[\delta(\varphi+\omega+\mu)+\omega\alpha]}{(\lambda^*_h+\lambda^*_s+\mu)(\phi+\alpha+\delta+\mu)(\varphi+\omega+\mu)(\nu+\mu+\xi)},$

$H^* = \frac{\lambda_s^*(\pi+\vartheta R^*)\varepsilon[(\tau+\eta+\epsilon+\mu)+\eta\gamma]}{(\lambda_h^*+\lambda_s^*+\mu)(\psi+\varepsilon+\gamma+\kappa+\mu)(\tau+\eta+\epsilon+\mu)(\chi+\pi+\mu+\xi)}$, $(AH)^* = \frac{\rho I_{uhs}^*+\sigma I_{shs}^*+\nu A^*+\chi H^*}{(\mu+\xi)}$,

$R^* = \frac{\kappa I_{us}^* + \epsilon I_{ss}^* + \pi H}{(\vartheta+\mu)}$

After substituting the variables, we see that the endemic equilibrium point is very long and complicated. We have therefore decided to use numerical simulation of the co-infection dynamics considering when $\Re_{hs} < 1$ and$\Re_{hs} > 1$.

4. Optimal control formulation of the model

This section discusses the optimal control problem formulation and analysis for the transmission dynamics of HIV-HSV-II coinfection model in [19] is considered. In this model, we introduce five control intervention; $u_1(t)$ prevention effort of HIV infection that protect susceptible from contacting the HIV infection, $u_2(t)$ prevention effort of HSV-II infection that protect susceptible from contacting the HSV-II infection, $u_3(t)$ screening effort of unawared HIV infected that used to screen unawared HIV infected , $u_4(t)$ screening effort of unawared HIV infected that used to screen unawared HSV-II infected and $u_5(t)$ treatment effort of HSV-II that help to treat HSV-II infectious individuals. Time is specified and is relatively short and is given by $t \in [0, T]$, T is the terminal time.

After incorporating all control functions $u_1(t), u_2(t), u_3(t)$, $u_4(t)$ and $u_5(t)$ in HIV-HSV-II coinfection model, we obtain the following state system;

$$\frac{dS}{dt} = \Pi + \vartheta R - ((1-u_1)\lambda_h + (1-u_2)\lambda_s + \mu)S$$

$$\frac{dI_{uh}}{dt} = (1-u_1)\lambda_h S - (1-u_2)\phi\lambda_s I_{uh} - (u_3+\alpha)I_{uh} - (\delta+\mu)I_{uh}$$

$$\frac{dI_{us}}{dt} = (1-u_2)\lambda_s S - (1-u_1)\psi\lambda_h I_{us} - (u_4+\gamma)I_{us} - (u_5+\kappa)I_{us} - (\varepsilon+\mu)I_{us}$$

$$\frac{dI_{uhs}}{dt} = (1-u_2)\phi\lambda_s I_{uh} + (1-u_1)\psi\lambda_h I_{us} - (u_3+u_4+\theta)I_{uhs} - (\rho+\mu)I_{uhs}$$

$$\frac{dI_{sh}}{dt} = (1-u_3)\alpha I_{uh} - (\varphi+\omega+\mu)I_{sh} \qquad (11)$$

$$\frac{dI_{ss}}{dt} = (1-u_4)\gamma I_{us} - (u_5+\epsilon)I_{ss} - (\tau+\eta+\mu)I_{ss}$$

$$\frac{dI_{shs}}{dt} = (1-u_3)(1-u_4)\theta I_{uhs} + \varphi I_{sh} + \tau I_{ss} - (\sigma+\mu)I_{shs}$$

$$\frac{dA}{dt} = \delta I_{uh} + \omega I_{sh} - (\nu+\mu+\xi)A$$

$$\frac{dH}{dt} = \varepsilon I_{us} + \eta I_{ss} - (u_5+\pi)H - (\chi+\mu+\xi)H$$

$$\frac{dR}{dt} = (u_5+\kappa)I_{us} + (u_5+\epsilon)I_{ss} + (u_5+\pi)H - (\vartheta+\mu)R$$

$$\frac{dAH}{dt} = \rho I_{uhs} + \sigma I_{shs} + \nu A + \chi H - (\mu+\xi)AH$$

With initial condition

$$S(0) = S_0,\ I_{uh}(0) = I_{uh0},\ I_{us}(0) = I_{us0},\ I_{uhs}(0) = I_{uhs0},\ I_{sh}(0) = I_{sh0},\ I_{ss}(0) = I_{ss0},\ I_{shs}(0) = I_{shs0},\ A(0) = A_0,\ H(0) = H_0,\ AH(0) = AH_0,\ R(0) = R_0. \qquad (12)$$

In formulation, cost function is quadratic with respect to the control terms and proposed as in [14, 18, 20, 21]. Thus, the objective functional is given as

$$J(u) = \int_0^T [A_1 I_{uh}(t) + A_2 I_{us}(t) + A_3 I_{uhs}(t) + A_4 I_{sh}(t) + A_5 I_{ss}(t) + A_6 A_{shs}(t) + \frac{1}{2}\sum_{i=1}^{5} B_i u_i^2]dt \rightarrow min \quad (13)$$

Subject to the dynamical system, equation (11) with initial conditions (12). Where T is the final time for control administration, and the control set Ω, with $u_i \leq 1-\epsilon,\ i = 1,2,3,4,5$ where $\epsilon \ll 1$ is defined as

$$\Omega = \left\{(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)) \in \left(L^{\infty}(0,T)\right)^5 : 0 \leq u_i(t) \leq 1-\epsilon, \forall_t \in [0,T]\right\} (14)$$

The controls are bounded between 0 and 1. The choice of $u_i \leq 1-\epsilon$ is based on the fact that the intervention is not 100% perfectly implemented.

The specification of objective functional in (13) involves minimization of the sizes of unawared infectious and screened infectious individuals and the costs associated with implementing the controls $u_1(t), u_2(t), u_3(t), u_4(t)$ and $u_5(t)$. In the integrand of objective functional, the coefficients $A_i > 0, i = 1,2,3,4,5,6$ are the weight constants associated with unawared HIV infected, unawared HSV-II infected, unawared HIV-HSV-II coinfected, screened HIV infected, screened HSV-II infected and screened HIV-HSV-II coinfected individuals respectively. They are used to balance each term of the integrand so that none of them dominates. The quantities $B_i > 0, i = 1,2,3,4,5$ are weight constants for prevention of (HIV, HSV-II, and HIV-HSV-II), screening of (HIV, HSV-II HIV-HSV-II) and treatment of HSV-II infectious controls respectively.

The term $\frac{B_1 u_1^2}{2}, \frac{B_2 u_2^2}{2}, \frac{B_3 u_3^2}{2}, \frac{B_4 u_4^2}{2}$ and $\frac{B_5 u_5^2}{2}$ denotes the costs related to the implementation of controls $u_1(t), u_2(t), u_3(t), u_4(t)$ and $u_5(t)$ respectively. Also, $A_i,\ i = 1,2,3,4,5,6$ measures the importance of reducing the size of the HIV infectious individuals, HSV-II infectious individuals and the disease burden, while $B_i, i = 1,2,3,4,5$ are the relative measures of the costs or efforts required to implement the respective controls. Additionally, the functional J corresponds the total cost due to HIV and HSV-II outbreak and its control strategies. Further, the integrand function

$$L(\emptyset, u) = A_1 I_{uh}(t) + A_2 I_{us}(t) + A_3 I_{uhs}(t) + A_4 I_{sh}(t) + A_5 I_{ss}(t) + A_6 A_{shs}(t) + \frac{1}{2}\sum_{i=1}^{5} B_i u_i^2 \quad (15)$$

measures the current cost at time t. Finally, the fixed constant T denotes the terminal innervations time.

The goal is to find an optimal control value $u^* = (u_1^*,\ u_2^*,\ u_3^*,\ u_4^*, u_5^*)$ of the controls $u = (u_1, u_2, u_3, u_4, u_5)$ such that the optimal control problem can be defined as

$$J(u_1^*,\ u_2^*,\ u_3^*,\ u_4^*, u_5^*) = \min_{\Omega} J\left(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)\right) \quad (16)$$

Satisfying model equation (11).

4.1.Existence of optimal controls

In this subsection, we prove the existence of such optimal control functions which minimize the cost function in the finite intervention period. The following result guarantees the

existence of optimal control functions. A detail and similar analysis on existence of optimal control can be obtained in [20, 21].

Theorem 5: There exists an optimal control $u^* = \left(u_1^*,\ u_2^*,\ u_3^*,\ u_{4,}^* u_5^*\right)$ in Ω and a corresponding solution vector $\bar{X} = (\bar{S}, \overline{I_{uh}, I_{us}},\ \overline{I_{uhs}},\ \overline{I_{sh}},\ \overline{I_{ss}},\ \overline{I_{shs}}, \bar{A}, \bar{H}, \bar{R}, \overline{AH})$ to the initial value problem (12) such that

$$J\left(u_1^*, u_2^*,\ u_3^*, u_{4,}^* u_5^*\right) = \min_{\Omega} J\left(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)\right)$$

Proof: The entire state variables involved in the model are continuously differentiable. Therefore, we need to verify the following four conditions as given in [20]

(i) The set of solutions to the system (11) with control variables are non-empty.
(ii) The set Ω is convex and closed.
(iii) The state system can be written as linear function of control variables with coefficients depending on time and state variables.
(iv) The integrand L of (15) is convex on Ω and $L(\emptyset, u) \geq g(u)$, where g continuous and $\|u\|^{-1} g(u) \to +\infty$ as $\|u\| \to \infty$.

Since the total population in (11) is defined as

$N(t) = S(t) + I_p(t) + I_h(t) + I_{ph}(t) + C(t) + I_{hc}(t) + R_s(t) + R_p(t) + R_h(t) + R_{Ph}(t)$

From governing system (11) it follows that

$$dN/dt = \Pi - \mu N$$

It follows that the solutions of the state system are continuous and bounded for each admissible control functions in Ω. Further, the right-hand side functions of the model equations (11) satisfy the Lipschitz condition with respect to state variables. Therefore, the initial value problem (11) has a unique solution corresponding to each admissible control function $u \in \Omega$. Thus, condition (i) is proved.

To prove (ii), consider

$\Omega = \{u \in \mathbb{R}^5 : \|u\| \leq 1 - \epsilon\}$.

Let $u_1, u_2 \in \Omega$ such that $\|u_1\| \leq 1 - \epsilon$ and $\|u_2\| \leq 1 - \epsilon$. Then for any $\lambda \in [0,1]$,

$\|\lambda u_1 + (1-\lambda) u_2\| \leq \lambda \|u_1\| + (1-\lambda)\|u_2\| \leq 1 - \epsilon$.

This implies that Ω is convex and closed. The state system (11) is linear in control variables u_1, u_2, u_3, u_4 and u_5 with coefficients depending on state variables. With this condition (iii) is satisfied. The integrand of the cost functional is the sum of convex function and hence convex with respect to control variables. Furthermore,

$L(\emptyset, u) = A_1 I_{uh}(t) + A_2 I_{us}(t) + A_3 I_{uhs}(t) + A_4 I_{sh}(t) + A_5 I_{ss}(t) + A_6 A_{shs}(t) + \frac{1}{2}\sum_{i=1}^{5} B_i u_i^2$

Let $\chi = \min\left(\frac{1}{2}\sum_{i=1}^{5} B_i u_i^2\right) > 0$ and define a continuous function $g(u) = \chi \|u\|^{-1}$. Then from equation (15) we have $L(\emptyset, u) \geq g(u)$. Clearly, $\|u\|^{-1} g(u) \to +\infty$ as $\|u\| \to \infty$. Thus, condition (iv) is achieved. Therefore, the existence of an optimal control pair $(\bar{X}, u^*)$ is satisfying (11) and (15) is assured by results given in [20]. Hence the proof.

4.2. Characterization of optimal control

In order to derive the necessary condition for the optimal control, we use pontryagin's maximum principle [14]. This principle converts the system and the objective functional into a problem minimizing point wise a Hamiltonian H with respect to u as in [15]. The Hamiltonian associated to our problem is defined by

$$H = \mathrm{L}(\emptyset, \mathrm{u}) + \frac{1}{2}\sum_{\mathrm{i}=1}^{5} \mathrm{B_i u_i^2} + \sum_{\mathrm{i}=1}^{11} \lambda_{\mathrm{i}}(\mathrm{t})\mathrm{g_i}(\mathrm{t}, \emptyset, \mathrm{u}) \quad (17)$$

Where, L is as defined in equation (15). Now, in order to obtain the optimal solution to optimal control problem involving the non-autonomous HIV-HSV-II model equation (11) with the state's initial conditions (12) and cost function in equation (13), Pontryagin's Maximum Principle [14] is applied in the following way. Suppose (X^*, u^*) is an optimal solution of the optimal control problem presented in equation (11) and (13). Then, there exists a non-trivial adjoint variable $\lambda = (\lambda_1(t),\ \lambda_2(t),\ \lambda_3(t),\ \lambda_4(t),\ \lambda_5(t),\ \lambda_6(t),\ \lambda_7(t), \lambda_8(t),\ \lambda_9(t),\ \lambda_{10}(t), \lambda_{11}(t))$ which satisfies the following equations.

$$\frac{d\lambda}{dt} = -\frac{\partial H(t,\emptyset,u,\lambda)}{\partial \emptyset} \quad (18)$$

$$0 = \frac{\partial H(t,\emptyset,u,\lambda)}{\partial u} \quad (19)$$

$$0 = \lambda(T) \quad (20)$$

Next, the necessary condition given by equation (18-20) is applied to the Hamiltonian, H in equation (17), to obtain the following result.

Theorem 6: Let $u^* = (u_1^*,\ u_2^*,\ u_3^*,\ u_4^*, u_5^*) \in \Omega$ be an optimal control with the corresponding optimal states $\bar{S}, \overline{I_{uh}}, I_{us},\ \overline{I_{uhs}},\ \overline{I_{sh}},\ \overline{I_{ss}},\ \overline{I_{shs}}, \bar{A}, \bar{H}, \bar{R}$ and $\overline{AH}$. Then, there exist the costate variables $\lambda_1(t),\ \lambda_2(t),\ \lambda_3(t),\ \lambda_4(t),\ \lambda_5(t),\ \lambda_6(t),\ \lambda_7(t), \lambda_8(t),\ \lambda_9(t),\ \lambda_{10}(t)$ and $\lambda_{11}(t)$ that satisfy

$$\frac{d\lambda_1}{dt} = \lambda_1[(1-u_1)\lambda_h + (1-u_2)\lambda_s + \mu] - \lambda_2(1-u_1)\lambda_h - \lambda_3(1-u_2)\lambda_s,$$

$$\frac{d\lambda_2}{dt} = -A_1 + \lambda_1\left[(1-u_1)\frac{\beta S}{N}\right] - \lambda_2\left[(1-u_1)\frac{\beta S}{N} - ((1-u_2)\phi\lambda_s + u_3 + \alpha + \delta + \mu)\right] +$$

$$\lambda_3\left[(1-u_1)\psi\frac{\beta S}{N}I_{us}\right] - \lambda_4[(1-u_2)\phi\lambda_s + (1-u_1)\psi\frac{\beta S}{N}I_{us}] - \lambda_5(1-u_3)\alpha - \lambda_8\delta,$$

$$.\frac{d\lambda_3}{dt} = -A_2 + \lambda_1\left[(1-u_2)\frac{\beta S}{N}\right] + \lambda_2\left[(1-u_2)\phi\frac{\beta S}{N}I_{us}\right] - \lambda_3\left[(1-u_2)\frac{\beta S}{N} - ((1-u_1)\psi\lambda_h + u_4 + u_5 + \gamma + \kappa + \varepsilon + \mu)\right] +$$

$$\lambda_4\left[(1-u_2)\phi\frac{\beta S}{N}I_{uh} + (1-u_1)\psi\lambda_h\right] + \lambda_6(1-u_4)\gamma + \lambda_9\varepsilon + \lambda_{10}(u_5 + \kappa),$$

$$\frac{d\lambda_4}{dt} = -A_3 + \lambda_4(u_3 + u_4 + \theta + \rho + \mu) - \lambda_7[(1-u_3)(1-u_4)\theta] - \lambda_{11}\rho,$$

$$\frac{d\lambda_5}{dt} = -A_4 + \lambda_1\left[(1-u_1)\frac{\beta q_1 S}{N}\right] - \lambda_2\left[(1-u_1)\frac{\beta q_1 S}{N}\right] + \lambda_3\left[(1-u_1)\psi\frac{\beta q_1 S}{N}I_{us}\right] -$$

$$\lambda_4\left[(1-u_1)\psi\frac{\beta q_1 S}{N}I_{us}\right] + \lambda_5(\varphi + \omega + \mu) - \lambda_7\varphi - \lambda_8\omega,$$

$$\frac{d\lambda_6}{dt} = -A_5 + \lambda_1\left[(1-u_2)\frac{\beta q_2 S}{N}\right] + \lambda_2\left[(1-u_2)\phi\frac{\beta q_2 S}{N}I_{uh}\right] - \lambda_3\left[(1-u_2)\frac{\beta q_2 S}{N}\right] -$$

$$\lambda_4\left[(1-u_2)\phi\frac{\beta q_2 S}{N}I_{uh}\right]+\lambda_6(u_5+\epsilon+\eta+\tau+\mu)-\lambda_7\tau-\lambda_9\eta-\lambda_{10}(u_5+\epsilon),$$

$$\frac{d\lambda_7}{dt}=-A_6+\lambda_7(\sigma+\mu)-\lambda_{11}\sigma$$

$$\frac{d\lambda_8}{dt}=\lambda_8(\nu+\mu+\xi)-\lambda_{11}\nu, \tag{21}$$

$$\frac{d\lambda_9}{dt}=\lambda_9(u_5+\pi+\chi+\mu+\xi)-\lambda_{10}(u_5+\pi)-\lambda_{11}\chi,$$

$$\frac{d\lambda_{10}}{dt}=\lambda_{10}(\vartheta+\mu)-\lambda_1\vartheta$$

$$\frac{d\lambda_{11}}{dt}=\lambda_{11}(\mu+\xi)$$

And the transversality or boundary conditions expressed by equation (22) as

$$\lambda_i(T)=0,\ \ i=1,2,\dots,11 \tag{22}$$

With optimal controls given as

$$u_1^*(t)=min\left\{max\left\{0,\ \frac{(\lambda_2-\lambda_1)\lambda_h S+(\lambda_4-\lambda_3)\psi\lambda_h I_{us}}{B_1}\right\},\ 1-\epsilon\right\}$$

$$u_2^*(t)=min\left\{max\left\{0,\ \frac{(\lambda_3-\lambda_1)\lambda_s S+(\lambda_4-\lambda_2)\phi\lambda_s I_{uh}}{B_2}\right\},\ 1-\epsilon\right\}$$

$$u_3^*(t)=min\left\{max\left\{0,\ \frac{(\lambda_2+\alpha\lambda_5)I_{uh}+(\lambda_4+\theta\lambda_7)I_{uhs}}{B_3}\right\},\ 1-\epsilon\right\}$$

$$u_4^*(t)=min\left\{max\left\{0,\ \frac{(\lambda_3+\gamma\lambda_6)I_{us}+(\lambda_4+\theta\lambda_7)I_{uhs}}{B_4}\right\},\ \ 1-\epsilon\right\}$$

$$u_5^*(t)=min\left\{max\left\{0,\ \frac{(\lambda_3-\lambda_{10})I_{us}+(\lambda_6-\lambda_{10})I_{ss}+(\lambda_9-\lambda_{10})H}{B_5}\right\},\ \ 1-\epsilon\right\} \tag{23}$$

Proof: Let the Hamiltonian, H, be as defined in equation (17). Then, using Pontryagin's Maximum Principle, equation (21) is obtained from equation (18). It is clear that the boundary conditions have the form equation (22), since all the states are at terminal time T. The Hamiltonian, H, is minimized with respect to the controls at $u^*=(u_1^*,\ u_2^*,\ u_3^*,\ u_4^*,u_5^*)$ by solving equation (19) at $u^*=(u_1^*,\ u_2^*,\ u_3^*,\ u_4^*,u_5^*)$, respectively. Hence,

$$\frac{\partial H}{\partial u_1}=B_1u_1-(\lambda_2-\lambda_1)\lambda_h S-(\lambda_4-\lambda_3)\psi\lambda_h I_{us}=0,\text{ at }\ u_1=u_1^*$$

$$\frac{\partial H}{\partial u_2}=B_2u_2-(\lambda_3-\lambda_1)\lambda_s S-(\lambda_4-\lambda_2)\phi\lambda_s I_{uh}=0,\text{ at }\ u_2=u_2^*$$

$$\frac{\partial H}{\partial u_3}=B_3u_3-(\lambda_2+\alpha\lambda_5)I_{uh}-(\lambda_4+\theta\lambda_7)I_{uhs}=0,\text{ at }\ u_3=u_3^* \tag{24}$$

$$\frac{\partial H}{\partial u_4}=B_4u_4-(\lambda_3+\gamma\lambda_6)I_{us}-(\lambda_4+\theta\lambda_7)I_{uhs}=0,\text{ at }\ u_4=u_4^*$$

$$\frac{\partial H}{\partial u_5}=B_5u_5-(\lambda_3-\lambda_{10})I_{us}-(\lambda_6-\lambda_{10})I_{ss}-(\lambda_9-\lambda_{10})H=0,\text{ at }\ u_5=u_5^*$$

Therefore, solving for $u_1^*,\ u_2^*,\ u_3^*,\ u_4^*$ and u_5^* from equation (24) we obtain

$$u_1^*(t)=min\left\{max\left\{0,\ \frac{(\lambda_2-\lambda_1)\lambda_h S+(\lambda_4-\lambda_3)\psi\lambda_h I_{us}}{B_1}\right\},\ 1-\epsilon\right\}$$

$$u_2^*(t) = min\left\{max\left\{0, \frac{(\lambda_3-\lambda_1)\lambda_s S+(\lambda_4-\lambda_2)\phi\lambda_s I_{uh}}{B_2}\right\}, 1-\epsilon\right\}$$
$$u_3^*(t) = min\left\{max\left\{0, \frac{(\lambda_2+\alpha\lambda_5)I_{uh}+(\lambda_4+\theta\lambda_7)I_{uhs}}{B_3}\right\}, 1-\epsilon\right\} \quad (25)$$
$$u_4^*(t) = min\left\{max\left\{0, \frac{(\lambda_3+\gamma\lambda_6)I_{us}+(\lambda_4+\theta\lambda_7)I_{uhs}}{B_4}\right\}, 1-\epsilon\right\}$$
$$u_5^*(t) = min\left\{max\left\{0, \frac{(\lambda_3-\lambda_{10})I_{us}+(\lambda_6-\lambda_{10})I_{ss}+(\lambda_9-\lambda_{10})H}{B_5}\right\}, 1-\epsilon\right\}$$

Finally, optimal controls and optimal state are found by solving numerically the optimality system which consists of the non-autonomous equation (11) with initial condition (12) and the costate system (21) with its boundary conditions coupled with the control characterization given in equation (25).

4.3. Uniqueness of the optimality system

In order to successively discuss uniqueness of the optimality system we notice that the adjoint system is also linear in λ_i for $i = 1,2,3,4,5,6,\dots,11$ with bounded coefficients. Thus, there exists a $M > 0$ such that $|\lambda_i(t)| < M$ for $i = 1,2,3,4,5,6,\dots,11$ on $[0,T]$.

Theorem 7. [18] For T sufficiently small the solution to the optimality system is unique.

5. Numerical Simulation

In this section, the numerical solutions of optimality system are discussed. Using the initial conditions $S(0) = 200,\ I_{uh}(0) = 180, I_{us}(0) = 175,\ I_{uhs}(0) = 170, I_{sh}(0) = 150, I_{ss}(0) = 140, I_{shs}(0) = 120,\ A(0) = 60, H(0) = 50, AH(0) = 40, R(0) = 30$ and also coefficients of the state and controls that we used are $A1 = 40, \mathrm{A2} = 35, A3 = 30, \mathrm{A4} = 25, \mathrm{A5} = 20, \mathrm{A6} = 15, B1 = 5\ , B2 = 5, B3 = 5, \mathrm{B4} = 5, \mathrm{B5} = 5$ a simulation study is conducted. Finally, an optimal control strategy is designed and discussed using different control strategies.To solve the optimal controls and states, we use the Runge-Kutta numerical method using MATLAB program. The solution of the optimal control problem is obtained by solving the optimality system which consists of the state and adjoint systems. For computational illustration, the values of parameters in Table 2 were employed and the solution is obtained by using the following iterative scheme.

Step1: Make a guess of the controls.

Step 2: Use the values of the controls together with the initial conditions to solve the state equations, using a forward numerical scheme.

Step 3: Using the current solution of the state system together with the transversality conditions, solve the adjoint equations using a backward numerical scheme. We use a backward scheme for the costate system because the transversality conditions are final time conditions.

Step 4: Update the controls using the characterizations in (25).

Step 5: Repeat Steps 2 to 4 until the values of the unknowns at the current iteration are very close to those of the previous iteration [24].

Table 2: Parameter values used in simulations

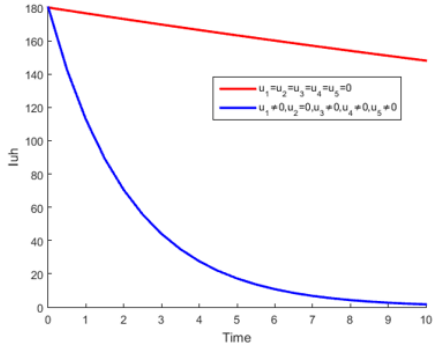
Parameter	Value	Source	Parameter	Value	Source
Π	0.004	[19]	κ	0.02	[19]
q_1	0.002	[19]	θ	0.003	[19]
q_2	0.00197	[19]	φ	0.003	[19]
μ	0.02	[19]	ω	0.054	[19]
ϑ	0.0031	[19]	τ	0.003	[19]
ϕ	0.003	[19]	η	0.011	[19]
α	0.003	[19]	ϵ	0.02	[19]
ρ	0.064	[19]	σ	0.017	[19]
δ	0.016	[19]	ν	0.001	[19]
ψ	0.003	[19]	χ	0.001	[19]
ε	0.039	[19]	π	0.0041	[19]
γ	0.003	[19]	ξ	0.0001	[19]
β	0.068	[19]			

a) Strategy I: Optimal use of HIV prevention, HIV and HSV-II screening and HSV-II treatment

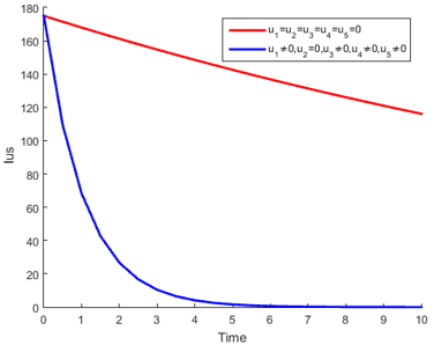
This intervention combines prevention effort for HIV, both screening effort for HIV and HSV-II and HSV-II treatment are used to optimize objective functional while setting prevention effort for HSV-II equal to zero. Results illustrate that the size of infectious population reduce sharply with controls more than the case without controls as shown in Figure 1.

b) Strategy II: Optimal use of HIV and HSV-II prevention, HSV-II screening and HSV-II treatment

This intervention strategy combines both prevention effort for HIV- HSV-II, screening effort for HSV-II and treatment effort HSV-II are used to optimize objective functional while setting screening effort for HIV equal to zero. As shown in Figure 2, the magnitudes of infectious population reduce more when controls are in use than the case without controls.



(1a)



(1b)

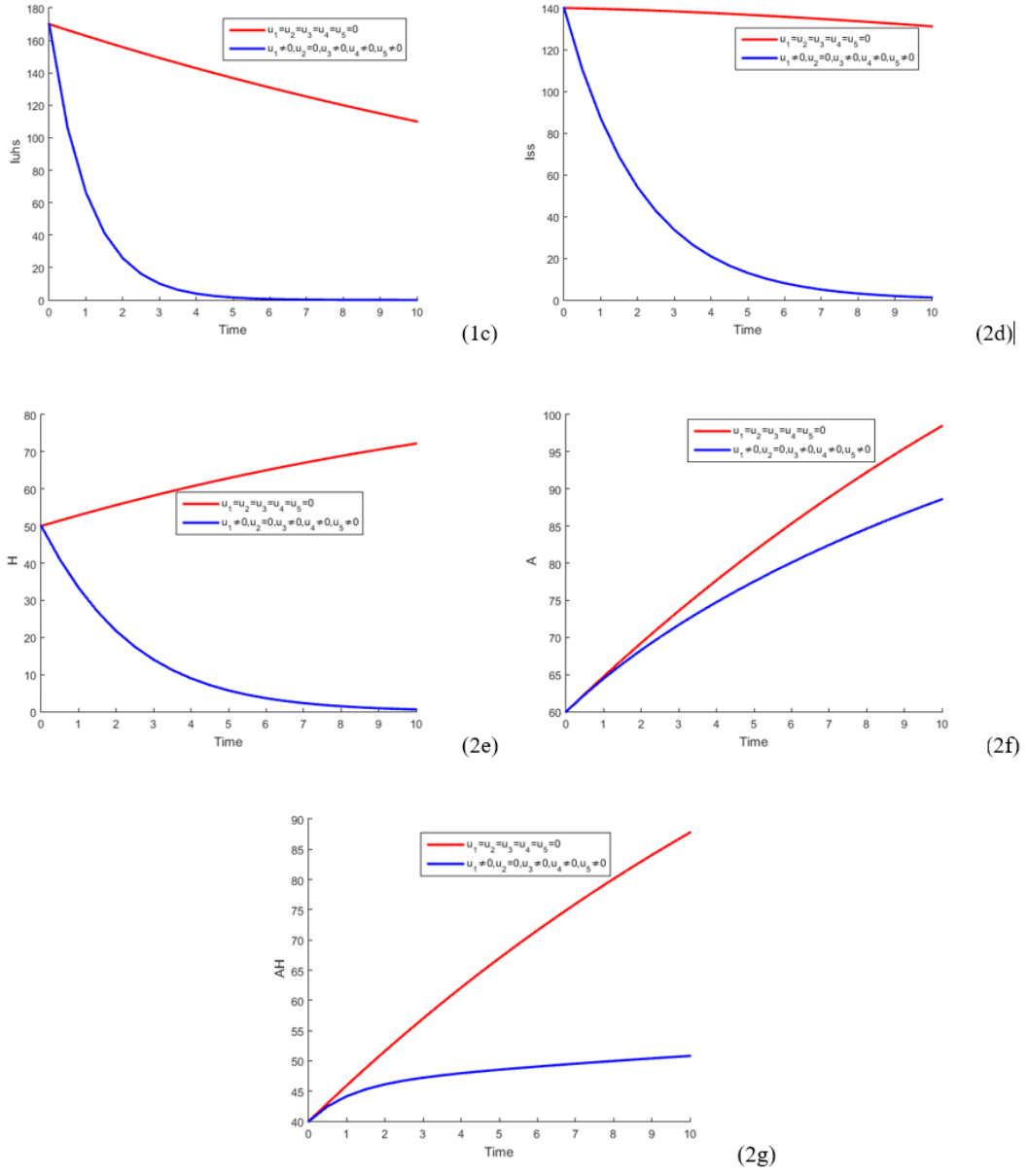


Figure 1: Simulations of using HIV prevention, HIV and HSV-II screening and HSV-II treatment

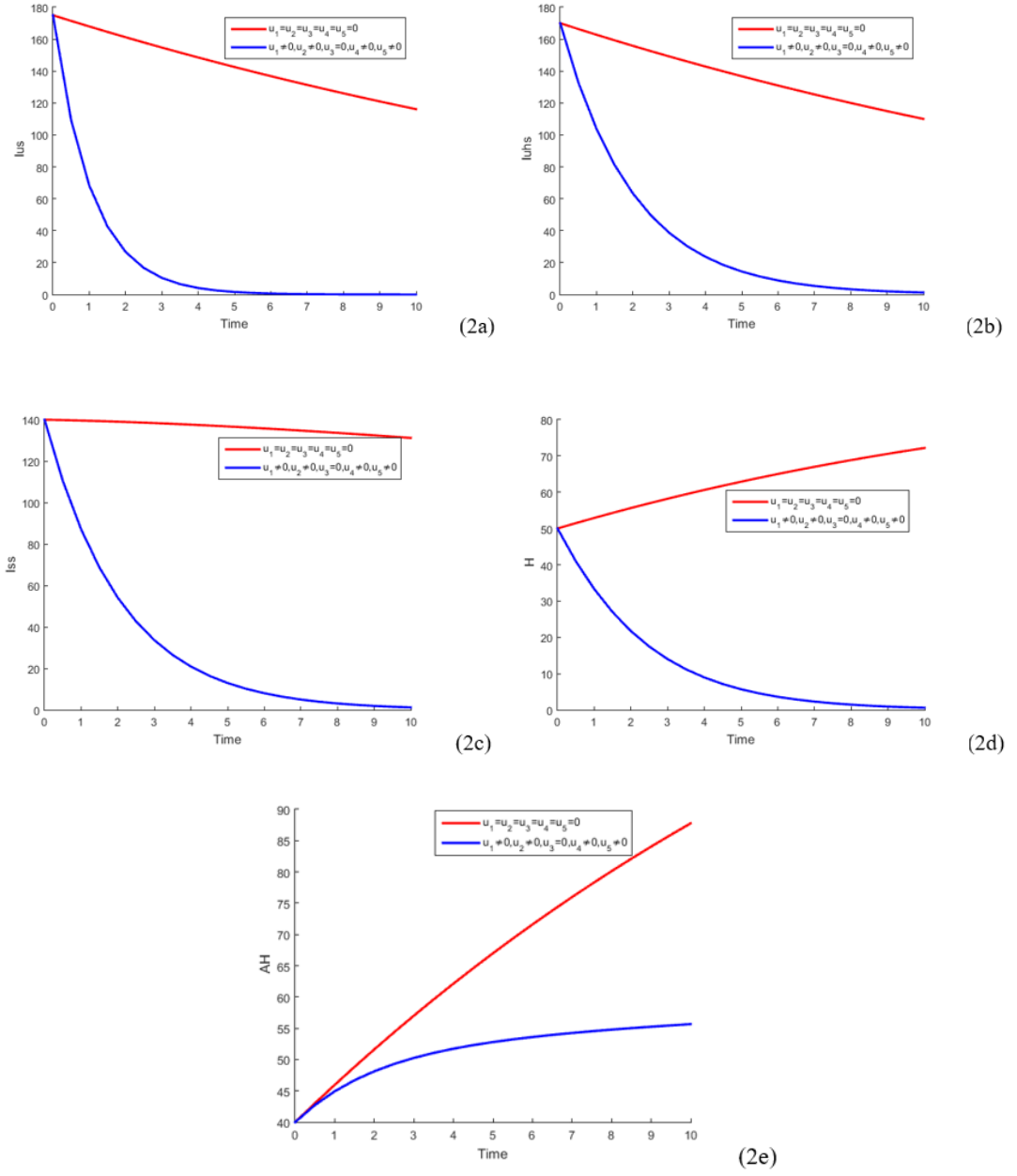


Figure 2: Simulations of using HIV and HSV-II prevention, HSV-II screening and HSV-II treatment

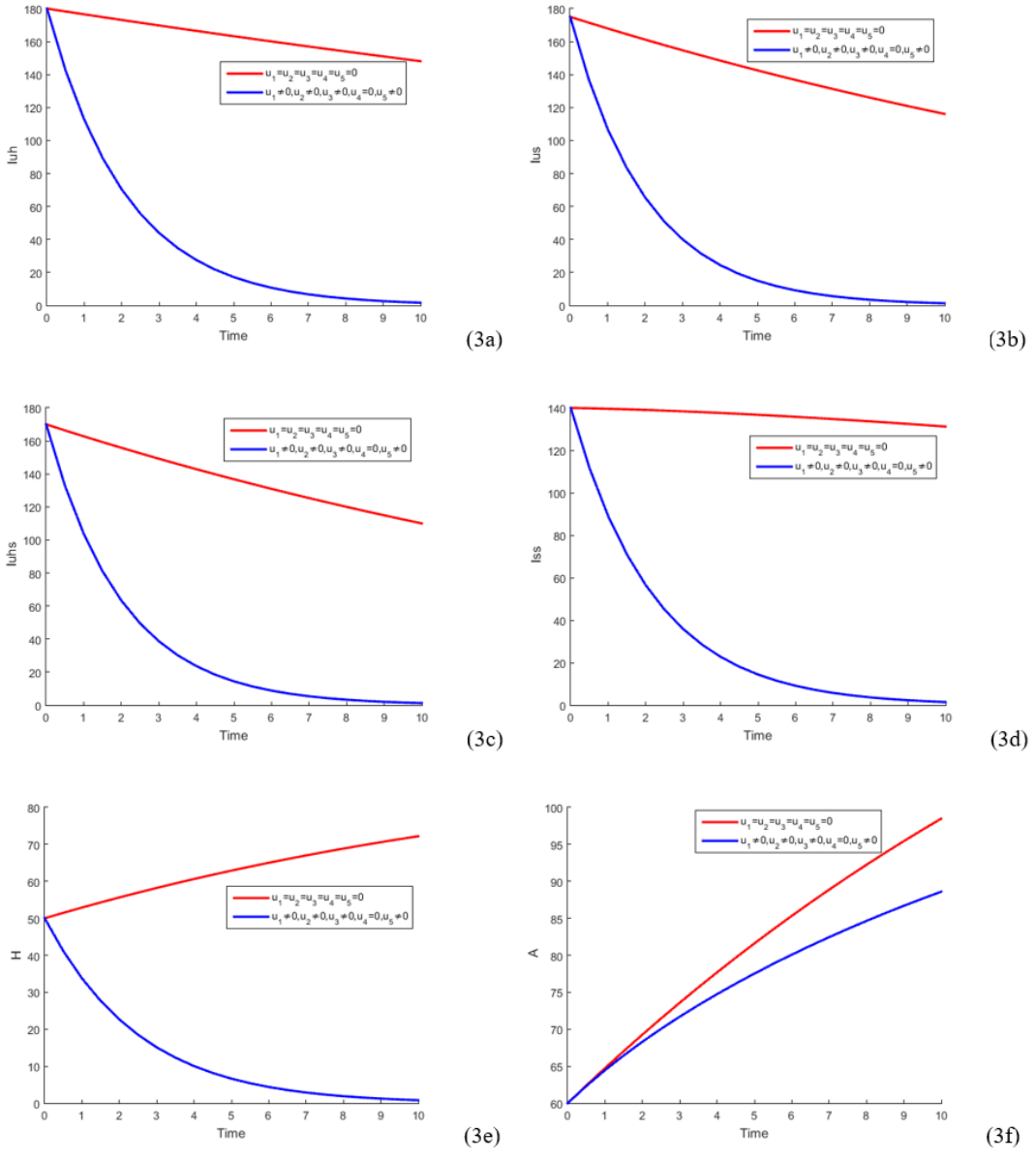
c) Strategy III: Optimal use of HIV and HSV-II prevention, HIV screening and HSV-II treatment

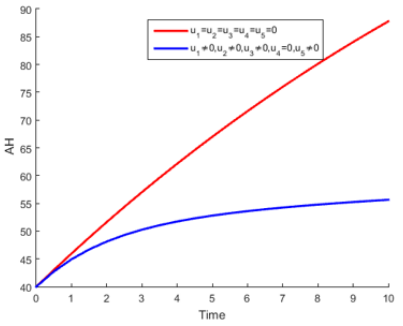
This strategy illustrates effect of prevention effort for both HIV and HSV-II, screening effort for HIV and treatment effort for HSV-II are used to optimize objective functional while setting screening effort for HSV-II equal to zero. As expected, the number of infectious

populations diminishes more rapidly with controls than the case without controls as illustrated in Figure 3.

d) Strategy IV: Optimal use of both HIV and HSV-II prevention and both HIVand HSV-II screening

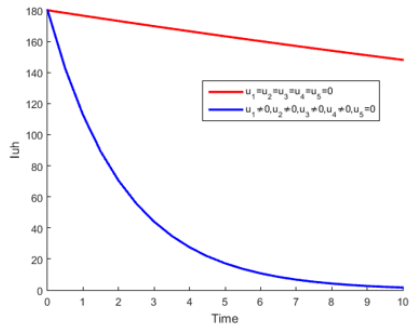
This strategy shows effect of both prevention effort for HIV and HSV-II and both screening effort for HIV-HSV-II are used to optimize objective functional while setting treatment effort for HSV-II equal to zero. Results describe that, the number of infectious populations decreases more rapidly with controls than the case without controls as illustrated in Figure 4.



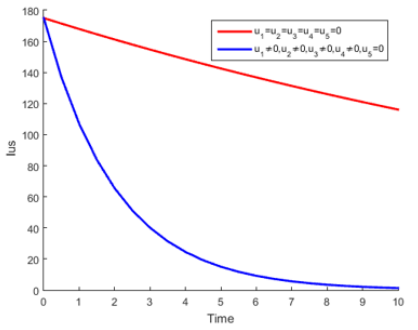


(3g)

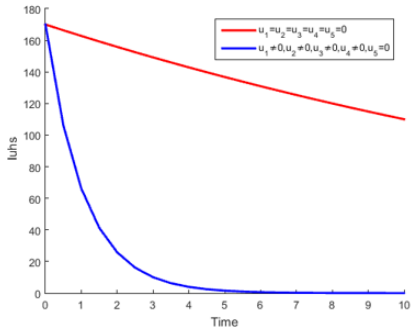
Figure 3: Simulations of using both HIV and HSV-II prevention, HIV screening and HSV-II treatment



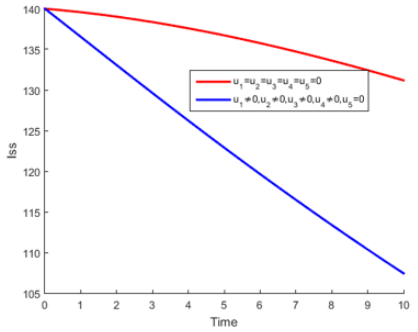
(4a)



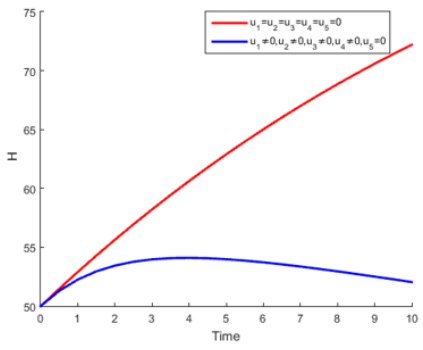
(4b)



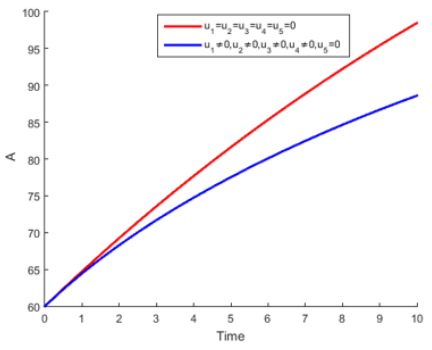
(4c)



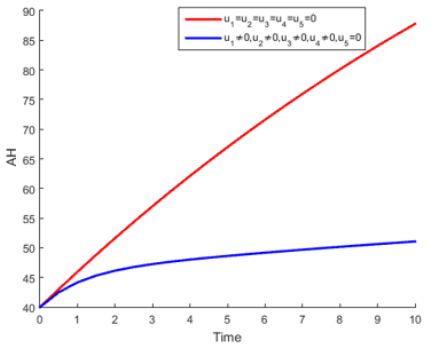
(4d)



(4e)



(4f)

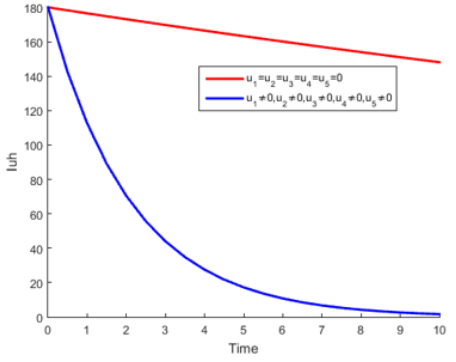


(4g)

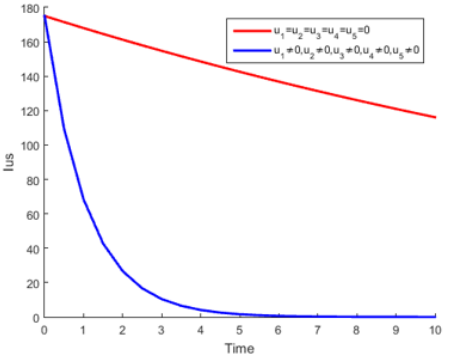
Figure 4: Simulations of using both HIV and HSV-II prevention and both HIVand HSV-II screening

e) Strategy V: Optimal use of all controls

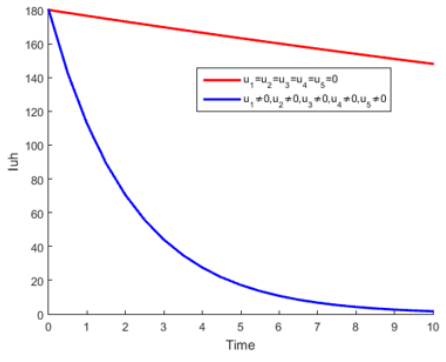
This strategy shows effect of prevention effort for both HIV and HSV-II and treatment effort for both HIV and HSV-II and HSV-II treatment are used to optimize objective functional. Results describe that, the number of infectious populations decreases more rapidly with controls than the case without controls as illustrated in Figure 5.



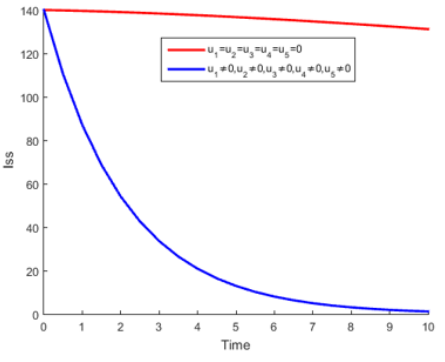
(5a)



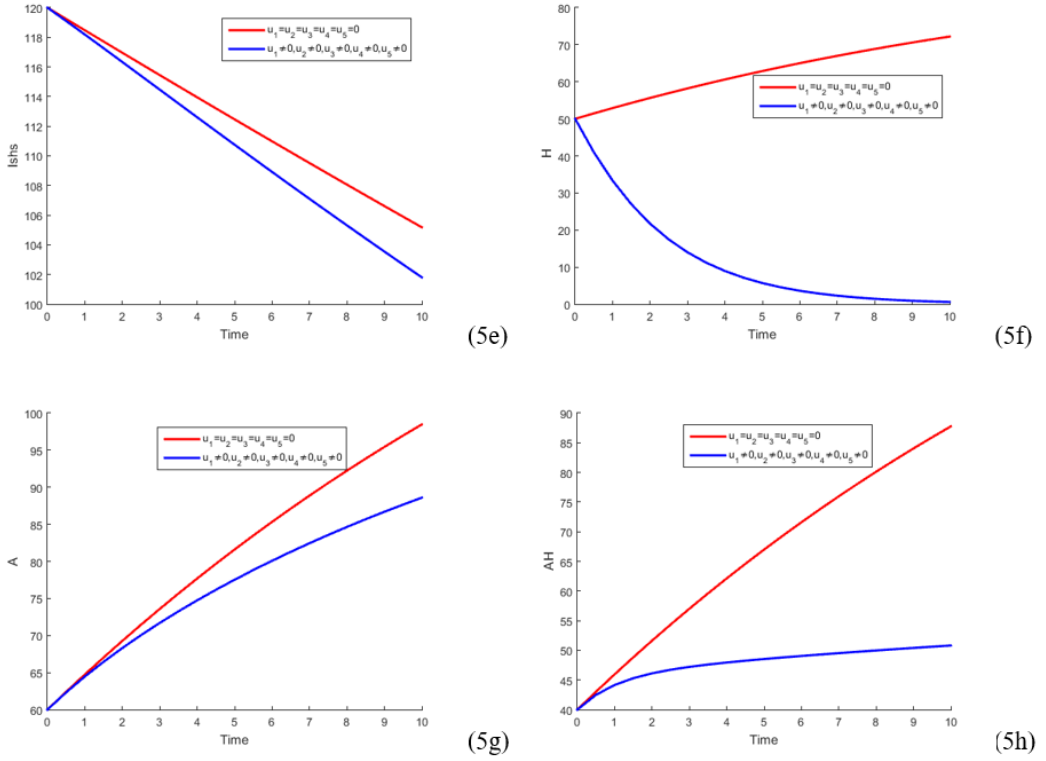
(5b)



(5c)



(5d)



(5e) (5f) (5g) (5h)

Figure 5: Simulations of using all controls

6. Conclusion

In this paper, an optimal control problem was formulated to study the effects of combining different control strategies on HIV-HSV-II coinfection model in [19]. In this study, we formulated an optimal control strategy that minimizes the cost for implementation of the controls while also minimizing the infectious individuals over the intervention interval.The existence of optimal controls and characterization was done using Pontryagin's Maximum Principle. The results shows that the size of infectious population are minimized by using different control strategies.

HIV-HSV-II coinfection remain a challenge especially in developing countries, but from results of this study we recommend that, the government should introduce education programmers on the importance of voluntary and routinely screening on HIV-HSV-II coinfection. In future work, we plan to extend the study by incorporating protected and treatment class to HIV- HSV-II transmission dynamics.

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