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Research Article

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Semi-analytical Method to Solve the Non-linear System of Equations to Model of Evolution for Smoking Habit in Spain

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Abstract

An epidemiological model of smoking habit is studied by using one of flexible and accurate semi-analytical methods. For this reason, the homotopy analysis transform method (HATM) is applied. Convergence theorem is studied and several $\hbar$-curves are demonstrated to show the convergence regions. Also, the optimal convergence regions are obtained by demonstrating the residual error functions versus $\hbar$. The numerical tables are presented to show the precision of method.

Keywords : Homotopy analysis method; Laplace transformation; Non-linear model of smoking habit.

1 Introduction

In last decades, many mathematical models have been presented to study the various phenommena such as the epidemiological model of computer viruses [32, 33, 40, 43, 45, 48], model of HIV infection for CD4$^+$T and CD8$^+$T cells [18, 31, 34, 35, 42, 56], model of malaria viruses transmission [55], model of migratory birds population [15] and other useful models.

Recently, number of killed people by tobacco consumption reported by World Health Organization (WHO). Every year, over five million people killed because they applied the tobacco consumption continually. It means that every six seconds, one human killed. Also, WHO informs that up to fifty percent of tobacco users will be died by a tobacco-related disease.

In this research, the model of smoking habit [19, 47, 53] is studied for constant population with equal birth and death rates in Spain. The presented model depends on four individuals, non-smokers who has never smoked, normal smokers who smoked less than 20 cigarettes per day, excessive smokers who smoked more than 20 cigarettes per day and ex-smokers who had smoked in the past which are shown by variables X, Y, S and B. The graphical form of this model is demonstrated in Fig. 1. Consider the following non-linear system

$$\frac{dx}{dt} = \mu - (d_0 + \mu)x(t) + d_0x^2(t)$$

$$+(d_f - \beta)x(t)(y(t) + s(t))$$

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$$
\begin{aligned}
&+\left(\frac{d_0+d_f}{2}\right)x(t)b(t),\\
\frac{dy}{dt} &= \beta x(t)(y(t)+s(t)) + \rho b(t) + \alpha s(t)\\
&-(\gamma+\lambda+\mu+d_f)y(t) + d_0 x(t)y(t)\\
&+d_f y(t)(y(t)+s(t))\\
&+\left(\frac{d_0+d_f}{2}\right)y(t)b(t),\\
\frac{ds}{dt} &= \gamma y(t) - (\alpha+\delta+\mu+d_f)s(t)\\
&+d_0 x(t)s(t) + d_f s(t)(y(t)+s(t))\\
&+\left(\frac{d_0+d_f}{2}\right)s(t)b(t),\\
\frac{db}{dt} &= \lambda y(t) + \delta s(t)\\
&-\left(\rho+\mu+\frac{d_0+d_f}{2}\right)b(t)\\
&+d_0 x(t)b(t) + d_f b(t)(y(t)+s(t))\\
&+\left(\frac{d_0+d_f}{2}\right)b^2(t)
\end{aligned}
\tag{1.1}
$$

with initial conditions

$$x(0)=0.5045,\ y(0)=0.2059,$$

$$s(0)=0.1559,\ b(0)=0.1337,$$

where are scaled by

$$x=\frac{X}{P},\ y=\frac{Y}{P},\ s=\frac{S}{P},\ b=\frac{B}{P},$$

which P explains the total population. List of parameters and their values are presented in Table 1. Since the constant population has been normalized to unity, we get

$$x(t)+y(t)+s(t)+b(t)=1.$$

Existence of solution and convergence theorems of presented model were studied in different cases [19, 47, 53]. We should note that in order to solve theses kinds of models the numerical methods can be applied. But they may raise the numerical instabilities, oscillations or false equilibrium states [24]. It means that the numerical solution may not correspond to the real solution of the original system of differential equations. Thus we are interesting to obtain a continuous solution using semi-analytical methods.

In last decades, several methods such as Adomian decomposition method [9, 20, 42], Homotopy analysis method [18, 19, 49, 54], Homotopy perturbation method [8, 30], Differential transform method [50, 51], Variational iteration method [16, 17, 32], collocation method [7, 44, 56] and many others [21, 31, 45, 47, 48, 53] have been applied for solving differential equations.

The HAM was introduced firstly by Liao [25, 26, 27, 28] and generalized by many authors to solve the mathematical and engineering problems [6, 10, 11, 12, 29, 36, 41]. Also, recently the stochastic arithmetic and the CESTAC method [13, 14, 37, 38, 39] was combined to the HAM for solving integral equations [36].

In this study, by using the HATM [11, 22, 23, 41] we calculate the approximate solution of model (1.1). Convergence theorem is illustrated to theoretical guarantee of presented method for solving Eq. (1.1). According to [11, 25, 26, 41, 52], the solution of HATM depends on the convergence parameter $\hbar$. Convergence regions can be found by plotting $\hbar$-curves. Also, the residual error functions are illustrated to show the performance of method. Furthermore, graphs of error functions are plotted and the obtained results are presented in some tables.

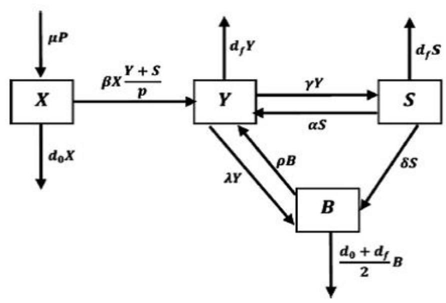

Figure 1: Diagram of smoking model.

Table 1: Parameters of model (1.1).

Parameters	Meaning	Values
μ	Natural birth rate	0.01
d_0	Natural death rate	0.0087
d_f	Rate of increased mortality by smoking	0.0132
β	Rate of transmission to accept smoking habit	0.0381
ρ	Rate of returning an ex-smoker to smoking	0.0425
α	Rate of transforming an excessive smoker to a normal smoker	0.1244
γ	Rate of transforming a normal smoker to an excessive smoker	0.1175
λ	Stopping rate for normal smokers	0.0498
δ	Stopping rate for excessive smokers	0.0498

Table 2: Residual errors of $x_5(t), y_5(t), s_5(t)$ and $b_5(t)$ for different values of $\hbar$.

function	t	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar=-1$	$\hbar=-0.9$
$x_5(t)$	0.0	3.2837×10^{-7}	1.0261×10^{-8}	1.7347×10^{-18}	1.0261×10^{-8}
	0.2	7.7643×10^{-7}	3.8607×10^{-8}	9.3574×10^{-14}	5.0180×10^{-9}
	0.4	1.3368×10^{-6}	8.0005×10^{-8}	2.9833×10^{-12}	1.4546×10^{-8}
	0.6	2.0231×10^{-6}	1.3733×10^{-7}	2.2572×10^{-11}	1.9468×10^{-8}
	0.8	2.8499×10^{-6}	2.1378×10^{-7}	9.4772×10^{-11}	2.0798×10^{-8}
	1.0	3.8325×10^{-6}	3.1290×10^{-7}	2.8818×10^{-10}	1.9425×10^{-8}
$y_5(t)$	0.0	1.6015×10^{-6}	5.0049×10^{-8}	3.4694×10^{-18}	5.0049×10^{-8}
	0.2	2.7421×10^{-6}	1.2446×10^{-7}	2.2034×10^{-12}	1.4192×10^{-8}
	0.4	4.2678×10^{-6}	2.4678×10^{-7}	7.05239×10^{-11}	6.1533×10^{-9}
	0.6	6.2628×10^{-6}	4.3592×10^{-7}	5.3565×10^{-10}	1.6015×10^{-8}
	0.8	8.8225×10^{-6}	7.1515×10^{-7}	2.2576×10^{-9}	1.9188×10^{-8}
	1.0	0.00001205	1.1125×10^{-6}	6.8912×10^{-9}	1.8385×10^{-8}
$s_5(t)$	0.0	1.6570×10^{-6}	5.1781×10^{-8}	0	5.1781×10^{-8}
	0.2	2.4365×10^{-6}	9.7323×10^{-8}	2.1217×10^{-12}	1.9413×10^{-8}
	0.4	3.2507×10^{-6}	1.4285×10^{-7}	6.7895×10^{-11}	5.9433×10^{-9}
	0.6	4.05921×10^{-6}	1.7827×10^{-7}	5.1557×10^{-10}	2.3033×10^{-8}
	0.8	4.8119×10^{-6}	1.8989×10^{-7}	2.1726×10^{-9}	3.1550×10^{-8}
	1.0	5.4493×10^{-6}	1.5999×10^{-7}	6.6302×10^{-9}	3.1988×10^{-8}
$b_5(t)$	0.0	3.5057×10^{-6}	1.0955×10^{-7}	8.6736×10^{-18}	1.0955×10^{-7}
	0.2	5.8402×10^{-6}	2.5571×10^{-7}	1.8010×10^{-13}	2.7173×10^{-8}
	0.4	8.6982×10^{-6}	4.6179×10^{-7}	5.7664×10^{-12}	2.7382×10^{-8}
	0.6	0.00001213	7.3928×10^{-7}	4.3808×10^{-11}	5.8930×10^{-8}
	0.8	0.00001621	1.1007×10^{-6}	1.8469×10^{-10}	7.1877×10^{-8}
	1.0	0.00002098	1.5596×10^{-6}	5.6389×10^{-10}	7.0235×10^{-8}

2 Solution of smoking habit model by HATM

HATM is an important and flexible technique to solve many problems [11, 22, 23, 41]. This method is obtained by combining the Laplace transformation $\mathcal{L}$ and the HAM. Let

$$L_x = L_y = L_s = L_b = \mathcal{L}, \qquad (2.2)$$

be the linear operators for functions $x(t), y(t), s(t), b(t)$. By applying Laplace transformation $\mathcal{L}$ for both sides of non-linear

Table 2. Continue

function	t	$\hbar=-0.8$	$\hbar=-0.7$	$\hbar=-0.6$
$x_5(t)$	0.0	3.2837×10^{-7}	2.4936×10^{-6}	0.000010508
	0.2	8.5847×10^{-8}	1.3802×10^{-6}	7.4379×10^{-6}
	0.4	1.1594×10^{-7}	3.7622×10^{-7}	4.5624×10^{-6}
	0.6	2.8059×10^{-7}	5.2401×10^{-7}	1.8749×10^{-6}
	0.8	4.1150×10^{-7}	1.3260×10^{-6}	6.3110×10^{-7}
	1.0	5.1189×10^{-7}	2.0352×10^{-6}	2.9620×10^{-6}
$y_5(t)$	0.0	1.6015×10^{-6}	0.00001216	0.000051250
	0.2	1.0182×10^{-6}	9.4600×10^{-6}	0.000043766
	0.4	5.5360×10^{-7}	7.0844×10^{-6}	0.000036867
	0.6	1.9052×10^{-7}	5.0076×10^{-6}	0.000030522
	0.8	8.6550×10^{-8}	3.2035×10^{-6}	0.000024701
	1.0	2.9145×10^{-7}	1.6477×10^{-6}	0.000019373
$s_5(t)$	0.0	1.6570×10^{-6}	0.00001258	0.000053024
	0.2	1.1729×10^{-6}	0.000010407	0.000047089
	0.4	7.2633×10^{-7}	8.3207×10^{-6}	0.000041303
	0.6	3.2324×10^{-7}	6.3343×10^{-6}	0.000035678
	0.8	3.1708×10^{-8}	4.4571×10^{-6}	0.000030227
	1.0	3.3521×10^{-7}	2.6969×10^{-6}	0.00002495
$b_5(t)$	0.0	3.5057×10^{-6}	0.000026621	0.00011218
	0.2	2.2137×10^{-6}	0.000020713	0.00009592
	0.4	1.1154×10^{-6}	0.000015321	0.000080574
	0.6	1.9638×10^{-7}	0.000010422	0.000066114
	0.8	5.5742×10^{-7}	5.9969×10^{-6}	0.000052516
	1.0	1.1594×10^{-6}	2.0218×10^{-6}	0.00003975

system of Eqs. (1.1) we get

$$\begin{aligned}\mathcal{L}[x(t)] &= \frac{x(0)}{z}+\frac{\mathcal{L}[\mu]}{z} -\frac{d_0+\mu}{z}\mathcal{L}[x(t)]+\frac{d_0}{z}\mathcal{L}[x^2(t)]\\ &+\frac{d_f-\beta}{z}\mathcal{L}[x(t)(y(t)+s(t))] +\frac{d_0+d_f}{2z}\mathcal{L}[x(t)b(t)],\end{aligned}$$

$$\begin{aligned}\mathcal{L}[s(t)] &= \frac{s(0)}{z}+\frac{\gamma}{z}\mathcal{L}[y(t)]\\ &-\frac{\alpha+\delta+\mu+d_f}{z}\mathcal{L}[s(t)] +\frac{d_0}{z}\mathcal{L}[x(t)s(t)]\\ &+\frac{d_f}{z}\mathcal{L}[s(t)(y(t)+s(t))]\\ &+\frac{d_0+d_f}{2z}\mathcal{L}[s(t)b(t)],\end{aligned}$$

$$\begin{aligned}\mathcal{L}[y(t)] &= \frac{y(0)}{z}+\frac{\beta}{z}\mathcal{L}[x(t)(y(t)+s(t))]\\ &+\frac{\rho}{z}\mathcal{L}[b(t)]+\frac{\alpha}{z}\mathcal{L}[s(t)]\\ &-\frac{\gamma+\lambda+\mu+d_f}{z}\mathcal{L}[y(t)]\\ &+\frac{d_0}{z}\mathcal{L}[x(t)y(t)]\\ &+\frac{d_f}{z}\mathcal{L}[y(t)(y(t)+s(t))]\\ &+\frac{d_0+d_f}{2z}\mathcal{L}[y(t)b(t)],\end{aligned}$$

$$\begin{aligned}\mathcal{L}[b(t)] &= \frac{b(0)}{z}+\frac{\lambda}{z}\mathcal{L}[y(t)]+\frac{\delta}{z}\mathcal{L}[s(t)]\\ &-(\rho+\mu+\frac{d_0+d_f}{2z})\mathcal{L}[b(t)]+\frac{d_0}{z}\mathcal{L}[x(t)b(t)]\\ &+\frac{d_f}{z}\mathcal{L}[b(t)(y(t)+s(t))]+\frac{d_0+d_f}{2z}\mathcal{L}[b^2(t)].\end{aligned} \tag{2.3}$$

According to the traditional homotopy [25, 26, 27, 28] we can define the homotopy maps as fol-

Table 3: Residual errors of $x_{10}(t), y_{10}(t), s_{10}(t)$ and $b_{10}(t)$ for different values of $\hbar$.

function	t	$\hbar=-1.3$	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar=-1$	$\hbar=-0.9$
$x_{10}(t)$	0.0	6.05948×10^{-9}	1.05081×10^{-10}	1.02482×10^{-13}	3.29597×10^{-17}	1.02562×10^{-13}
	0.2	1.8959×10^{-8}	4.38952×10^{-10}	8.46536×10^{-13}	3.64292×10^{-17}	1.35546×10^{-13}
	0.4	3.73288×10^{-8}	9.71934×10^{-10}	2.44133×10^{-12}	3.46945×10^{-17}	1.88521×10^{-13}
	0.6	6.25156×10^{-8}	1.77258×10^{-9}	5.43742×10^{-12}	5.0307×10^{-17}	1.47198×10^{-13}
	0.8	9.60972×10^{-8}	2.92591×10^{-9}	1.0632×10^{-11}	3.64292×10^{-17}	6.97324×10^{-14}
	1.0	1.39912×10^{-7}	4.53636×10^{-9}	1.91506×10^{-11}	4.33681×10^{-17}	8.93903×10^{-15}
$y_{10}(t)$	0.0	2.95536×10^{-8}	5.12505×10^{-10}	5.00847×10^{-13}	2.19443×10^{-16}	5.00569×10^{-13}
	0.2	6.32655×10^{-8}	1.40867×10^{-9}	2.68284×10^{-12}	2.18575×10^{-16}	4.04798×10^{-14}
	0.4	1.17498×10^{-7}	3.10584×10^{-9}	9.08249×10^{-12}	2.26381×10^{-16}	1.64365×10^{-13}
	0.6	2.01924×10^{-7}	6.14417×10^{-9}	2.53321×10^{-11}	2.43729×10^{-16}	1.3162×10^{-13}
	0.8	3.29821×10^{-7}	1.13429×10^{-8}	6.23648×10^{-11}	2.59341×10^{-16}	3.8268×10^{-14}
	1.0	5.19065×10^{-7}	1.9905×10^{-8}	1.39876×10^{-10}	7.92769×10^{-16}	7.78379×10^{-14}
$s_{10}(t)$	0.0	3.05766×10^{-8}	5.30245×10^{-10}	5.17583×10^{-13}	5.20417×10^{-17}	5.17763×10^{-13}
	0.2	5.15865×10^{-8}	1.03661×10^{-9}	1.37162×10^{-12}	5.11743×10^{-17}	3.90096×10^{-14}
	0.4	7.20628×10^{-8}	1.45678×10^{-9}	9.00647×10^{-13}	5.72459×10^{-17}	2.53903×10^{-13}
	0.6	8.63425×10^{-8}	1.44691×10^{-9}	5.12283×10^{-12}	6.93889×10^{-17}	2.02016×10^{-13}
	0.8	8.56199×10^{-8}	4.14266×10^{-10}	2.58793×10^{-11}	1.33574×10^{-16}	2.71727×10^{-14}
	1.0	5.69718×10^{-8}	2.58619×10^{-9}	7.89578×10^{-11}	6.9042×10^{-16}	1.48265×10^{-13}
$b_{10}(t)$	0.0	6.46909×10^{-8}	1.12184×10^{-9}	1.09457×10^{-12}	4.54498×10^{-16}	1.0952×10^{-12}
	0.2	1.3134×10^{-7}	2.83297×10^{-9}	4.81777×10^{-12}	4.96131×10^{-16}	2.23458×10^{-13}
	0.4	2.23017×10^{-7}	5.4424×10^{-9}	1.22195×10^{-11}	5.11743×10^{-16}	6.11568×10^{-13}
	0.6	3.44983×10^{-7}	9.21036×10^{-9}	2.52269×10^{-11}	4.38885×10^{-16}	4.8749×10^{-13}
	0.8	5.03186×10^{-7}	1.44436×10^{-8}	4.63741×10^{-11}	6.00214×10^{-16}	1.44849×10^{-13}
	1.0	7.04316×10^{-7}	2.15002×10^{-8}	7.89121×10^{-11}	6.245×10^{-16}	2.23536×10^{-13}

lows

$$
\begin{aligned}
&H_x[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)]\\
&=(1-q)L_x[\bar{x}(t;q)-x_0(t)]\\
&-q\hbar H_x(t)N_x[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)],\\
&H_y[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)]\\
&=(1-q)L_y[\bar{y}(t;q)-y_0(t)]\\
&-q\hbar H_y(t)N_y[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)],\\
&H_s[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)]\\
&=(1-q)L_s[\bar{s}(t;q)-s_0(t)]\\
&-q\hbar H_s(t)N_s[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)],\\
&H_b[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)]\\
&=(1-q)L_b[\bar{b}(t;q)-b_0(t)]\\
&-q\hbar H_b(t)N_b[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)],
\end{aligned}
\tag{2.4}
$$

where $0 \leq q \leq 1$ is an embedding parameter, L_x, L_y, L_s, L_b show the linear operators and $\hbar$ is a convergence control parameter which we apply to find the convergence regions. These regions are the parallel parts of $\hbar$-curves with axiom x. Choosing the proper value of $\hbar$, we will have the best approximate solution. Thus, the parameter $\hbar$ has the main role in the HATM. Also, $H_x(t), H_y(t), H_s(t), H_b(t)$ are the auxiliary functions that we are free to choose these func-

Table 3 Continue.

function	t	$\hbar = -0.8$	$\hbar = -0.7$	$\hbar = -0.6$	$\hbar = -0.5$
$x_{10}(t)$	0.0	1.05081×10^{-10}	6.05948×10^{-9}	1.07603×10^{-7}	1.00213×10^{-6}
	0.2	3.43332×10^{-11}	9.73683×10^{-10}	4.7172×10^{-8}	6.17855×10^{-7}
	0.4	1.23074×10^{-10}	3.01504×10^{-9}	4.79838×10^{-9}	2.69695×10^{-7}
	0.6	1.72486×10^{-10}	6.0521×10^{-9}	4.90393×10^{-8}	4.44668×10^{-8}
	0.8	1.92046×10^{-10}	8.26855×10^{-9}	8.62337×10^{-8}	3.26646×10^{-7}
	1.0	1.89608×10^{-10}	9.78224×10^{-9}	1.17019×10^{-7}	5.78766×10^{-7}
$y_{10}(t)$	0.0	5.12504×10^{-10}	2.95536×10^{-8}	5.24804×10^{-7}	4.88762×10^{-6}
	0.2	1.8456×10^{-10}	1.73887×10^{-8}	3.78919×10^{-7}	3.95456×10^{-6}
	0.4	8.60841×10^{-12}	8.32821×10^{-9}	2.57716×10^{-7}	3.12865×10^{-6}
	0.6	1.12964×10^{-10}	1.71807×10^{-9}	1.57768×10^{-7}	2.39984×10^{-6}
	0.8	1.59731×10^{-10}	2.97049×10^{-9}	7.60726×10^{-8}	1.75893×10^{-6}
	1.0	1.69497×10^{-10}	6.16112×10^{-9}	1.00072×10^{-8}	1.19747×10^{-6}
$s_{10}(t)$	0.0	5.30246×10^{-10}	3.05766×10^{-8}	5.42972×10^{-7}	5.05682×10^{-6}
	0.2	2.36091×10^{-10}	2.02829×10^{-8}	4.23371×10^{-7}	4.30687×10^{-6}
	0.4	6.34624×10^{-12}	1.11278×10^{-8}	3.11495×10^{-7}	3.5871×10^{-6}
	0.6	1.51286×10^{-10}	3.29463×10^{-9}	2.0848×10^{-7}	2.90111×10^{-6}
	0.8	2.39341×10^{-10}	3.12708×10^{-9}	1.15129×10^{-7}	2.25187×10^{-6}
	1.0	2.66757×10^{-10}	8.12291×10^{-9}	3.19594×10^{-8}	1.64172×10^{-6}
$b_{10}(t)$	0.0	1.12184×10^{-9}	6.46909×10^{-8}	1.14876×10^{-6}	0.0000106987
	0.2	3.7059×10^{-10}	3.75229×10^{-8}	8.27329×10^{-7}	8.65996×10^{-6}
	0.4	1.34754×10^{-10}	1.56074×10^{-8}	5.46079×10^{-7}	6.7917×10^{-6}
	0.6	4.42766×10^{-10}	1.65791×10^{-9}	3.02059×10^{-7}	5.08557×10^{-6}
	0.8	5.95818×10^{-10}	1.48314×10^{-8}	9.24511×10^{-8}	3.53349×10^{-6}
	1.0	6.30593×10^{-10}	2.44291×10^{-8}	8.54255×10^{-8}	2.12765×10^{-6}

tions and finally N_x, N_y, N_s, N_b demonstrate the non-linear operators which are defined as

$$
\begin{aligned}
&N_x[\bar{x}(t;q), \bar{y}(t;q), \bar{s}(t;q), \bar{b}(t;q)] \\
&= \tfrac{\partial \bar{x}(t;q)}{\partial t} - \mu + (d_0 + \mu)\bar{x}(t;q) \\
&-d_0\bar{x}^2(t;q) - (d_f - \beta)\bar{x}(t;q)(\bar{y}(t;q) \\
&+\bar{s}(t;q)) - \left(\tfrac{d_0+d_f}{2}\right)\bar{x}(t;q)\bar{b}(t;q), \\
&N_y[\bar{x}(t;q), \bar{y}(t;q), \bar{s}(t;q), \bar{b}(t;q)] \\
&= \tfrac{\partial \bar{y}(t;q)}{\partial t} - \beta\bar{x}(t;q)(\bar{y}(t;q) + \bar{s}(t;q)) \\
&-\rho\bar{b}(t;q) - \alpha\bar{s}(t;q) \\
&+(\gamma + \lambda + \mu + d_f)\bar{y}(t;q) - d_0\bar{x}(t;q)\bar{y}(t;q) \\
&-d_f\bar{y}(t;q)(\bar{y}(t;q) + \bar{s}(t;q)) \\
&- \left(\tfrac{d_0+d_f}{2}\right)\bar{y}(t;q)\bar{b}(t;q), \\
&N_s[\bar{x}(t;q), \bar{y}(t;q), \bar{s}(t;q), \bar{b}(t;q)] \\
&= \tfrac{\partial \bar{s}(t;q)}{\partial t} - \gamma\bar{y}(t;q) + (\alpha + \delta + \mu + d_f)\bar{s}(t;q) \\
&-d_0\bar{x}(t;q)\bar{s}(t;q) - d_f\bar{s}(t;q)(\bar{y}(t;q) + \bar{s}(t;q)) \\
&- \left(\tfrac{d_0+d_f}{2}\right)\bar{s}(t;q)\bar{b}(t;q), \\
&N_b[\bar{x}(t;q), \bar{y}(t;q), \bar{s}(t;q), \bar{b}(t;q)] \\
&= \tfrac{\partial \bar{b}(t;q)}{\partial t} - \lambda\bar{y}(t;q) - \delta\bar{s}(t;q) \\
&+ \left(\rho + \mu + \tfrac{d_0+d_f}{2}\right)\bar{b}(t;q) \\
&-d_0\bar{x}(t;q)\bar{b}(t;q) - d_f\bar{b}(t;q)(\bar{y}(t;q) + \bar{s}(t;q)) \\
&- \left(\tfrac{d_0+d_f}{2}\right)\bar{b}^2(t;q).
\end{aligned} \tag{2.5}
$$

Table 4: Residual errors of $x_{15}(t), y_{15}(t), s_{15}(t)$ and $b_{15}(t)$ for different values of $\hbar$.

function	t	$\hbar=-1.4$	$\hbar=-1.3$	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar=-1$
$x_{15}(t)$	0.0	1.10185×10^{-9}	1.4722×10^{-11}	3.69062×10^{-14}	4.04364×10^{-15}	9.35016×10^{-16}
	0.2	4.09311×10^{-9}	6.71925×10^{-11}	2.22357×10^{-13}	5.53724×10^{-15}	1.10155×10^{-15}
	0.4	8.71797×10^{-9}	1.55693×10^{-10}	5.8902×10^{-13}	6.87297×10^{-15}	1.33227×10^{-15}
	0.6	1.55193×10^{-8}	2.95221×10^{-10}	1.25499×10^{-12}	9.15934×10^{-15}	6.80012×10^{-16}
	0.8	2.51743×10^{-8}	5.05461×10^{-10}	2.34807×10^{-12}	1.07431×10^{-14}	1.51615×10^{-15}
	1.0	3.85209×10^{-8}	8.1187×10^{-10}	4.12658×10^{-12}	1.27381×10^{-14}	1.5838×10^{-15}
$y_{15}(t)$	0.0	5.37385×10^{-9}	7.1722×10^{-11}	1.41945×10^{-13}	2.14802×10^{-14}	9.04572×10^{-15}
	0.2	1.33431×10^{-8}	2.14663×10^{-10}	6.78569×10^{-13}	1.73594×10^{-14}	1.13052×10^{-14}
	0.4	2.78383×10^{-8}	5.10487×10^{-10}	2.09082×10^{-12}	2.01054×10^{-14}	1.17909×10^{-14}
	0.6	5.31889×10^{-8}	1.08914×10^{-9}	5.45508×10^{-12}	2.0651×10^{-14}	9.83848×10^{-15}
	0.8	9.6084×10^{-8}	2.16956×10^{-9}	1.29787×10^{-11}	1.42525×10^{-14}	1.37711×10^{-14}
	1.0	1.6658×10^{-7}	4.10592×10^{-9}	2.85675×10^{-11}	8.13759×10^{-15}	1.03823×10^{-14}
$s_{15}(t)$	0.0	5.56021×10^{-9}	7.43699×10^{-11}	2.17635×10^{-13}	2.57884×10^{-14}	8.02483×10^{-15}
	0.2	1.01937×10^{-8}	1.50765×10^{-10}	4.66823×10^{-13}	2.77755×10^{-14}	7.83835×10^{-15}
	0.4	1.41789×10^{-8}	2.02043×10^{-10}	4.54616×10^{-13}	3.63798×10^{-14}	7.9034×10^{-15}
	0.6	1.47372×10^{-8}	1.3955×10^{-10}	6.09461×10^{-13}	3.95135×10^{-14}	9.76996×10^{-15}
	0.8	6.9682×10^{-9}	2.06529×10^{-10}	4.57045×10^{-12}	3.60285×10^{-14}	9.35536×10^{-15}
	1.0	1.7148×10^{-8}	1.13374×10^{-9}	1.49774×10^{-11}	2.81376×10^{-14}	1.03723×10^{-14}
$b_{15}(t)$	0.0	1.17635×10^{-8}	1.57206×10^{-10}	4.03563×10^{-13}	3.40804×10^{-14}	5.65867×10^{-15}
	0.2	2.71312×10^{-8}	4.24994×10^{-10}	1.33798×10^{-12}	2.67234×10^{-14}	3.00801×10^{-15}
	0.4	4.98825×10^{-8}	8.53711×10^{-10}	3.04097×10^{-12}	1.9252×10^{-14}	4.60222×10^{-15}
	0.6	8.20743×10^{-8}	1.49893×10^{-9}	5.9432×10^{-12}	2.52558×10^{-14}	5.98306×10^{-15}
	0.8	1.26137×10^{-7}	2.42823×10^{-9}	1.0496×10^{-11}	2.13406×10^{-14}	7.95024×10^{-15}
	1.0	1.84915×10^{-7}	3.7237×10^{-9}	1.73877×10^{-11}	2.44683×10^{-14}	1.07431×10^{-14}

When the homotopy maps (2.4) are equal to zero, the following deformation equations can be constructed as

$$
\begin{aligned}
&(1-q)L_x[\bar{x}(t;q)-x_0(t)]\\
&-q\hbar H_x(t)N_x[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)]=0,\\
&(1-q)L_y[\bar{y}(t;q)-y_0(t)]\\
&-q\hbar H_y(t)N_y[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)]=0,\\
&(1-q)L_s[\bar{s}(t;q)-s_0(t)]\\
&-q\hbar H_s(t)N_s[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)]=0,\\
&(1-q)L_b[\bar{b}(t;q)-b_0(t)]\\
&-q\hbar H_b(t)N_b[\bar{x}(t;q),\bar{y}(t;q),\bar{s}(t;q),\bar{b}(t;q)]=0,
\end{aligned}
\tag{2.6}
$$

which are called the zero order deformation equations. We know that by changing q from 0 to 1 the HATM can be leaded to the exact solution from the initial functions $x_0(t), y_0(t), s_0(t), b_0(t)$.

The Taylor series are constructed with respect to q as

$$
\begin{aligned}
\bar{x}(t;q)&=x_0(t)+\textstyle\sum_{m=1}^{\infty}x_m(t)q^m,\\
\bar{y}(t;q)&=y_0(t)+\textstyle\sum_{m=1}^{\infty}y_m(t)q^m,\\
\bar{s}(t;q)&=s_0(t)+\textstyle\sum_{m=1}^{\infty}s_m(t)q^m,\\
\bar{b}(t;q)&=b_0(t)+\textstyle\sum_{m=1}^{\infty}b_m(t)q^m,
\end{aligned}
\tag{2.7}
$$

where

$$
x_m=\frac{1}{m!}\frac{\partial^m\bar{x}(t;q)}{\partial q^m}\Big|_{q=0},\quad y_m=\frac{1}{m!}\frac{\partial^m\bar{y}(t;q)}{\partial q^m}\Big|_{q=0},
$$

$$
s_m=\frac{1}{m!}\frac{\partial^m\bar{s}(t;q)}{\partial q^m}\Big|_{q=0},\quad b_m=\frac{1}{m!}\frac{\partial^m\bar{b}(t;q)}{\partial q^m}\Big|_{q=0}.
$$

It is important that by choosing the suitable value of convergence control parameter $\hbar$, func-

Table 4 Continue.

function	t	$\hbar=-0.9$	$\hbar=-0.8$	$\hbar=-0.7$	$\hbar=-0.6$	$\hbar=-0.5$
$x_{15}(t)$	0.0	2.68882×10^{-16}	3.34819×10^{-14}	1.47243×10^{-11}	1.10185×10^{-9}	3.13165×10^{-8}
	0.2	2.8276×10^{-16}	2.65152×10^{-14}	2.69185×10^{-12}	2.09883×10^{-10}	1.37738×10^{-8}
	0.4	2.39392×10^{-16}	5.38198×10^{-14}	1.43934×10^{-11}	4.90785×10^{-10}	1.23345×10^{-9}
	0.6	1.99493×10^{-16}	5.98549×10^{-14}	2.15801×10^{-11}	1.0264×10^{-9}	1.39408×10^{-8}
	0.8	2.498×10^{-16}	5.31363×10^{-14}	2.52575×10^{-11}	1.42039×10^{-9}	2.45661×10^{-8}
	1.0	4.38885×10^{-16}	3.95517×10^{-14}	2.62632×10^{-11}	1.69365×10^{-9}	3.33113×10^{-8}
$y_{15}(t)$	0.0	4.60482×10^{-15}	1.65809×10^{-13}	7.18153×10^{-11}	5.374×10^{-9}	1.52738×10^{-7}
	0.2	5.22325×10^{-15}	2.73826×10^{-14}	3.06816×10^{-11}	3.23998×10^{-9}	1.10411×10^{-7}
	0.4	5.83214×10^{-15}	3.08677×10^{-14}	4.90109×10^{-12}	1.64565×10^{-9}	7.54487×10^{-8}
	0.6	6.59368×10^{-15}	4.82574×10^{-14}	1.04135×10^{-11}	4.74479×10^{-10}	4.67596×10^{-8}
	0.8	7.61457×10^{-15}	4.40264×10^{-14}	1.86377×10^{-11}	3.66116×10^{-10}	2.34031×10^{-8}
	1.0	9.20878×10^{-15}	2.90939×10^{-14}	2.20654×10^{-11}	9.49174×10^{-10}	4.57242×10^{-9}
$s_{15}(t)$	0.0	1.80758×10^{-15}	1.69611×10^{-13}	7.43005×10^{-11}	5.56003×10^{-9}	1.58026×10^{-7}
	0.2	1.8657×10^{-15}	3.713×10^{-14}	3.7912×10^{-11}	3.75541×10^{-9}	1.2326×10^{-7}
	0.4	1.93595×10^{-15}	4.47498×10^{-14}	8.57709×10^{-12}	2.15149×10^{-9}	9.08523×10^{-8}
	0.6	2.31846×10^{-15}	7.69402×10^{-14}	1.28533×10^{-11}	7.79468×10^{-10}	6.11526×10^{-8}
	0.8	2.09468×10^{-15}	7.27482×10^{-14}	2.65697×10^{-11}	3.47164×10^{-10}	3.43927×10^{-8}
	1.0	1.22732×10^{-15}	4.63189×10^{-14}	3.3405×10^{-11}	1.22848×10^{-9}	1.07057×10^{-8}
$b_{15}(t)$	0.0	4.99947×10^{-15}	3.62047×10^{-13}	1.57199×10^{-10}	1.17633×10^{-8}	3.34334×10^{-7}
	0.2	5.37764×10^{-15}	3.54508×10^{-14}	6.35448×10^{-11}	6.99873×10^{-9}	2.40999×10^{-7}
	0.4	7.41074×10^{-15}	1.29749×10^{-13}	2.43559×10^{-12}	3.15059×10^{-9}	1.59719×10^{-7}
	0.6	9.48373×10^{-15}	1.84682×10^{-13}	4.58686×10^{-11}	1.09763×10^{-10}	8.9537×10^{-8}
	0.8	1.19332×10^{-14}	1.69481×10^{-13}	7.12344×10^{-11}	2.22411×10^{-9}	2.95472×10^{-8}
	1.0	1.36315×10^{-14}	1.14127×10^{-13}	8.24228×10^{-11}	3.9431×10^{-9}	2.11066×10^{-8}

tions $H_x(t), H_y(t), H_s(t), H_b(t)$ and linear operators L_x, L_y, L_s, L_b, the Taylor series (2.7) will be convergent to the exact solution. For more analysis, we define the vectors

$$\bar{x}_m(t)=\Big\{x_0(t),x_1(t),\ldots,x_m(t)\Big\},$$

$$\bar{y}_m(t)=\Big\{y_0(t),y_1(t),\ldots,y_m(t)\Big\},$$

$$\bar{s}_m(t)=\Big\{s_0(t),s_1(t),\ldots,s_m(t)\Big\},$$

$$\bar{b}_m(t)=\Big\{b_0(t),b_1(t),\ldots,b_m(t)\Big\}.$$

Now, the following m-th order deformation equations

$$\begin{aligned}
&L_x\left[x_m(t)-\chi_m x_{m-1}(t)\right]\\
&=\hbar H_x(t)\Re_m^x\left(\vec{x}_{m-1},\vec{y}_{m-1},\vec{s}_{m-1},\vec{b}_{m-1}\right),\\
&L_y\left[y_m(t)-\chi_m y_{m-1}(t)\right]\\
&=\hbar H_y(t)\Re_m^y\left(\vec{x}_{m-1},\vec{y}_{m-1},\vec{s}_{m-1},\vec{b}_{m-1}\right),\\
&L_s\left[s_m(t)-\chi_m s_{m-1}(t)\right]\\
&=\hbar H_s(t)\Re_m^s\left(\vec{x}_{m-1},\vec{y}_{m-1},\vec{s}_{m-1},\vec{b}_{m-1}\right),\\
&L_b\left[b_m(t)-\chi_m b_{m-1}(t)\right]\\
&=\hbar H_b(t)\Re_m^b\left(\vec{x}_{m-1},\vec{y}_{m-1},\vec{s}_{m-1},\vec{b}_{m-1}\right),
\end{aligned} \tag{2.8}$$

can be obtained by differentiating Eqs. (2.6) with respect to q, dividing by $m!$ and putting $q=0$

where $\Re^x_m, \Re^y_m, \Re^s_m, \Re^b_m$ are defined as

$$\begin{aligned}
&\Re^x_m = \mathcal{L}[x_{m-1}(t)] - \tfrac{x_{m-1}(0)}{z} - (1-\chi_m)\tfrac{\mathcal{L}[\mu]}{z} \\
&+\tfrac{d_0+\mu}{z}\mathcal{L}[x_{m-1}(t)] - \tfrac{d_0}{z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} x_i(t)x_{m-1-i}(t)\right] \\
&-\tfrac{d_f-\beta}{z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} x_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t))\right] \\
&-\tfrac{d_0+d_f}{2z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} x_i(t)b_{m-1-i}(t)\right], \\
&\Re^y_m = \mathcal{L}[y_{m-1}(t)] - \tfrac{y_{m-1}(0)}{z} \\
&-\tfrac{\beta}{z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} x_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t))\right] \\
&-\tfrac{\rho}{z}\mathcal{L}[b_{m-1}(t)] - \tfrac{\alpha}{z}\mathcal{L}[s_{m-1}(t)] \\
&+\tfrac{\gamma+\lambda+\mu+d_f}{z}\mathcal{L}[y_{m-1}(t)] - \tfrac{d_0}{z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} x_i(t)y_{m-1-i}(t)\right] \\
&-\tfrac{d_f}{z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} y_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t))\right] \\
&-\tfrac{d_0+d_f}{2z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} y_i(t)b_{m-1-i}(t)\right], \\
&\Re^s_m = \mathcal{L}[y_{m-1}(t)] - \tfrac{y_{m-1}(0)}{z} \\
&-\tfrac{\gamma}{z}\mathcal{L}[y_{m-1}(t)] + \tfrac{\alpha+\delta+\mu+d_f}{z}\mathcal{L}[s_{m-1}(t)] \\
&-\tfrac{d_0}{z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} x_i(t)s_{m-1-i}(t)\right] \\
&-\tfrac{d_f}{z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} s_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t))\right] \\
&-\tfrac{d_0+d_f}{2z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} s_i(t)b_{m-1-i}(t)\right], \\
&\Re^b_m = \mathcal{L}[b_{m-1}(t)] - \tfrac{b_{m-1}(0)}{z} - \tfrac{\lambda}{z}\mathcal{L}[y_{m-1}(t)] \\
&-\tfrac{\delta}{z}\mathcal{L}[s_{m-1}(t)] + \tfrac{1}{z}(\rho+\mu+\tfrac{d_0+d_f}{2})\mathcal{L}[b_{m-1}(t)] \\
&-\tfrac{d_0}{z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} x_i(t)b_{m-1-i}(t)\right] \\
&-\tfrac{d_f}{z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} b_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t))\right] \\
&-\tfrac{d_0+d_f}{2z}\mathcal{L}\left[\textstyle\sum_{i=0}^{m-1} b_i(t)b_{m-1-i}(t)\right],
\end{aligned} \tag{2.9}$$

and parameter χ_m is presented as

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1. \end{cases} \tag{2.10}$$

By applying the inverse Laplace transformation $\mathcal{L}^{-1}$ for both sides of Eqs. (2.8) and putting $H_x(t) = H_y(t) = H_s(t) = H_b(t) = 1$ we get the following traditional equations as

$$\begin{aligned}
x_m &= \chi_m x_{m-1} + \hbar\mathcal{L}^{-1}\left[\Re^x_m\right], \\
y_m &= \chi_m y_{m-1} + \hbar\mathcal{L}^{-1}\left[\Re^y_m\right], \\
s_m &= \chi_m s_{m-1} + \hbar\mathcal{L}^{-1}\left[\Re^s_m\right], \\
b_m &= \chi_m b_{m-1} + \hbar\mathcal{L}^{-1}\left[\Re^b_m\right].
\end{aligned} \tag{2.11}$$

Finally, the N-th order approximate solutions can be obtained by using the following relations

$$\begin{aligned}
x_N(t) &= \textstyle\sum_{j=0}^{N} x_j(t), \\
y_N(t) &= \textstyle\sum_{j=0}^{N} y_j(t), \\
s_N(t) &= \textstyle\sum_{j=0}^{N} s_j(t), \\
b_N(t) &= \textstyle\sum_{j=0}^{N} b_j(t).
\end{aligned} \tag{2.12}$$

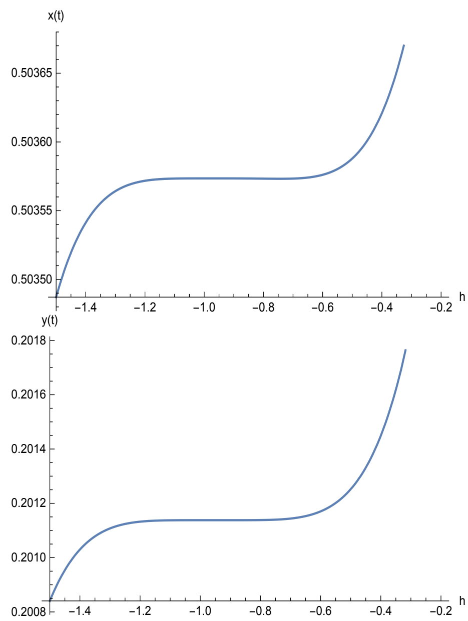

Figure 2: The $\hbar$-curves for $N = 5, t = 1$.

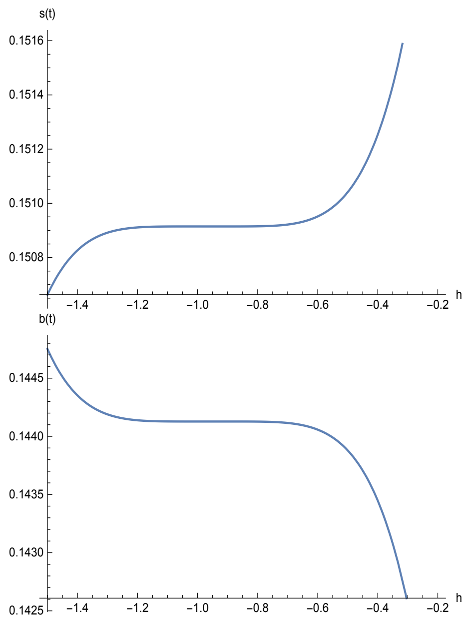

Figure 2, Continue.

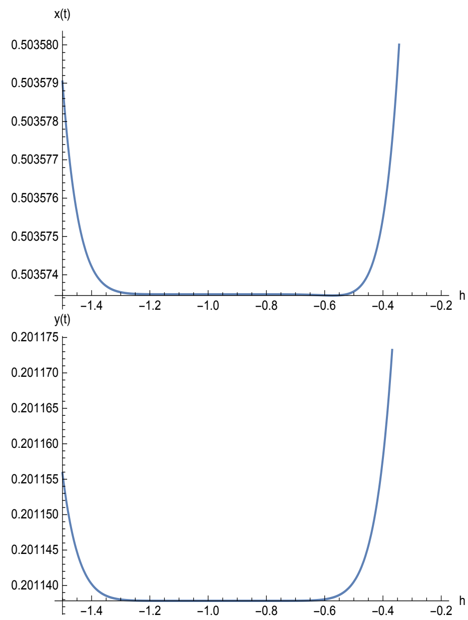

Figure 3: The $\hbar$-curves for $N = 10, t = 1$.

3 Convergence Theorem

In order to show the convergence of presented method the following theorem is presented. According to this theorem when $N \to \infty$, the HATM leads to the exact solution of non-linear problem (1.1).

Theorem 3.1 *As long as series solutions (2.12) are convergent where $x_j(t), y_j(t), s_j(t), b_j(t)$ are produced by the high order deformation equations (2.8) under definitions (2.9), they must be the exact solutions of non-linear system (1.1).*

Proof. If the series solutions

$$\begin{aligned} P_1(t) &= \textstyle\sum_{m=0}^{\infty} x_m(t), \\ P_2(t) &= \textstyle\sum_{m=0}^{\infty} y_m(t), \\ P_3(t) &= \textstyle\sum_{m=0}^{\infty} s_m(t), \\ P_4(t) &= \textstyle\sum_{m=0}^{\infty} b_m(t), \end{aligned} \tag{3.13}$$

are convergent then

$$\begin{aligned} \lim_{m\to\infty} x_m(t) &= 0, \\ \lim_{m\to\infty} y_m(t) &= 0, \\ \lim_{m\to\infty} s_m(t) &= 0, \\ \lim_{m\to\infty} b_m(t) &= 0. \end{aligned} \tag{3.14}$$

So, we can write

$$\begin{aligned} \textstyle\sum_{m=1}^{N} \Big[x_m(t) - \chi_m x_{m-1}(t)\Big] &= x_N(t), \\ \textstyle\sum_{m=1}^{N} \Big[y_m(t) - \chi_m y_{m-1}(t)\Big] &= y_N(t), \\ \textstyle\sum_{m=1}^{N} \Big[s_m(t) - \chi_m s_{m-1}(t)\Big] &= s_N(t), \\ \textstyle\sum_{m=1}^{N} \Big[b_m(t) - \chi_m b_{m-1}(t)\Big] &= b_N(t), \end{aligned} \tag{3.15}$$

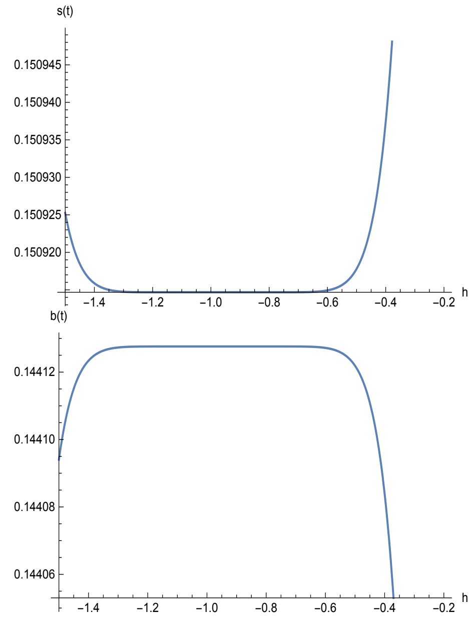

Figure 3, Continue.

that by Eqs. (3.14) and (3.15) we have

$$
\begin{aligned}
&\sum_{m=1}^{N}\Big[x_m(t)-\chi_m x_{m-1}(t)\Big]\\
&=\lim_{N\to\infty} x_N(t)=0,\\
&\sum_{m=1}^{N}\Big[y_m(t)-\chi_m y_{m-1}(t)\Big]\\
&=\lim_{N\to\infty} y_N(t)=0,\\
&\sum_{m=1}^{N}\Big[s_m(t)-\chi_m s_{m-1}(t)\Big]\\
&=\lim_{N\to\infty} s_N(t)=0,\\
&\sum_{m=1}^{N}\Big[b_m(t)-\chi_m b_{m-1}(t)\Big]\\
&=\lim_{N\to\infty} b_N(t)=0.
\end{aligned}
\tag{3.16}
$$

By applying the linear operators L_x, L_y, L_s and L_b for Eqs. (3.16), the following relations can be written as

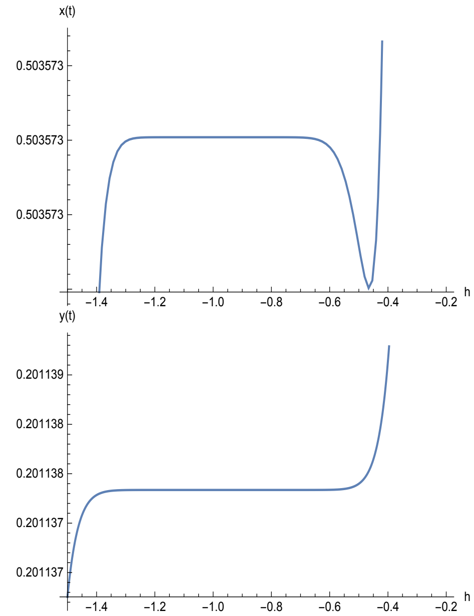

Figure 4: The $\hbar$-curves for $N = 15, t = 1$.

$$
\begin{aligned}
&\sum_{m=1}^{\infty} L_x\Big[x_m(t)-\chi_m x_{m-1}(t)\Big]\\
&=L_x\Big[\sum_{m=1}^{\infty} x_m(t)-\chi_m x_{m-1}(t)\Big]=0,\\
&\sum_{m=1}^{\infty} L_y\Big[y_m(t)-\chi_m y_{m-1}(t)\Big]\\
&=L_y\Big[\sum_{m=1}^{\infty} y_m(t)-\chi_m y_{m-1}(t)\Big]=0,\\
&\sum_{m=1}^{\infty} L_s\Big[s_m(t)-\chi_m s_{m-1}(t)\Big]\\
&=L_s\Big[\sum_{m=1}^{\infty} s_m(t)-\chi_m s_{m-1}(t)\Big]=0,\\
&\sum_{m=1}^{\infty} L_b\Big[b_m(t)-\chi_m b_{m-1}(t)\Big]\\
&=L_b\Big[\sum_{m=1}^{\infty} b_m(t)-\chi_m b_{m-1}(t)\Big]=0.
\end{aligned}
\tag{3.17}
$$

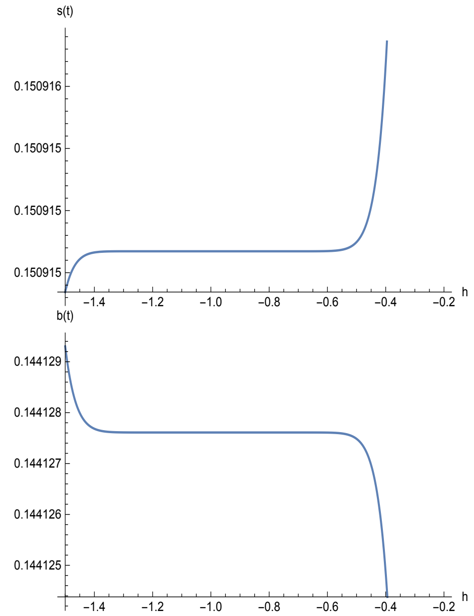

Figure 4, Continue.

Thus the right hand side of m-th order deformation Eqs. (2.8) are equaled to zero as follows

$$\hbar H_x(t) \sum_{m=1}^{\infty} \Re_m^x(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{s}_{m-1}, \vec{b}_{m-1}) = 0,$$

$$\hbar H_y(t) \sum_{m=1}^{\infty} \Re_m^y(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{s}_{m-1}, \vec{b}_{m-1}) = 0,$$

$$\hbar H_s(t) \sum_{m=1}^{\infty} \Re_m^s(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{s}_{m-1}, \vec{b}_{m-1}) = 0,$$

$$\hbar H_b(t) \sum_{m=1}^{\infty} \Re_m^b(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{s}_{m-1}, \vec{b}_{m-1}) = 0. \tag{3.18}$$

But based on the assumptions of HATM, in Eqs. (3.18) we get $\hbar, H_x(t), H_y(t), H_s(t), H_b(t) \neq 0$, thus

$$\sum_{m=1}^{\infty} \Re_m^x(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{s}_{m-1}, \vec{b}_{m-1}) = 0,$$

$$\sum_{m=1}^{\infty} \Re_m^y(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{s}_{m-1}, \vec{b}_{m-1}) = 0,$$

$$\sum_{m=1}^{\infty} \Re_m^s(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{s}_{m-1}, \vec{b}_{m-1}) = 0,$$

$$\sum_{m=1}^{\infty} \Re_m^b(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{s}_{m-1}, \vec{b}_{m-1}) = 0. \tag{3.19}$$

By putting $\Re_m^x, \Re_m^y, \Re_m^s$ and $\Re_m^b$ into Eqs. (3.19) and assuming $(.)' = \frac{d}{dt}$ we get,

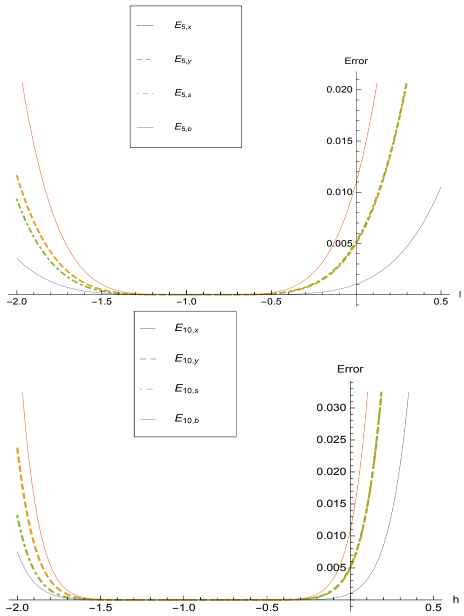

Figure 5: Averaged residual errors versus $\hbar$ for $N = 5, 10, 15$ and $t = 1$.

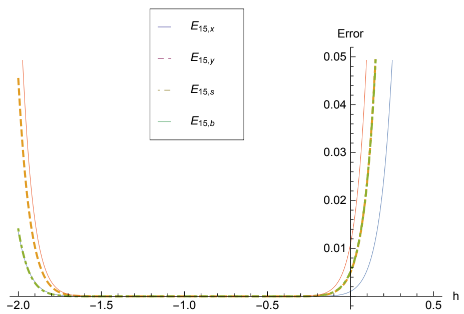

Figure 5, Continue.

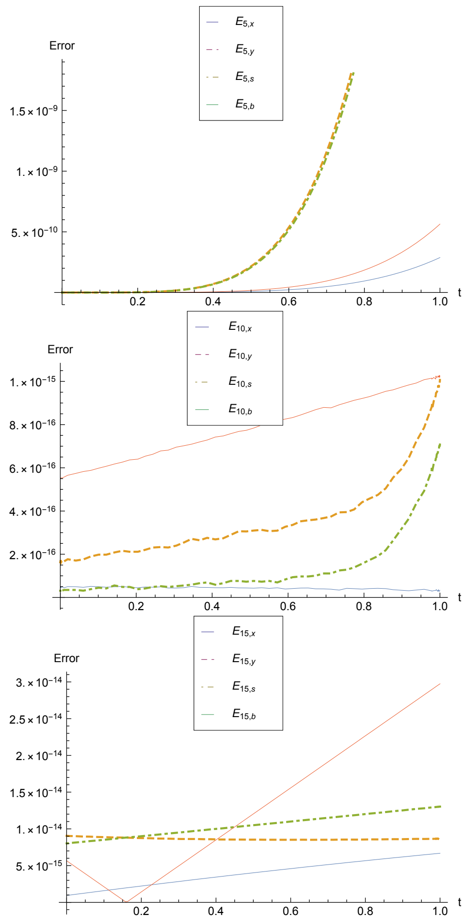

Figure 6: Residual errors for $\hbar = -1$ and $N = 5, 10, 15$.

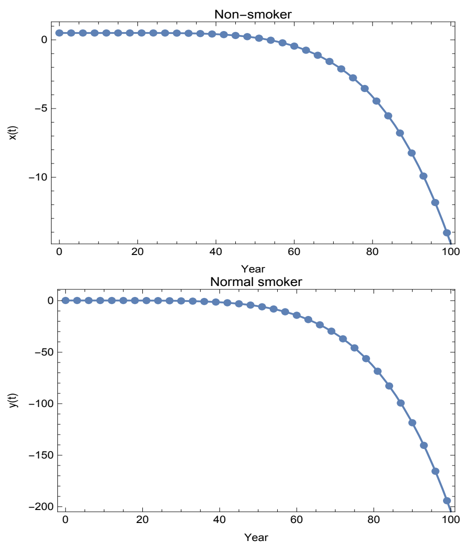

Figure 7: Plot of numerical solutions of $x_5(t), y_5(t), s_5(t), b_5(t)$ for $\hbar = -1$.

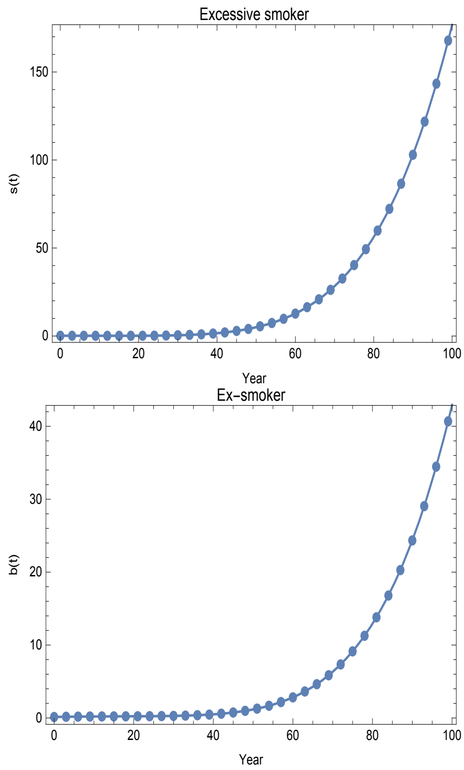

Figure 7, Continue.

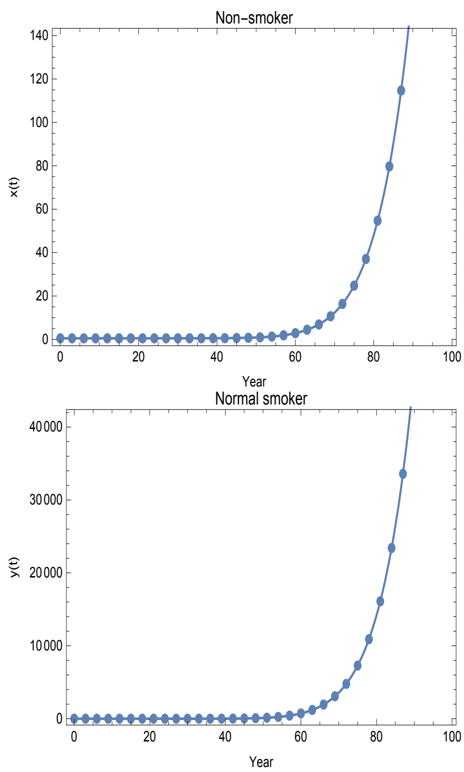

Figure 8: Plot of numerical solutions of $x_{10}(t), y_{10}(t), s_{10}(t), b_{10}(t)$ for $\hbar = -1$.

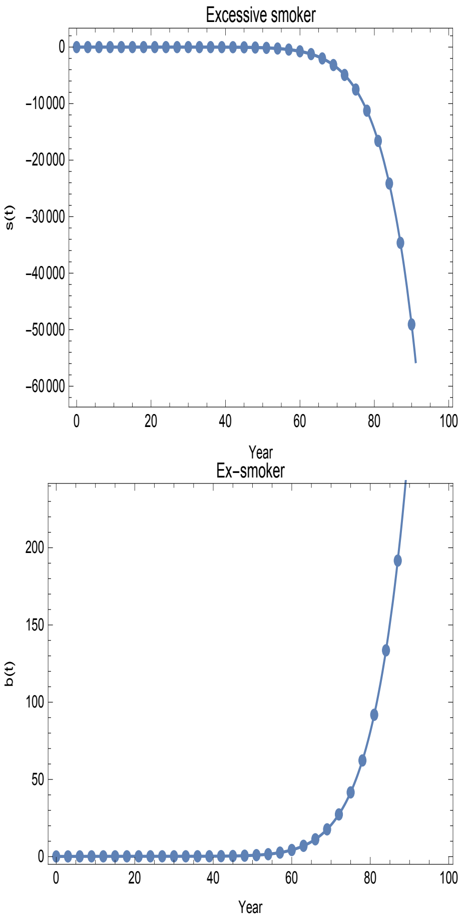

Figure 8, Continue.

$$
\begin{aligned}
\sum_{m=1}^{\infty} \Re_m^I &= \sum_{m=1}^{\infty} \Bigg[x'_{m-1}(t) - (1-\chi_m)\mu + (d_0+\mu)x_{m-1}(t) - d_0 \sum_{i=0}^{m-1} x_i(t)x_{m-1-i}(t) \\
&\quad - (d_f-\beta)\sum_{i=0}^{m-1} x_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t)) - (\frac{d_0+d_f}{2})\sum_{i=0}^{m-1} x_i(t)b_{m-1-i}(t) \Bigg] \\
&= \sum_{m=0}^{\infty} x'_m(t) - \mu + (d_0+\mu)\sum_{m=0}^{\infty} x_m(t) - d_0 \sum_{m=1}^{\infty}\sum_{i=0}^{m-1} x_i(t)x_{m-1-i}(t) \\
&\quad - (d_f-\beta)\sum_{m=1}^{\infty}\sum_{i=0}^{m-1} x_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t)) - (\frac{d_0+d_f}{2})\sum_{m=1}^{\infty}\sum_{i=0}^{m-1} x_i(t)b_{m-1-i}(t) \\
&= \sum_{m=0}^{\infty} x'_m(t) - \mu + (d_0+\mu)\sum_{m=0}^{\infty} x_m(t) - d_0 \sum_{i=0}^{\infty}\sum_{m=i+1}^{\infty} x_i(t)x_{m-1-i}(t) \\
&\quad - (d_f-\beta)\sum_{i=0}^{\infty}\sum_{m=i+1}^{\infty} x_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t)) - (\frac{d_0+d_f}{2})\sum_{i=0}^{\infty}\sum_{m=i+1}^{\infty} x_i(t)b_{m-1-i}(t) \\
&= \sum_{m=0}^{\infty} x'_m(t) - \mu + (d_0+\mu)\sum_{m=0}^{\infty} x_m(t) - d_0 \sum_{i=0}^{\infty} x_i(t)\sum_{m=0}^{\infty} x_m(t) \\
&\quad - (d_f-\beta)\sum_{i=0}^{\infty} x_i(t)\sum_{m=0}^{\infty}(y_m(t)+s_m(t)) - (\frac{d_0+d_f}{2})\sum_{i=0}^{\infty} x_i(t)\sum_{m=0}^{\infty} b_m(t) \\
&= P'_1(t) - \mu + (d_0+\mu)P_1(t) - d_0P_1^2(t) - (d_f-\beta)P_1(t)(P_2(t)+P_3(t)) - (\frac{d_0+d_f}{2})P_1(t)P_4(t),
\end{aligned} \tag{3.20}
$$

and

$$
\begin{aligned}
\sum_{m=1}^{\infty} \Re_m^y &= \sum_{m=1}^{\infty} \Bigg[y'_{m-1} - \beta \sum_{i=0}^{m-1} x_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t)) - \rho b_{m-1}(t) - \alpha s_{m-1}(t) \\
&\quad + (\gamma+\lambda+\mu+d_f)y_{m-1}(t) - d_0 \sum_{i=0}^{m-1} x_i(t)y_{m-1-i}(t) \\
&\quad - d_f \sum_{i=0}^{m-1} y_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t)) - (\frac{d_0+d_f}{2})\sum_{i=0}^{m-1} y_i(t)b_{m-1-i}(t) \Bigg]
\end{aligned} \tag{3.21}
$$

$$
= \sum_{m=0}^{\infty} y'_m - \beta \sum_{m=1}^{\infty}\sum_{i=0}^{m-1} x_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t)) - \rho \sum_{m=0}^{\infty} b_m(t) - \alpha \sum_{m=0}^{\infty} s_m(t)
$$

$$+ (\gamma+\lambda+\mu+d_f)\sum_{m=0}^{\infty} y_m(t) - d_0\sum_{m=1}^{\infty}\sum_{i=0}^{m-1} x_i(t)y_{m-1-i}(t)$$

$$- d_f\sum_{m=1}^{\infty}\sum_{i=0}^{m-1} y_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t)) - (\frac{d_0+d_f}{2})\sum_{m=1}^{\infty}\sum_{i=0}^{m-1} y_i(t)b_{m-1-i}(t)$$

$$= \sum_{m=0}^{\infty} y'_m - \beta\sum_{i=0}^{\infty}\sum_{m=i+1}^{\infty} x_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t)) - \rho\sum_{m=0}^{\infty} b_m(t) - \alpha\sum_{m=0}^{\infty} s_m(t)$$

$$+ (\gamma+\lambda+\mu+d_f)\sum_{m=0}^{\infty} y_m(t) - d_0\sum_{i=0}^{\infty}\sum_{m=i+1}^{\infty} x_i(t)y_{m-1-i}(t)$$

$$- d_f\sum_{i=0}^{\infty}\sum_{m=i+1}^{\infty} y_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t)) - (\frac{d_0+d_f}{2})\sum_{i=0}^{\infty}\sum_{m=i+1}^{\infty} y_i(t)b_{m-1-i}(t) \tag{3.22}$$

$$= \sum_{m=0}^{\infty} y'_m - \beta\sum_{i=0}^{\infty} x_i(t)\sum_{m=0}^{\infty}(y_m(t)+s_m(t)) - \rho\sum_{m=0}^{\infty} b_m(t) - \alpha\sum_{m=0}^{\infty} s_m(t)$$

$$+ (\gamma+\lambda+\mu+d_f)\sum_{m=0}^{\infty} y_m(t) - d_0\sum_{i=0}^{\infty} x_i(t)\sum_{m=0}^{\infty} y_m(t)$$

$$- d_f\sum_{i=0}^{\infty} y_i(t)\sum_{m=0}^{\infty}(y_m(t)+s_m(t)) - (\frac{d_0+d_f}{2})\sum_{i=0}^{\infty} y_i(t)\sum_{m=0}^{\infty} b_m(t)$$

$$= P'_2(t) - \beta P_1(t)(P_2(t)+P_3(t)) - \rho P_4(t) - \alpha P_3(t) + (\gamma+\lambda+\mu+d_f)P_2(t)$$

$$- d_0P_1(t)P_2(t) - d_fP_2(t)(P_2(t)+P_3(t)) - (\frac{d_0+d_f}{2})P_2(t)P_4(t),$$

and

$$\sum_{m=1}^{\infty}\Re_m^s = \sum_{m=1}^{\infty}\Big[s'_{m-1}(t) - \gamma y_{m-1}(t) + (\alpha+\delta+\mu+d_f)s_{m-1}(t) - d_0\sum_{i=0}^{m-1} x_i(t)s_{m-1-i}(t)$$

$$- d_f\sum_{i=0}^{m-1} s_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t)) - (\frac{d_0+d_f}{2})\sum_{i=0}^{m-1} s_i(t)b_{m-1-i}(t)\Big] \tag{3.23}$$

$$= \sum_{m=0}^{\infty} s'_m(t) - \gamma\sum_{m=0}^{\infty} y_m(t) + (\alpha+\delta+\mu+d_f)\sum_{m=0}^{\infty} s_m(t) - d_0\sum_{m=1}^{\infty}\sum_{i=0}^{m-1} x_i(t)s_{m-1-i}(t)$$

$$- d_f\sum_{m=1}^{\infty}\sum_{i=0}^{m-1} s_i(t)(y_{m-1-i}(t)+s_{m-1-i}(t)) - (\frac{d_0+d_f}{2})\sum_{m=1}^{\infty}\sum_{i=0}^{m-1} s_i(t)b_{m-1-i}(t) \tag{3.24}$$

$$= \sum_{m=0}^{\infty} s'_m(t) - \gamma\sum_{m=0}^{\infty} y_m(t) + (\alpha+\delta+\mu+d_f)\sum_{m=0}^{\infty} s_m(t) - d_0\sum_{i=0}^{\infty}\sum_{m=i+1}^{\infty} x_i(t)s_{m-1-i}(t)$$

$$- d_f \sum_{i=0}^{\infty} \sum_{m=i+1}^{\infty} s_i(t)(y_{m-1-i}(t) + s_{m-1-i}(t)) - (\frac{d_0 + d_f}{2}) \sum_{i=0}^{\infty} \sum_{m=i+1}^{\infty} s_i(t) b_{m-1-i}(t)$$

$$= \sum_{m=0}^{\infty} s'_m(t) - \gamma \sum_{m=0}^{\infty} y_m(t) + (\alpha + \delta + \mu + d_f) \sum_{m=0}^{\infty} s_m(t) - d_0 \sum_{i=0}^{\infty} x_i(t) \sum_{m=0}^{\infty} s_m(t)$$

$$- d_f \sum_{i=0}^{\infty} s_i(t) \sum_{m=0}^{\infty} (y_m(t) + s_m(t)) - (\frac{d_0 + d_f}{2}) \sum_{i=0}^{\infty} s_i(t) \sum_{m=0}^{\infty} b_m(t)$$

$$= P'_3(t) - \gamma P_2(t) + (\alpha + \delta + \mu + d_f) P_3(t) - d_0 P_1(t) P_3(t)$$

$$- d_f P_3(t)(P_2(t) + P_3(t)) - (\frac{d_0 + d_f}{2}) P_3(t) P_4(t),$$

$andfinally$

$$\sum_{m=1}^{\infty} \Re_m^b = \sum_{m=1}^{\infty} \Bigg[b'_{m-1}(t) - \lambda y_{m-1}(t) - \delta s_{m-1}(t) + \left(\rho + \mu + \frac{d_0 + d_f}{2}\right) b_{m-1}(t)$$

$$- d_0 \sum_{i=0}^{m-1} x_i(t) b_{m-1-i}(t) - d_f \sum_{i=0}^{m-1} b_i(t)(y_{m-1-i}(t) + s_{m-1-i}(t))$$

$$- \left(\frac{d_0 + d_f}{2}\right) \sum_{i=0}^{m-1} b_i(t) b_{m-1-i}(t) \Bigg]$$

$$= \sum_{m=0}^{\infty} b'_m(t) - \lambda \sum_{m=0}^{\infty} y_m(t) - \delta \sum_{m=0}^{\infty} s_m(t) + \left(\rho + \mu + \frac{d_0 + d_f}{2}\right) \sum_{m=0}^{\infty} b_m(t)$$

$$- d_0 \sum_{m=1}^{\infty} \sum_{i=0}^{m-1} x_i(t) b_{m-1-i}(t) - d_f \sum_{m=1}^{\infty} \sum_{i=0}^{m-1} b_i(t)(y_{m-1-i}(t) + s_{m-1-i}(t))$$

$$- \left(\frac{d_0 + d_f}{2}\right) \sum_{m=1}^{\infty} \sum_{i=0}^{m-1} b_i(t) b_{m-1-i}(t)$$

$$= \sum_{m=0}^{\infty} b'_m(t) - \lambda \sum_{m=0}^{\infty} y_m(t) - \delta \sum_{m=0}^{\infty} s_m(t) + \left(\rho + \mu + \frac{d_0 + d_f}{2}\right) \sum_{m=0}^{\infty} b_m(t)$$

$$- d_0 \sum_{i=0}^{\infty} \sum_{m=i+1}^{\infty} x_i(t) b_{m-1-i}(t) - d_f \sum_{i=0}^{\infty} \sum_{m=i+1}^{\infty} b_i(t)(y_{m-1-i}(t) + s_{m-1-i}(t))$$

$$- \left(\frac{d_0 + d_f}{2}\right) \sum_{i=0}^{\infty} \sum_{m=i+1}^{\infty} b_i(t) b_{m-1-i}(t)$$

$$= \sum_{m=0}^{\infty} b'_m(t) - \lambda \sum_{m=0}^{\infty} y_m(t) - \delta \sum_{m=0}^{\infty} s_m(t) + \left(\rho + \mu + \frac{d_0 + d_f}{2}\right) \sum_{m=0}^{\infty} b_m(t)$$

$$- d_0 \sum_{i=0}^{\infty} x_i(t) \sum_{m=0}^{\infty} b_m(t) - d_f \sum_{i=0}^{\infty} b_i(t) \sum_{m=0}^{\infty} (y_m(t) + s_m(t))$$

$$- \left(\frac{d_0 + d_f}{2}\right) \sum_{i=0}^{\infty} b_i(t) \sum_{m=0}^{\infty} b_m(t)$$

$$= P'_4(t) - \lambda P_2(t) - \delta P_3(t) + \left(\rho + \mu + \frac{d_0 + d_f}{2}\right) P_4(t)$$

$$- d_0 P_1(t) P_4(t) - d_f P_4(t)(P_2(t) + P_3(t)) - \left(\frac{d_0 + d_f}{2}\right) P_4^2(t).$$

Eqs. (3.20), (3.21), (3.23) and (3.24) show the series solutions (2.12) must be the exact solutions of problem (1.1).

4 Numerical Illustration

In this section, the approximate solution of Eqs. (1.1) is obtained. For $N = 5$ we get

$$x_5(t) = 0.5045 + 0.00513089ht$$
$$+ \cdots + 3.82918 \times 10^{-7} h^5 t^4$$
$$+2.22461 \times 10^{-9} h^5 t^5,$$
$$y_5(t) = 0.2059 + 0.0250246ht$$
$$+ \cdots + 2.05093 \times 10^{-6} h^5 t^4$$
$$+2.46846 \times 10^{-8} h^5 t^5,$$
$$s_5(t) = 0.1559 + 0.0258909ht$$
$$+ \cdots - 8.41777 \times 10^{-7} h^5 t^4$$
$$-1.98603 \times 10^{-8} h^5 t^5,$$
$$b_5(t) = 0.1337 - 0.0547773ht$$
$$+ \cdots + 1.55965 \times 10^{-6} h^5 t^4$$
$$-6.91485 \times 10^{-9} h^5 t^5,$$

and for $N = 10$ is in the following form

$$x_{10}(t) = 0.5045 + 0.0102618ht$$
$$+ \cdots + 3.91031 \times 10^{-15} h^{10} t^9$$
$$+7.71978 \times 10^{-18} h^{10} t^{10},$$
$$y_{10}(t) = 0.2059 + 0.0500492ht$$
$$+ \cdots + 5.5466 \times 10^{-13} h^{10} t^9$$
$$+1.87646 \times 10^{-15} h^{10} t^{10},$$
$$s_{10}(t) = 0.1559 + 0.0517818ht$$
$$+ \cdots - 5.70543 \times 10^{-13} h^{10} t^9$$
$$-1.93383 \times 10^{-15} h^{10} t^{10},$$
$$b_{10}(t) = 0.1337 - 0.109555ht$$
$$+ \cdots + 2.99605 \times 10^{-15} h^{10} t^9$$
$$+1.08046 \times 10^{-17} h^{10} t^{10},$$

and finally for $N = 15$ we have

$$x_{15}(t) = 0.5045 + 0.0153927ht$$
$$+ \cdots - 2.29444 \times 10^{-23} h^{15} t^{14}$$
$$-6.15273 \times 10^{-26} h^{15} t^{15},$$
$$y_{15}(t) = 0.2059 + 0.0750739ht$$
$$+ \cdots + 9.54277 \times 10^{-21} h^{15} t^{14}$$
$$+1.40286 \times 10^{-23} h^{15} t^{15},$$
$$s_{15}(t) = 0.1559 + 0.0776727ht$$
$$+ \cdots - 1.00308 \times 10^{-20} h^{15} t^{14}$$
$$-1.48472 \times 10^{-23} h^{15} t^{15},$$
$$b_{15}(t) = 0.1337 - 0.164332ht$$
$$+ \cdots + 4.12203 \times 10^{-23} h^{15} t^{14}$$
$$+4.84391 \times 10^{-26} h^{15} t^{15}.$$

By using the obtained numerical solutions, we plot some $\hbar$-curves which are applied to find the convergence intervals. Figs. 2, 3 and 4 show the convergence regions based on the HATM for $N = 5, 10, 15$ and $t = 1$. According to these figures the convergence intervals for $N = 5$ is $-1.2 \leq \hbar_x, \hbar_y, \hbar_s, \hbar_b \leq -0.6$, for $N = 10$ is $-1.3 \leq \hbar_x, \hbar_y, \hbar_s, \hbar_b \leq -0.5$ and finally for $N = 15$ are $-1.3 \leq \hbar_x \leq -0.6$ and $-1.4 \leq \hbar_y, \hbar_s, \hbar_b \leq -0.5$.

The following residual error functions

$$
\begin{aligned}
&E_{N,x}(t)=x'_N(t)-\mu+(d_0+\mu)x_N(t)\\
&-d_0x_N^2(t)-(d_f-\beta)x_N(t)(y_N(t)+s_N(t))\\
&-\left(\tfrac{d_0+d_f}{2}\right)x_N(t)b_N(t),\\
&E_{N,y}(t)=y'_N(t)-\beta x_N(t)(y_N(t)+s_N(t))\\
&-\rho b_N(t)-\alpha s_N(t)+(\gamma+\lambda+\mu+d_f)y_N(t)\\
&-d_0x_N(t)y_N(t)-d_fy_N(t)(y_N(t)+s_N(t))\\
&-\left(\tfrac{d_0+d_f}{2}\right)y_N(t)b_N(t),\\
&E_{N,s}(t)=s'_N(t)-\gamma y_N(t)\\
&+(\alpha+\delta+\mu+d_f)s_N(t)-d_0x_N(t)s_N(t)\\
&-d_fs_N(t)(y_N(t)+s_N(t))\\
&-\left(\tfrac{d_0+d_f}{2}\right)s_N(t)b_N(t),\\
&E_{N,b}(t)=b'_N(t)-\lambda y_N(t)\\
&-\delta s_N(t)+\left(\rho+\mu+\tfrac{d_0+d_f}{2}\right)b_N(t)\\
&-d_0x_N(t)b_N(t)-d_fb_N(t)(y_N(t)+s_N(t))\\
&-\left(\tfrac{d_0+d_f}{2}\right)b_N^2(t),
\end{aligned}
\tag{4.25}
$$

are presented to show the accuracy and efficiency of method. The averaged residual errors of HATM versus $\hbar$ are demonstrated in Fig. 5 for various N and $t=1$. By applying the averaged residual errors and minimizing them we can find the optimal values of $\hbar$. According to this figure the obtained optimal value of $\hbar$ is $\hbar^*\simeq-1$. Now, we can plot the residual error functions based on presented method. Fig. 6 is the comparative figure to exhibit the accuracy of method. Also, the approximate solutions in non-smoker, normal smoker, excessive smoker and ex-smoker cases are demonstrated in Figs. 7 and 8 for $N=5,10$ and $\hbar=-1$. Furthermore the residual errors of $x(t), y(t), s(t)$ and $b(t)$ for different values of $\hbar$ and $N=5,10,15$ are presented in Tables 2-4.

5 Conclusion

The mathematical models can help to scientists for tracking and controlling the phenomena and behaviors. These events may be related to human life. Every day, we see many people that they are smoking and we cross from front of them without any attention. WHO says every year many people are killed because they smoked. So studying the mathematical model of smoking habit is very important to rescue the humans life. In this paper, the HATM was applied to solve the nonlinear mathematical model of smoking habit in a constant population which was modeled on Spain people. We know that the approximate solution of this method depends on convergence control parameter $\hbar$. So some $\hbar$-curves were demonstrated to find the convergence intervals. The plots and the results of error functions showed the accuracy and efficiency of method. By using this model we can predict the evolution of a social habit. Since the social behaviors are continuously changing and in our model the parameters are constant, we do not need a very large range of validity.

References

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