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Optimal Feedback Control of Fractional Semilinear Integro-differential Equations in The Banach Spaces

S. Valizadeh *, A. Borhanifar †‡, M. R. Abdollahpour §

Abstract

Recently, there has been significant development in the existence of mild solutions for fractional semilinear integro-differential equations but optimal control is not provided. The aim of this paper is studying optimal feedback control for fractional semilinear integro-differential equations in an arbitrary Banach space associated with operators generating compact semigroup.

Keywords : Optimal feedback control; Fractional integro-differential equations; Mild solutions; Feasible pairs.

1 Introduction

Fractional order dynamics appear in several problems in science and engineering such as viscoelasticity, bioengineering, dynamics of interfaces between nanoparticles and substrates, etc [9, 10]. Fractional-order models seem to be more adequate than integer-order models because fractional derivatives provide an excellent tool for the description of memory and heredity effects of various materials and processes, including gas diffusion and heat conduction, in fractal porous media [5, 19]. Considerable work has been done in this area in recent years, both in theory and its applications [2, 7, 8, 17, 23]. Optimal control problems with integer and fractional order have been widely studied and many techniques have been developed for solving these problems. For integer order problems, Naderi et al used hyperchaotic complex T-system for master and slave systems by constructing the optimal adaptive sliding mode controller [15]. Niknam et al proposed and analyzed a three-axis optimal control of the six-dimensional system which describes the kinetic and kinematic equations of a satellite subjected to deterministic external perturbations which induce chaotic motion [16]. In [12], it is shown that under certain conditions for the active set the measure representing the Lagrange multiplier associated with the state constraint can be decomposed in a distributed part in L_2 with support on the active set and a boundary mea-

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*Department of Mathematics and Applications, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran.

†Corresponding author. borhani@uma.ac.ir, Tel:+98(914)3548213.

‡Department of Mathematics and Applications, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran.

§Department of Mathematics and Applications, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran.

sure concentrated on the interface between active and inactive sets. For fractional order problems, Mophou firstly studied the existence and the uniqueness of the solution of the fractional diffusion equation in a Hilbert space and then he is shown that the considered optimal control problem has a unique solution [13]. Malmir is presented the fractional integration(operational matrix) based on new Chebyshev wavelet methods for finding solutions of linear-quadratic optimal control problems and analysis of linear fractional time-delay systems [11]. Soradi Zeid et al reviewed the Legendre operational, Ritz method and the Jacobi, Bernoulli and Legendre polynomials to extend as numerical methods for fractional optimal control problems accordingly [18]. Karite et al considered the Hilbert uniqueness method to prove exact enlarged controllability of Riemann-Liouville Fractional Differential Equations for both cases of the zone and pointwise actuators [6]. Yıldız et al derived optimality system for formulation of time fractional optimal control problems governed by Caputo-Fabrizio fractional derivative [24]. Mophou and N'Gurkata proved the existence of a Lagrange multiplier and a decoupled optimality condition for the fractional diffusion via a penalization method [14]. Also, there has been significant development in the existence of mild solutions for fractional neutral evolution equations [3]. Wang et al studied optimal feedback controls of a system governed by semilinear fractional evolution equations via a compact semigroup in Banach spaces [22].

In this paper, we study the optimal feedback control of semilinear fractional integro-differential systems by ideas of [1, 22] and twice using the Mazur and Filippov theorems.

Consider the following fractional integro-differential control system with initial condition

$$D_t^\alpha x(t) = Ax(t) + f(t, x(t), u(t))$$

$$+ \int_0^t K(t-\tau)g(\tau, x(\tau), u(\tau))d\tau, \qquad (1.1)$$

$$x(0) = x_0, \quad t \in J = [0, T], \quad 0 < \alpha < 1 \quad (1.2)$$

where D_t^α, $0 < \alpha < 1$, is the Caputo fractional derivative of order α, and $(A, D(A))$ is the infinitesimal generator of a compact c_0-semigroup

$\{T(t), t \geq 0\}$ in a reflexive Banach space X. The control u takes its value from $U(J)$ which is a control set, and $f, g : J \times X \times U \longrightarrow X$, $K : J \longrightarrow X$ will be determinate in the section 2.

The remainder of this paper is organized as follows. In section 2, some definitions, lemmas, and assumptions are introduced to be used in the sequel. Section 3 is discussed about the existence of mild solution for fractional integro-differential equations. Section 4 will involve the main results and proofs of existence of feasible pairs and optimal Pairs. The Existence of Optimal feedback control pairs theorem for the problem (1.1)-(1.2) and its proof are arranged in Section 5.

2   Preliminaries and notations

In this section, we present some necessary definitions and a number of assumptions for existence of feasible pairs for fractional integro-differential equations.

Definition 2.1 *Let E and F be metric spaces, a multi function $\Pi : E \longrightarrow F$ is said to be pseudo continuous at $x \in E$ if*

$$\bigcap_{\epsilon > 0} \overline{\Pi(O_\epsilon(x))} = \Pi(x).$$

We say that Π is pseudo continuous on E if it is pseudo continuous at each point $x \in E$.

Lemma 2.1 *(Mazur's lemma [22]) Let $(X, \| \cdot \|)$ be a Banach space and let $(u_n)_{n \in N}$ be a sequence in X that converges weakly to some u_0 in X:*

$$u_n \longrightarrow u_0, \quad as \quad n \longrightarrow \infty.$$

That is, for every continuous linear functional f in X^*, the continuous dual space of X,

$$f(u_n) \longrightarrow f(u_0), \quad as \quad n \longrightarrow \infty.$$

Then there exists a function $N : N \longrightarrow N$ and a sequence of sets of real numbers

$$\{\alpha(n)_k \mid k = n, ..., N(n)\}$$

such that $\alpha(n)_k \geq 0$ and such that the sequence $(v_n)_{n \in N}$ defined by the convex combination

$$v_n = \sum_{k=n}^{N(n)} \alpha(n)_k u_k$$

converges strongly in X to u_0, i.e.

$$\| v_n - u_0 \| \longrightarrow 0, \quad as \quad n \longrightarrow \infty.$$

Definition 2.2 *A family $\{S(t); t \geq 0\}$ in $L(X)$ is a semigroup of linear operators on X, if*
(a) $S(0) = I$
(b) $S(t + s) = S(t)S(s)$ for each $t, s \geq 0$
If $\lim_{t\downarrow 0} S(t) = I$, then the semigroup is called uniformly continuous. [4]

Definition 2.3 *The infinitesimal generator, or generator of the semigroup of linear operators $\{S(t); t \geq 0\}$ is the operator $A : D(A) \subseteq X \longrightarrow X$, defined by*

$$D(A) = \{x \in X; \ \exists \lim_{t\downarrow 0} \frac{1}{t}\{S(t)x - x\}\}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}\{S(t)x - x\}$$

We say that A generates $\{S(t); t \geq 0\}$. [4]

Definition 2.4 *A semigroup of linear operators $\{S(t); t \geq 0\}$ is called a C_0-semigroup if for each $x \in X$ we have*

$$\lim_{t\downarrow 0} S(t)x = x$$

and

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}\{S(t)x - x\}$$

We say that A generates $\{S(t); t \geq 0\}$. [4]

Definition 2.5 *For m to the smallest integer that exceeds α, the Caputo fractional derivative operator of order $\alpha > 0$ is defined as [17]*

$$D_t^\alpha x(t) = \frac{1}{\Gamma(m - \alpha)} \int_0^t (t - \tau)^{m-\alpha-1} \frac{\partial^m x(\tau)}{\partial \tau^m}$$

$d\tau$ for $\quad m - 1 < \alpha < m$

We consider following assumptions for existence of feasible pairs for fractional integro-differential equations:
[P1] X is a reflexive Banach space and U is a Polish space.

[P2] A is the infinitesimal generator of a compact c_0-semigroup $\{T(t), t \geq 0\}$ on X.
[P3] $f, g : J \times X \times U \longrightarrow X$ are Borel measurable in (t, x, u) and continuous in (x, u).
[P4] f and g satisfy local Lipschitz continuity with respect to x, i.e. for any constant $\rho > 0$, there are two constants $L_f(\rho) > 0$ and $L_g(\rho) > 0$ such that

$$\| f(t, x, u) - f(t, y, u) \| \leq L_f(\rho) \| x - y \|$$

and

$$\| g(t, x, u) - g(t, y, u) \| \leq L_g(\rho) \| x - y \|$$

for every $x, y \in X$, $t \in J$ and uniformly $u \in U$ provided with $\| x \|$, $\| y \| \leq \rho$.
[P5] $K : J \longrightarrow X$ is an integrable function on J.
[P6] for arbitrary $t \in J$, $x \in X$ and $u \in U$, there exist a positive constant $M > 0$ such that

$$\| f(t, x, u) \| \leq \frac{M}{2}(1 + \| x \|)$$

and

$$\| g(t, x, u) \| \leq \frac{M}{2 \| K \|}(1 + \| x \|)$$

[P7] for almost all $t \in J$, the sets $f(t, x, \Pi(t, x))$ and $g(t, x, \Pi(t, x))$ satisfy the following relations

$$\bigcap_{\delta > 0} \overline{co} f(t, O_\delta(x), \Pi(O_\delta(t, x))) = f(t, x, \Pi(t, x))$$

and

$$\bigcap_{\delta > 0} \overline{co} g(t, O_\delta(x), \Pi(O_\delta(t, x))) = g(t, x, \Pi(t, x))$$

[P8] $\Pi : J \times X \longrightarrow 2^U$ is pseudo continuous.

Definition 2.6 *Let $U[0, T] = \{u : [0, T] \longrightarrow U \mid u(.) \text{ is measurable}\}$ Then, any element in the set $U[0, T]$ is a control on J.*

3 Mild solution for fractional integro-differential equations

Equations (1.1)-(1.2) have only a solution if and only if the following equation has been one solution

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, x(s), u(s))$$

$$ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s K(s-\tau) \times$$

$$g(\tau, x(\tau), u(\tau)) d\tau ds, \quad t \in J, \ 0 < \alpha < 1. \quad (3.3)$$

Lemma 3.1 *if (3.3) hold, then we have mild solution of fractional differential equation in the following form*

$$x(t) = Q(t)x_0 + \int_0^t (t-s)^{\alpha-1} R(t-s)$$

$$f(s, x(s), u(s)) ds + \int_0^t (t-s)^{\alpha-1} R(t-s)$$

$$\int_0^s K(s-\tau) g(\tau, x(\tau), u(\tau)) d\tau ds,$$

$$t \in J, \quad 0 < \alpha < 1. \quad (3.4)$$

where

$$Q(t) = \int_0^\infty \xi_\alpha(\theta) T(t^\alpha \theta) d\theta,$$

$$R(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha \theta) d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \eta_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\eta_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} \times$$

$$\sin(n\pi\alpha), \quad \theta \in (0, \infty).$$

ξ_α *is a probability density function defined on* $(0, \infty)$.

3.1 Properties of operators Q and R

(a) $Q(t)$ and $R(t)$ are strongly continuous, linear and bounded operators for any $t \geq 0$,
(b) $Q(t)$ and $R(t)$ are compact operators for any $t > 0$ if $T(t)$ is compact.

4 Existence of Feasible Pairs and Optimal Pairs

Lemma 4.1 *Let* $[P1]$-$[P6]$ *hold. There is a unique mild solution* $x \in C(J, X)$ *of the system (1.1) for any* $x_0 \in X$ *and* $u \in U$, *and* $\| x \|_{C(J,X)} \leq C$, *for some constant* $C > 0$.

Lemma 4.2 *if* $[P1]$ *and* $[P5]$ *are hold and* $f, g \in L^p(J, X)$ *for* $p > \frac{1}{\alpha}$, *then the operators for* $t \in J, 0 < \alpha < 1$

$$\int_0^t (t-s)^{\alpha-1} R(t-s) f(s, x(s), u(s)) ds,$$

$$\int_0^t (t-s)^{\alpha-1} R(t-s) \int_0^s K(s-\tau) \times$$

$$g(\tau, x(\tau), u(\tau)) d\tau ds,$$

and

$$x(t) = Q(t)x_0 + \int_0^t (t-s)^{\alpha-1} R(t-s) \times$$

$$f(s, x(s), u(s)) ds + \int_0^t (t-s)^{\alpha-1} R(t-s) \times$$

$$\int_0^s K(s-\tau) g(\tau, x(\tau), u(\tau)) d\tau ds$$

are compact.

Lemma 4.3 *Let* $[P1]$-$[P6]$ *hold. There is a unique mild solution* $x \in C(J, X)$ *of the system (1.1) for any* $x_0 \in X$ *and* $u \in U$, *and* $\| x \|_{C(J,X)} \leq C$, *for some constant* $C > 0$.

Theorem 4.1 *Let* $[P1]$-$[P8]$ *hold. Then for any* $x_0 \in X$ *and for* $p > \frac{1}{\alpha}$, *the set*

$$\{(x, u) \in C([0, T], X) \times U[0, T] \mid (x, u)$$
$$\text{is measurable}\}$$

is nonempty.

Proof. For any $k \geq 1$, let $t_j = \frac{j}{k}T$, $0 \leq j \leq k-1$, we set

$$u_k = \sum_{j=1}^{k-1} u^j \chi_{[t_j, t_{j+1})}(t).$$

The constructing of u^j is explained in the [22], where

$$u^j \in \Pi(t_j, x(t_j)), t \in [t_j, t_{j+1}), 0 \le j \le k - 1.$$

Therefore, we have

$$x_k(t) = Q(t)x_0 + \int_0^t (t - s)^{\alpha-1} R(t - s) \times$$

$$f(s, x_k(s), u_k(s))ds + \int_0^t (t - s)^{\alpha-1} R(t - s)$$

$$\int_0^s K(s - \tau)g(\tau, x_k(\tau), u_k(\tau))d\tau ds,$$

$$t \in J, \quad 0 < \alpha < 1. \tag{4.5}$$

$$u_k(t) \in \Pi(t_j, x_k(t_j)), \quad t \in [t_j, t_{j+1}),$$

$$0 \le j \le k - 1.$$

By Gronwall's inequality, $[P4]$ and $[P7]$, we see that for some absolute constant $C > 0$ and for $k \ge 1$

$$\begin{cases} \| x_k(t) \| \le C, & \forall \quad t \in J; \\ \| f(s, x_k(s), u_k(s) \| \le C, & a.e. \quad s \in J; \\ \| f(s, x_k(\tau), u_k(\tau) \| \le C, & a.e. \quad \tau \in J. \end{cases}$$

It follows from lemma (3.1) that sequence $\{x_k(.)\}_{k \ge 1}$ is relatively compact in $C(J, X)$. Thus, we assume

$$x_k(t) \longrightarrow \overline{x}(t) \quad \text{in} \quad C(J, X) \tag{4.6}$$

for some $\overline{x}(t) \in C(J, X)$. Also, we may let

$$f(t, x_k(t), u_k(t)) \longrightarrow \overline{f}(t) \quad \text{in} \quad L_p(J, X) \tag{4.7}$$

$$g(t, x_k(t), u_k(t)) \longrightarrow \overline{g}(t) \quad \text{in} \quad L_p(J, X) \tag{4.8}$$

for some $\overline{f}(t)$ and $\overline{g}(t) \in L^p(J, X)$, By compactness of the operators

$$\int_0^t (t - s)^{\alpha-1} R(t - s)(.)ds,$$

and

$$\int_0^t (t - s)^{\alpha-1} R(t - s) \int_0^s K(s - \tau)(.)d\tau ds,$$

and by (4.5), we have

$$\overline{x}(t) = Q(t)x_0 + \int_0^t (t - s)^{\alpha-1} R(t - s) \overline{f}(s)ds$$

$$+ \int_0^t (t - s)^{\alpha-1} R(t - s) \int_0^s K(s - \tau)\overline{g}(\tau)d\tau ds,$$

$$t \in J, \quad 0 < \alpha < 1.$$

by (4.6), for $\delta > 0$ there exist a k_0, such that

$$x_k(t) \in O_\delta(\overline{x}(t)), \quad \forall t \in J, \quad k \ge k_0. \tag{4.9}$$

On the other hand, by the definition of $u_k(.)$, for k large, one has

$$u_k(t) \in \Pi(t_j, x_k(t_j)) \subset \Pi(O_\delta(t, \overline{x}(t))),$$

$$\forall t \in [t_j, t_{j+1}), \quad 0 \le j \le k - 1. \tag{4.10}$$

By (4.7), (4.8) and Mazur theorem, we may let $\alpha_{i,j} \ge 0$, $\beta_{s,l} \ge 0$ and $\sum_{i \ge 1} \alpha_{i,j} = \sum_{m \ge 1} \beta_{m,n} = 1$, Such that for $p > 1$

$$\varphi_j(t) = \sum_{i \ge 1} \alpha_{i,j} f(., x_{i+j}(t), u_{i+j}(t)) \longrightarrow \overline{f}(t)$$

in $L^p(J, X)$ and

$$\psi_n(t) = \sum_{s \ge 1} \alpha_{s,n} g(., x_{s+n}(t), u_{s+n}(t)) \longrightarrow \overline{g}(t)$$

in $L^p(J, X)$. Then we can assume

$$\begin{cases} \varphi_j(t) \longrightarrow \overline{f}(t), & \text{in X } a.e. \quad t \in J; \\ \psi_n(t) \longrightarrow \overline{g}(t), & \text{in X } a.e. \quad t \in J. \end{cases}$$

On the other hand, by (4.9) and (4.10), we see that for j and n large, we have

$$\varphi_j(t) \in cof(t, O_\delta(\overline{x}(t)), \Pi(O_\delta(t, \overline{x}(t)))),$$

a.e. $t \in J$, and

$$\psi_n(t) \in cog(t, O_\delta(\overline{x}(t)), \Pi(O_\delta(t, \overline{x}(t)))),$$

a.e. $t \in J$. Thus, for any $\delta > 0$

$$\overline{f}(t) \in \overline{co}f(t, O_\delta(\overline{x}(t)), \Pi(O_\delta(t, \overline{x}(t)))),$$

a.e. $t \in J$ and

$$\overline{g}(t) \in \overline{co}g(t, O_\delta(\overline{x}(t)), \Pi(O_\delta(t, \overline{x}(t)))),$$

a.e. $t \in J$. By $[P7]$, we have

$$\overline{f}(t) \in f(t, \overline{x}(t), \Pi(t, \overline{x}(t))), \quad a.e. \quad t \in J,$$

and

$$\overline{g}(t) \in g(t, \overline{x}(t), \Pi(t, \overline{x}(t))), \quad a.e. \quad t \in J.$$

We know that $\Pi(.\overline{x}(.))$ is Souslin measurable. Therefore there exist a $\overline{u}(.) \in H[0, T]$, such that

$$\begin{cases} \overline{u}(t) \in \Pi(t, \overline{x}(t)), & a.e. \quad t \in J; \\ \overline{f}(t) \in f(t, \overline{x}(t), \overline{u}(t)), & a.e. \quad t \in J; \\ \overline{g}(t) \in g(t, \overline{x}(t), \overline{u}(t)), & a.e. \quad t \in J. \end{cases}$$

Therefore, $(\overline{x}, \overline{u})$ is a feasible pair in $H[0, T]$.

5 Existence of Optimal feedback control Pairs

5.1 Lagrange problem (P):

The problem is finding $(\overline{x}(.), \overline{u}(.)) \in H[0, T]$ such that

$$J(\overline{x}(t), \overline{u}(t)) = Min \int_0^T (L(t, x(t), u(t)) +$$

$$\int_0^t R(s, x(s), u(s)) ds) dt.$$

We consider

$$\psi(x, u) = \int_0^t R(s, x(s), u(s)) ds$$

and

$$\varphi(x, u) = \int_0^T (L(t, x(t), u(t)) + \psi(t, x(t), u(t))) dt$$

We make the following assumptions on L and R:
$[A1]$ the functionals $L, R : J \times X \times U \longrightarrow \mathbb{R} \cup \{\infty\}$ are Borel measurable in (t, x, u).
$[A2]$ $L(t, x(t), u(t))$ and $R(t, x(t), u(t))$ are sequentially linear semicontinuous on $X \times U$ for almost all $t \in J$ and there are constants $K_1, K_2 > 0$ such that

$$L(t, x(t), u(t)) \geq -K_1$$

and

$$R(t, x(t), u(t)) \geq -K_2,$$

for $(t, x, u) \in J \times X \times U$. For any $(t, x) \in J \times X$, we consider the following sets

$$\epsilon_1(t, x) = \{(y^*, y) \in \mathbb{R} \times X \mid y^* \geq L(t, x, u),$$

$$y = f(t, x, u), \quad u \in \Pi(t, x)\}$$

and

$$\epsilon_2(t, x) = \{(z^*, z) \in \mathbb{R} \times X \mid z^* \geq R(t, x, u),$$

$$z = g(t, x, u), \quad u \in \Pi(t, x)\}$$

To prove the existence of optimal control pairs for Problem (P), we assume that:
$[M]$ for almost all $t \in J$, the maps $\epsilon_j(t, .) : X \longrightarrow 2^{\mathbb{R} \times X}$ for $j = 1, 2$ have the Cesari properties.

Theorem 5.1 *Suppose that the hypothesis $[P1]$-$[P8]$, $[A1]$, $[A2]$ and $[M]$ hold. Then Lagrange Problem (P) admits at least one optimal control pair.*

Proof. If $in\{J(x, u) \mid (x, u) \in H[0, T]\} = +\infty$, there is nothing to prove, some assume that $inf\{J(x, u) \mid (x, u) \in H[0, T]\} = \lambda < \infty$. By $[A]$, we have $\psi(u) \geq \mu \geq -K_2 > -\infty$ and $\varphi(u) \geq \lambda \geq -K_1 - \tau K_2 > -\infty$, $(\tau > 0)$ so there exists a sequence $\{x^n, u^n\} \in H[0, T]$ such that

$$\varphi(x^n, u^n) \longrightarrow \lambda$$

and

$$\psi(x^n, u^n) \longrightarrow \mu$$

We denote

$$\psi(x^n, u^n) = \int_0^t R(s, x^n(s), u^n(s)) ds,$$

and

$$\varphi(x^n, u^n) = \int_0^T (L(t, x^n(t), u^n(t)) +$$

$$\psi(t, x^n(t), u^n(t))) dt,$$

and we have

$$\lim_{n \longrightarrow \infty} inf \psi(x^n, u^n) = \mu,$$

and

$$\lim_{n \longrightarrow \infty} inf\varphi(x^n, u^n) = \lambda.$$

By the bounded growth condition of f and g [P6], one can obtain that $\{f(t, x^n(t), u^n(t))\}$ and $\{g(t, x^n(t), u^n(t))\}$ are bounded in $L^p(J, X)$ and we may assume that (without loss of generality)

$$f^n(t) = f(t, x^n(t), u^n(t)) \longrightarrow f(t),$$

in $L^p(J, X)$ and

$$g^n(t) = g(t, x^n(t), u^n(t)) \longrightarrow g(t),$$

in $L^p(J, X)$. For some $f(t)$ and $g(t)$ in $L^p(J, X)$, By Lemma (4.2), we obtain

$$x^n(t) = Q(t)x_0 + \int_0^t (t-s)^{\alpha-1}R(t-s)\times$$

$$f(s, x^n(s), u^n(s))ds + \int_0^t (t-s)^{\alpha-1}R(t-s)\times$$

$$\int_0^s K(s-\tau)g(\tau, x^n(\tau), u^n(\tau))d\tau ds,$$

$$\longrightarrow \overline{x}(t) = Q(t)x_0 + \int_0^t (t-s)^{\alpha-1}R(t-s)\overline{f}(s)ds$$

$$+ \int_0^t (t-s)^{\alpha-1}R(t-s)\int_0^s K(s-\tau)\overline{g}(\tau)d\tau ds,$$

Uniformly in $t \in J$, a.e., By the Mazur theorem let $\alpha_{i,j} \geq 0$

$$\psi_n(t) = \sum_{s \geq 1} \xi_{s,n}g(t, x_{s+n}(t), u_{s+n}(t)) \longrightarrow \overline{g}(t),$$

in $L^p(J, X)$, and

$$\varphi_j(t) = \sum_{i \geq 1} \alpha_{i,j}f(t, x_{i+j}(t), u_{i+j}(t)) \longrightarrow \overline{f}(t),$$

in $L^p(J, X)$. By repeat the above process

$$\omega_q(t) = \sum_{p \geq 1} \gamma_{p,q}\psi_n(t, x_{p+q}(t), u_{p+q}(t)) \longrightarrow$$

$$\int_0^t K(t-\tau)\overline{g}(\tau)d\tau, \quad \text{in} \quad L^p(J, X).$$

Also, we can write following relation

$$\sum_{p \geq 1}\sum_{s \geq 1} \gamma_{p,q}\xi_{s,n} = \sum_{i \geq 1} \beta_{i,q(n)}$$

We construct follow equality

$$\nu_{q(n)}(t) = \sum_{k \geq 1} \beta_{i,q(n)}g(t, x_{k+n}(t), u_{k+n}(t))$$

We consider

$$\varphi_j^*(t) = \sum_{i \geq 1} \alpha_{i,j}L(t, x_{i+j}(t), u_{i+j}(t)),$$

and

$$\nu_{q(n)}^*(t) = \sum_{k \geq 1} \beta_{k,q(n)}R(t, x_{k+q(n)}(t), u_{k+q(n)}(t)),$$

Hence, we have

$$L^*(t) = \lim_{j \to \infty} inf\varphi_j^*(t) \geq -K_1 \quad a.e. \quad t \in J,$$

and

$$R^*(t) = \lim_{q \to \infty} inf\omega_q^*(t) \geq -K_2 \quad a.e. \quad t \in J.$$

For any $\delta > 0$ and $m > j, q$ large enough, we have

$$(\varphi_j(t), \varphi_j^*(t)) \in \epsilon_1(t, O_\delta(\overline{x}(t))),$$

and

$$(\nu_q(t), \nu_q^*(t)) \in \epsilon_2(t, O_\delta(\overline{x}(t))).$$

By [M], we have

$$(L^*(t), \overline{f}(t)) \in \epsilon_1(t, \overline{x}(t)), \quad a.e. \quad t \in J,$$

and

$$(R^*(t), \overline{g}(t)) \in \epsilon_2(t, \overline{x}(t)), \quad a.e. \quad t \in J,$$

This means that

$$\begin{cases} L^*(t) \geq L(t, \overline{x}(t), u), & t \in J; \\ \overline{f}(t) = f(t, \overline{x}(t), u), & t \in J; \\ R^*(t) \geq R(t, \overline{x}(t), u), & t \in J; \\ \overline{g}(t) = g(t, \overline{x}(t), u), & t \in J; \\ u \in \Pi(t, \overline{x}(t)). \end{cases}$$

By the Filippov theorem [20] again there is a measurable selection $\overline{u}(t)$ of $\Pi(t, \overline{x}(t))$ such that

$$\begin{cases} L^*(t) \geq L(t, \overline{x}(t), \overline{u}(t)), & t \in J; \\ \overline{f}(t) = f(t, \overline{x}(t), \overline{u}(t)), & a.e. \quad t \in J; \\ R^*(t) \geq R(t, \overline{x}(t), \overline{u}(t)), & t \in J; \\ \overline{g}(t) = g(t, \overline{x}(t), \overline{u}(t)), & a.e. \quad t \in J. \end{cases}$$

On the other hand, we have

$$\bar{x}(t) = Q(t)x_0 + \int_0^t (t-s)^{\alpha-1} R(t-s) \times$$

$$\bar{f}(s, \bar{x}(s), \bar{u}(s))ds + \int_0^t (t-s)^{\alpha-1} R(t-s) \times$$

$$\int_0^s K(s-\tau)\bar{g}(\tau, \bar{x}(\tau), \bar{u}(\tau))d\tau ds$$

for $t \in J$, and

$$(\bar{x}, \bar{u}) \in H[0, T]$$

We have

$$J(\bar{x}, \bar{u}) = \int_0^T (L^*(t) + R^*(t))dt =$$

$$\int_0^T (\lim_{m \to \infty} \inf \varphi_m^*(t) + \lim_{m \to \infty} \inf \omega_m^*(t))dt$$

$$\leq \int_0^T \lim_{m \to \infty} \inf (\varphi_m^*(t) + \omega_m^*(t))dt$$

and by Fatou's lemma

$$\int_0^T \lim_{m \to \infty} \inf (\varphi_m^*(t) + \omega_m^*(t))dt$$

$$\leq \lim_{m \to \infty} \inf \int_0^T (\varphi_m^*(t) + \omega_m^*(t))dt$$

Hence

$$J(\bar{x}, \bar{u}) = \int_0^T (L(t, \bar{x}(t), \bar{u}(t))$$

$$+ \int_0^t R(s, \bar{x}(s), \bar{u}(s))ds)dt$$

$$= \inf_{(x,u) \in H[0,T]} J(x, u) = \lambda$$

Thus, $(\bar{x}, \bar{u})$ is just an optimal pair.

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Sohrab Valizadeh got PhD degree in Applied Mathematics from University of Mohaghegh Ardabili in 2020. His research interests include numerical analysis, fractional partial differential equations and optimal control theory.

Abdollah Borhanifar has got PhD degree in nonlinear PDE from Moscow University in 2004 and Now he is a Professor in the department of mathematics and applications at the University of Mohaghegh Ardabili, Ardabil, Iran. He published more than 77 publications in journals on optics, numerical analysis and non-linear partial differential equations.

Mohammad Reza Abdollahpour is an associate professor in the department of mathematics and applications at the University of Mohaghegh Ardabili, Ardabil, Iran. He received his PhD degree in Pure Mathematics-Functional Analysis from Tabriz University in 2007. His area of interest includes G-frames and stability analysis.