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Research Article

On an Efficient Family with Memory with High Order of Convergence for Solving Nonlinear Equations

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Abstract

The primary goal of this work is to introduce general family Steffensen-like methods with memory of the high efficiency indices. To achieve this target two parameters are introduced which are calculated with the help of Newton's interpolatory polynomial. It is shown that the R-order convergence of the proposed methods has been increased from 2, 4, 8, $\cdots$ and 2^n to $3.5, 7, 14, \cdots$ and $3.5*2^{n-1}$, respectively without any extra evaluation. Computational results confirm the efficient and robust character of presented methods.

Keywords : Nonlinear equations; With memory methods; Acceleration of convergence; Efficiency index.

1 Introduction

One of the most important subjects in developing numerical algorithms is to establish optimal algorithms with economic complexity. For example, developing iterative methods for approximating zero(s) of a given nonlinear equation falls within this matter, and many studies have been devoted to it [30]. Iterative methods are used to solve nonlinear equations. They are divided into two categories without memory and with memory. Iterative with memory methods have higher the order of convergence and efficiency indexes. We recall the so-called efficiency index defined by A. M. Ostrowski [23], as $E(p,n) = p^{1/n}$, where p is the order of convergence and n is the total number function evaluations per iteration. One way to increase the order of convergence, is to apply self-acceleration parameters by using suitable Newton's interpolatory polynomials. Traub developed the first method with memory from Steffensen's method [29] as follows:

$$\begin{cases} w_k = x_k + \gamma_k f(x_k), \ k = 0, 1, 2, \ldots, \\ x_{k+1} = x_k - \frac{f(x_k)}{f[x_k, w_k]}, \ \gamma_k = \frac{1}{N'(x_{k+1})}, \end{cases} \tag{1.1}$$

where x_0 and γ_0 are given initially suitable values, and $N(t) = f(x_{k+1}) + (t - x_{k+1}) f[x_{k+1}, x_k]$ is the linear Newton's interpolation. The convergence order of the with memory method is $1 + \sqrt{2} \approx 2.4142$. The number of people using memory techniques can be called: Cordero et al. [3], Dzunic [7], Petkovic et al. [24], Sharma et al. [27] and Soleymani et al. [28]. They prosper to upgrade the convergence order of family, by approximating the parameter(s). In this work, we are going to suggest a new kind of with mem-

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ory methods (two parameters accelerators) for solving the nonlinear equations. To this end, it is attempted to consider two new accelerators in which one of them increase the convergence order to $2^{1/2} = 1.41421$, $4^{1/3} = 1.58740$, $8^{1/4} = 1.68179$, $16^{1/5} = 1.74110, \cdots$, while the other one increases it to $3.56^{1/2} = 1.8868$, $7^{1/3} = 1.9129$, $14^{1/4} = 1.9343$, $28^{1/5} = 1.9473, \cdots$. In Section 2, we construct iterative methods without memory. The acceleration of convergence speed is obtained in Section 3. A new family of iterative methods with memory is obtained by varying two free parameters in per full iteration. These self-accelerating parameters are calculated using information available from the current and previous iteration. These parameters are calculated by the Newton's interpolating polynomial, where the R-order convergence increased from $2, 4, 8$ and $16, \cdots, 2^n$ to $3.56, 7, 14$ and $28, \cdots, 3.5 * 2^{n-1}$, respectively. Numerical performances, comparisons and conclusions are given in Sections 4 and 5.

2 Preliminaries

In year 2011, Q. Zheng et al. [34] exhibitted Steffesen-type without memory optimal methods with orders 2, 4, 8 and 16. One-step iterative methods (ZLHM2):

$$\begin{cases} w_k = x_k + \gamma f(x_k), \ \gamma \in \mathbb{R}, \ k = 0, 1, 2, \ldots, \\ x_{k+1} = x_k - \frac{f(x_k)}{f[x_k, w_k]}, \end{cases} \tag{2.2}$$

two-step without memory methods (ZLHM4):

$$\begin{cases} w_k = x_k + \gamma f(x_k), \ \gamma \in \mathbb{R} \\ y_k = x_k - \frac{f(x_k)}{f[x_k, w_k]}, k = 0, 1, 2, \ldots, \\ x_{k+1} = y_k - \frac{f(y_k)}{f[y_k, x_k] + f[y_k, x_k, w_k](y_k - x_k)}, \end{cases} \tag{2.3}$$

Steffensen-like methods with order 8 (ZLHM8)

$$\begin{cases} w_k = x_k + \gamma f(x_k), \ \gamma \in \mathbb{R}, \ k = 0, 1, 2, \ldots, \\ y_k = x_k - \frac{f(x_k)}{f[x_k, w_k]}, \\ z_k = y_k - \frac{f(y_k)}{f[y_k, x_k] + f[y_k, x_k, w_k](y_k - x_k)}, \\ x_{k+1} = z_k - f(z_k)(f[z_k, y_k] + f[z_k, y_k, x_k] \\ (z_k - y_k) + f[z_k, y_k, x_k, w_k](z_k - y_k)(z_k - x_k))^{-1}, \end{cases} \tag{2.4}$$

four-point methods with order 16 (ZLHM16)

$$\begin{cases} w_k = x_k + \gamma f(x_k), \ \gamma \in \mathbb{R}, \ k = 0, 1, 2, \ldots, \\ y_k = x_k - \frac{f(x_k)}{f[x_k, w_k]}, \\ z_k = y_k - \frac{f(y_k)}{f[y_k, x_k] + f[y_k, x_k, w_k](y_k - x_k)}, \\ u_k = z_k - f(z_k)(f[z_k, y_k] + f[z_k, y_k, x_k] \\ (z_k - y_k) + f[z_k, y_k, x_k, w_k](z_k - y_k)(z_k - x_k))^{-1}, \\ x_{k+1} = u_k - f(u_k)(f[u_k, z_k] + f[u_k, z_k, y_k](u_k - \\ z_k) + f[u_k, z_k, y_k, x_k](u_k - z_k)(u_k - y_k) + \\ f[u_k, z_k, y_k, x_k, w_k](u_k - z_k)(u_k - y_k)(u_k - x_k))^{-1}. \end{cases} \tag{2.5}$$

Similarly, a family of iterative methods without memory 2^n-order is obtained as follows:

$$\begin{cases} y_{-1} = x_k + \gamma f(x_k), \ \gamma \in \mathbb{R}, \ y_0 = x_k, \\ y_1 = y_0 - \frac{f(y_0)}{f[y_0, y_{-1}]}, \\ y_2 = y_1 - \frac{f(y_1)}{f[y_1, y_0] + f[y_1, y_0, y_{-1}](y_1 - y_0)}, \\ y_k = y_{k-1} - f(y_{k-1})(f[y_{k-1}, y_{k-2}] + \cdots + \\ f[y_{k-1}, \cdots, y_{-1}](y_{k-1} - y_{k-2}) \cdots (y_{k-1} - y_{-1}))^{-1}. \end{cases} \tag{2.6}$$

The next theorem states of the error equations the methods (2.2), (2.3), (2.4), (2.5),and (2.6).

Theorem 2.1 *[34] Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be a scalar function which has a simple root α in the open interval I, and also the initail approximation x_0 is sufficiently close the simple zero, then, the iteration methods (2.2), (2.3), (2.4), (2.5), and (2.6) have convergence with two-, four-, eight-, sixteen-, and, 2^n-order and satisfy the following error equations, respectively :*

$$e_{k+1} = (1 + f'(\alpha)\gamma)c_2 e_k^2 + O(e_k^3), \gamma \in \mathbb{R}, \tag{2.7}$$

$$e_{k+1} = (1 + f'(\alpha)\gamma)^2 c_2(c_2^2 - c_3)e_k^4 + O(e_k^5), \tag{2.8}$$

$$\begin{cases} e_{k+1} = (1 + f'(\alpha)\gamma)^4 c_2^2(c_2^2 - c_3)(c_2(c_2^2 - c_3) \\ +c_4)e_k^8 + O(e_k^9), \end{cases} \tag{2.9}$$

$$\begin{cases} e_{k+1} = (1 + f'(\alpha)\gamma)^8 c_2^4(c_2^2 - c_3)^2(c_2(c_2^2 - c_3) \\ +c_4)(c_2(c_2^2 - c_3) + c_4) - c_5)e_k^{16} + O(e_k^{17}), \end{cases} \tag{2.10}$$

$$\begin{cases} e_{k+1} = D_{m-1} e_k^{2^m} + O(e_k^{2^m+1}), \\ m = 1, 2, 3, \ldots, \ k = 0, 1, 2, \ldots, \\ D_{-1} = (1 + f'(\alpha)\gamma), \ D_0 = 1, \\ D_1 = (1 + f'(\alpha)\gamma)c_2 \cdots, \ D_m = \\ [(c_2 D_{m-1} + (-1)^{m-1} c_{m+1} + D_{m-2} \cdots D_{-1}], \end{cases} \tag{2.11}$$

The above theorem was proved in [34]. Let us consider the following modification of without memory methods (2.2), (2.3), (2.4), (2.4)and (2.6) with an additional p parameter in the first step. The methods of one, two, three and four steps are obtained respectively.

Second-order family of two-parameter without memory methods

$$\begin{cases} w_k = x_k + \gamma f(x_k), \ \ \gamma \in \mathbb{R}, \ k = 0, 1, 2, \ldots, \\ x_{k+1} = x_k - \frac{f(x_k)}{f[x_k, w_k] + p f(w_k)}. \end{cases} \tag{2.12}$$

New optimal biparametric fourth-order family:

$$\begin{cases} w_k = x_k + \gamma f(x_k), \ \gamma \in \mathbb{R}, \\ y_k = x_k - \frac{f(x_k)}{f[x_k, w_k] + p f(w_k)}, \ k = 0, 1, 2, \ldots, \\ x_{k+1} = y_k - \frac{f(y_k)}{f[y_k, x_k] + f[y_k, x_k, w_k](y_k - x_k)}. \end{cases} \tag{2.13}$$

New optimal biparametric eight-order family:

$$\begin{cases} w_k = x_k + \gamma f(x_k), \gamma \in \mathbb{R} \\ y_k = x_k - \frac{f(x_k)}{f[x_k, w_k] + p f(w_k)}, \ k = 0, 1, 2, \ldots, \\ z_k = y_k - \frac{f(y_k)}{f[y_k, x_k] + f[y_k, x_k, w_k](y_k - x_k)}, \\ x_{k+1} = z_k - f(z_k)(f[z_k, y_k] + f[z_k, y_k, x_k] \\ (z_k - y_k) + f[z_k, y_k, x_k, w_k](z_k - y_k)(z_k - x_k))^{-1}, \end{cases} \tag{2.14}$$

new optimal biparametric 16-order family:

$$\begin{cases} w_k = x_k + \gamma f(x_k), \ \gamma \in \mathbb{R}, \\ y_k = x_k - \frac{f(x_k)}{f[x_k, w_k] + p f(w_k)}, \ k = 0, 1, 2, \ldots, \\ z_k = y_k - \frac{f(y_k)}{f[y_k, x_k] + f[y_k, x_k, w_k](y_k - x_k)}, \\ u_k = z_k - f(z_k)(f[z_k, y_k] + f[z_k, y_k, x_k](z_k - y_k) \\ + f[z_k, y_k, x_k, w_k](z_k - y_k)(z_k - x_k))^{-1}, \\ x_{k+1} = u_k - f(u_k)(f[u_k, z_k] + f[u_k, z_k, y_k] \\ (u_k - z_k) + f[u_k, z_k, y_k, x_k](u_k - z_k) \\ (u_k - y_k) + f[u_k, z_k, y_k, x_k, w_k](u_k - z_k) \\ (u_k - y_k)(u_k - x_k))^{-1}. \end{cases} \tag{2.15}$$

And so on new optimal biparametric 2^n-order family:

$$\begin{cases} y_{-1} = x_k + \gamma f(x_k), \ \gamma \in \mathbb{R}, \\ y_0 = x_k, \ y_1 = y_0 - \frac{f(y_0)}{f[y_0, y_{-1}] + p f(y_{-1})}, \\ y_2 = y_1 - \frac{f(y_1)}{f[y_1, y_0] + f[y_1, y_0, y_{-1}](y_1 - y_0)}, \\ \vdots \ k = 0, 1, 2, \ldots, \\ y_k = y_{k-1} - f(y_{k-1})(f[y_{k-1}, y_{k-2}] + \cdots + f[\\ y_{k-1}, \cdots, y_{-1}](y_{k-1} - y_{k-2}) \cdots (y_{k-1} - y_{-1}))^{-1}. \end{cases} \tag{2.16}$$

The main contribution of this section lies in the following theorem.

Theorem 2.2 *Assume that the function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a single root $\alpha \in D$, where D is an open interval. Then the convergence order of the derivative-free iterative methods defined by (2.12), (2.13), (2.14), (2.15) and (2.16) are two-, four-, eight-, sixteen- and, 2^n-order also satisfy the following error equations, respectively:*

$$e_{k+1} = (1 + f'(\alpha)\gamma)(p + c_2)e_k^2 + O(e_k^3), \gamma, \ p \in \mathbb{R}, \tag{2.17}$$

$$\begin{cases} e_{k+1} = (1 + f'(\alpha)\gamma)^2 (p + c_2)(c_2(p + c_2) - \\ c_3)e_k^4 + O(e_k^5), \end{cases} \tag{2.18}$$

$$\begin{cases} e_{k+1} = (1 + f'(\alpha)\gamma)^4 (p + c_2)^2 (c_2(p + c_2) \\ -c_3)(c_2(c_2(p + c_2) - c_3) + c_4)e_k^8 + O(e_k^9), \end{cases} \tag{2.19}$$

$$\begin{cases} e_{k+1} = (1 + f'(\alpha)\gamma)^8 (p + c_2)^4 (c_2(p + c_2) - \\ c_3)^2 (c_2(c_2(p + c_2) - c_3) + c_4)(c_2(c_2(p + c_2) \\ -c_3) + c_4) - c_5)e_k^{16} + O(e_k^{17}). \end{cases} \tag{2.20}$$

$$\begin{cases} e_{k+1} = (1 + f'(\alpha)\gamma)^{2^{n-1}} (p + c_2)^{2^{n-2}} \\ (\cdots)e_k^{2^n} + O(e_k^{2^{n+1}}). \end{cases} \tag{2.21}$$

The proof is completely similar to the proof of Theorem 2.1, ,hence it is omitted. The above without memory schemes are optimal in the sense of Kung and Traub. (The order of convergence of any multipoint method without memory cannot exceed the bound 2^n (called optimal order), where $n + 1$ is the number of function evaluations per iteration.)

3 Construction of with memory methods

In this section, we are going to construct new iterative with memory methods from (2.12), (2.13), (2.14), (2.15), and (2.16) using two self-accelerating parameters. It is clear from errors (2.17), (2.18), (2.19), (2.20), and (2.21) that the order of convergence of the family (2.12), (2.13), (2.14), (2.15), and (2.16) are two-, four-, eight-, sixteen- and, 2^n-order, when $\gamma \neq -1/f'(\alpha)$ and $p \neq -c_2$. Therefore, it is possible to increase the convergence speed of the proposed class (2.12), (2.13), (2.14), (2.15), and (2.16), if $\gamma = -1/f'(\alpha)$ and $p = -c_2 = -f''(\alpha)/2f'(\alpha)$. However, the values of $f'(\alpha)$ and $f''(\alpha)$ are not available in practice and such acceleration is not possible. Instead of that, we could use approximations $\tilde{f}'(\alpha) \approx f'(\alpha)$ and $\tilde{f}''(\alpha) \approx f''(\alpha)$, calculated by already available information. Therefore, by setting $\gamma = -1/\tilde{f}'(\alpha)$ and $p = -c_2 = -\tilde{f}''(\alpha)/2\tilde{f}'(\alpha)$ we can increase the convergence order without using any new functional evaluations. Hence, the main idea in constructing with memory methods consists of the calculation of the parameters $\gamma = \gamma_k$ and $p = p_k$ as the iteration proceeds by the formulas $\gamma_k = -1/\tilde{f}'(\alpha)$ and $p_k = -c_2 = -\tilde{f}''(\alpha)/2\tilde{f}'(\alpha)$ for $k = 1, 2, 3, \cdots$. Further, it is also assumed that the initial estimates γ_0 and p_0 should be chosen before starting the iterative process, for example, using one of the ways proposed in [30]. It is worth mentioning that the evaluation of the self-accelerating parameters γ_k and p_k depends on the data available from the current and the previous iterations. Therefore, order of convergence will be increased significantly without using an extra functional evaluation. Finally, replacing fixed parameters γ and p by the varying parameters γ_k and p_k , we shall obtain new methods with memory. Hence, with memory versions of derivative-free methods can be presented as follows: Two parameters family with memory is given by for given x_0, w_0 consider (DM),

$$\begin{cases} \gamma_k = -\frac{1}{N_2'(x_{k+1})}, \; p_k = \frac{N_2''(w_{k+1})}{-2N_2'(w_{k+1})}, \\ w_k = x_k + \gamma_k f(x_k), \; k = 0, 1, 2, \cdots, \\ x_{k+1} = x_k - \frac{f(x_k)}{f[x_k,w_k]+p_k f(w_k)}, \end{cases} \tag{3.22}$$

and a family of two-step with memory methods (CLBTM)

$$\begin{cases} \gamma_k = -\frac{1}{N_3'(x_{k+1})}, \; p_k = \frac{N_3''(w_{k+1})}{-2N_3'(w_{k+1})}, \\ w_k = x_k + \gamma_k f(x_k), \; , k = 0, 1, 2, \cdots, \\ y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]+p_k f(w_k)}, \\ x_{k+1} = y_k - \frac{f(y_k)}{f[y_k,x_k]+f[y_k,x_k,w_k](y_k-x_k)}. \end{cases} \tag{3.23}$$

Three-step schemes with memory (TKM14)

$$\begin{cases} \gamma_k = -\frac{1}{N_4'(x_{k+1})}, \; p_k = \frac{N_4''(w_{k+1})}{-2N_4'(w_{k+1})}, \\ w_k = x_k + \gamma_k f(x_k), \; k = 0, 1, 2, \cdots, \\ y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]+p_k f(w_k)}, \\ z_k = y_k - \frac{f(y_k)}{f[y_k,x_k]+f[y_k,x_k,w_k](y_k-x_k)}, \\ x_{k+1} = z_k - f(z_k)(f[z_k, y_k] + f[z_k, y_k, x_k] \\ (z_k - y_k) + f[z_k, y_k, x_k, w_k](z_k - y_k) \\ (z_k - x_k))^{-1}, \end{cases} \tag{3.24}$$

besides a family of 4-step with memory (TKM28)

$$\begin{cases} \gamma_k = -\frac{1}{N_5'(x_{k+1})}, \; p_k = \frac{N_5''(w_{k+1})}{-2N_5'(w_{k+1})} \\ w_k = x_k + \gamma_k f(x_k), \; k = 0, 1, 2, \cdots, \\ y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]+p_k f(w_k)}, \\ z_k = y_k - \frac{f(y_k)}{f[y_k,x_k]+f[y_k,x_k,w_k](y_k-x_k)}, \\ u_k = z_k - f(z_k)(f[z_k, y_k] + f[z_k, y_k, x_k] \\ (z_k - y_k) + f[z_k, y_k, x_k, w_k](z_k - y_k) \\ (z_k - x_k))^{-1}, \; x_{k+1} = u_k - f(u_k) \\ (f[u_k, z_k] + f[u_k, z_k, y_k](u_k - z_k)+ \\ f[u_k, z_k, y_k, x_k](u_k - z_k)(u_k - y_k) \\ +f[u_k, z_k, y_k, x_k, w_k](u_k - z_k) \\ (u_k - y_k)(u_k - x_k))^{-1} \end{cases} \tag{3.25}$$

and so on

$$\begin{cases} \gamma_k = -\frac{1}{N_{k+1}'(y_k)}, \; p_k = \frac{N_{k+1}''(y_0)}{-2N_{k+1}'(y_0)}, \; \gamma \in \mathbb{R}, \\ y_{-1} = x_k + \gamma_k f(x_k), \; k = 0, 1, 2, \cdots, \\ y_0 = x_k, \; y_1 = y_0 - \frac{f(y_0)}{f[y_0,y_{-1}]+p_k f(y_{-1})}, \; (TM2^n) \\ y_2 = y_1 - \frac{f(y_1)}{f[y_1,y_0]+f[y_1,y_0,y_{-1}](y_1-y_0)}, \\ \vdots \\ y_k = y_{k-1} - f(y_{k-1})(f[y_{k-1}, y_{k-2}] + \cdots + \\ f[y_{k-1} \ldots, y_{-1}](y_{k-1} - y_{k-2}) \cdots (y_{k-1} - y_{-1}))^{-1}. \end{cases} \tag{3.26}$$

where

$$N_2(t) = f(x_k) + f[x_k, x_{k-1}](t - x_k)+ f[x_k, w_{k-1}, x_{k-1}](t - x_k)(t - w_{k-1}),$$
$$N_3(t) = f(x_k) + f[x_k, x_{k-1}](t - x_k)+ f[x_k, w_{k-1}, x_{k-1}](t - x_k)(t - w_{k-1}) + f[x_k, y_{k-1}, w_{k-1}, x_{k-1}](t - x_k)(t - y_{k-1})(t - w_{k-1}),$$
$$N_4(t) = f(x_k) + f[x_k, x_{k-1}](t - x_k)+ f[x_k, w_{k-1}, x_{k-1}](t - x_k)(t - w_{k-1}) + f[x_k, y_{k-1}, w_{k-1}, x_{k-1}](t - x_k)(t - y_{k-1})(t - w_{k-1}) + f[x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1}](t - x_k)(t - z_{k-1})(t - y_{k-1})(t - w_{k-1}),$$
$$N_5(t) = f(x_k) + f[x_k, x_{k-1}](t - x_k)+ f[x_k, w_{k-1}, x_{k-1}](t - x_k)(t - w_{k-1})+ f[x_k, y_{k-1}, w_{k-1}, x_{k-1}](t - x_k)(t - y_{k-1})(t - w_{k-1}) + f[x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1}](t - x_k)(t - z_{k-1})(t - y_{k-1})(t - w_{k-1})+ f[x_k, u_{k-1}, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1}](t - x_k)(t - u_{k-1})(t - z_{k-1})(t - y_{k-1})(t - w_{k-1}).$$

In the convergence analysis of the new methods, we employ the notation used in Traub's book [15]: if m_k and n_k are null sequences and $m_k/n_k \to C$, where C is a non-zero constant, we shall write $m_k = O(n_k)$ or $m_k \sim Cn_k$. We also use the concept of R-order of convergence introduced by Ortega and Rheinboldt [22]. Next theorem fully describes the convergence order of general with memory methods (3.22), (3.23), (3.24), (3.25), and (3.26) along with the given errors Eqs. (2.17), (2.18), (2.19), (2.20), and (2.21). Now, we denote : $e_k = x_k - \alpha$, $e_{k,u} = u_k - \alpha$, $e_{k,z} = z_k - \alpha$, $e_{k,y} = y_k - \alpha$, $e_{k,w} = w_k - \alpha$. where α is the exact root. We first prove the following lemma:

Lemma 3.1 *The following estimates are satisfied:*

$$\begin{cases} (1+\gamma_k f'(\alpha)) \sim (p_k + c_2) \sim e_{k-1}e_{k-1,w}, \\ for \;\; Eq.\ (3.22). \\ (1+\gamma_k f'(\alpha)) \sim (p_k + c_2) \sim e_{k-1}e_{k-1,w} \\ e_{k-1,y},\ for \;\; Eq.\ (3.23) \\ (1+\gamma_k f'(\alpha)) \sim (p_k + c_2) \sim e_{k-1}e_{k-1,w} \\ e_{k-1,y}e_{k-1,z},\ \ for \;\; Eq.\ (3.24). \\ (1+\gamma_k f'(\alpha)) \sim (p_k + c_2) \sim e_{k-1}e_{k-1,w} \\ e_{k-1,y}e_{k-1,z}e_{k-1,u},\ \ for,\ Eq.\ (3.25). \\ (1+\gamma_k f'(\alpha)) \sim (p_k + c_2) \sim e_{0-1,y}e_{-1-1,y} \\ e_{1-1,y}e_{2-1,y}\cdots e_{k-1,y},\ for,\ Eq.\ (3.26). \end{cases} \tag{3.27}$$

Proof: Proof of relationships $(1 + \gamma_k f'(\alpha)) \sim e_{k-1}e_{k-1,w}$ and $(p_k + c_2) \sim e_{k-1}e_{k-1,w}$, for Eq. (2.18):

$$N_2'(x_k) = f[x_k, x_{k-1}] + f[x_k, w_{k-1}, x_{k-1}](x_k - x_{k-1}) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} + \frac{f(x_k) - f(w_{k-1})}{x_k - w_{k-1}} - \frac{f(w_{k-1}) - f(x_{k-1})}{w_{k-1} - x_{k-1}}$$
$$= f'(\alpha)((e_k - e_{k-1} + c_2(e_k^2 - e_{k-1}^2)+ c_3(e_k^3 - e_{k-1}^3) + ...)(e_k - e_{k-1})^{-1} + (e_k - e_{k-1,w} + c_2(e_k^2 - e_{k-1,w}^2)+ c_3(e_k^3 - e_{k-1,w}^3) + ...)(e_k - e_{k-1,w} - (e_{k-1,w} - e_{k-1} + c_2(e_{k-1,w}^2 - e_{k-1}^2)+ c_3(e_{k-1,w}^3 - e_{k-1}^3) + ...)(e_{k-1,w} - e_{k-1})^{-1}]$$
$$= f'(\alpha)(1 + 2c_2e_k + c_3(2e_k^2 - e_{k-1}e_{k-1,w} + e_ke_{k-1} + e_ke_{k-1,w}) + ...)$$
$$\sim f'(\alpha)(1 - c_3e_{k-1}e_{k-1,w}),$$

therefore, $(1 + \gamma_k f'(\alpha)) \sim e_{k-1}e_{k-1,w}$.
Suppose that there are $s + 1$ nodes $t_0, t_1, \cdots, t_s$ from the interval $D = [a, b]$, where a is the minimum and b is the maximum of these nodes, respectively. Then the error of Newton's interpolation polynomial $N_s(t)$ of degree s is given by

$$f(t) - N_s(t) = \frac{f^{(s+1)}(\alpha)}{(s+1)!}\prod_{j=0}^{s}(t - t_j). \tag{3.28}$$

For $s = 2$ the above equation assumes the form (keeping in the mind $t_0 = x_{k-1}$, $t_1 = w_{k-1}$, $t_2 =$

x_k)

$$f(t)-N_2(t)=\frac{f^{(3)}(\alpha)}{3!}(t-x_k)(t-x_{k-1})(t-w_{k-1}). \quad (3.29)$$

Using above result and with respect to t and putting $t=x_k$, we get

$$f'(x_k)-N_2'(x_k)=\frac{f'''(\alpha)}{3!}(x_k-x_{k-1})(x_k-w_{k-1}). \quad (3.30)$$

Now

$$x_k-x_{k-1}=(x_k-\alpha)-(x_{k-1}-\alpha)=e_k-e_{k-1}.$$

Similarly

$$x_k-w_{k-1}=e_k-e_{k-1,w}.$$

Substituting these relations in Eq. (3.30) and simplifying we get

$$N_2'(x_k)=f'(\alpha)(1+2c_2e_k+3c_3e_k^2+...)- \frac{f'''(\alpha)}{3!}(e_k-e_{k-1})(e_k-x_{k-1,w}) \sim f'(\alpha)(1+2c_2e_k-c_3e_{k-1}e_{k-1,w}). \quad (3.31)$$

And thus

$$1+f'(\alpha)\gamma_k\sim 1-\frac{1}{1+2c_2e_k-c_3e_{k-1}e_{k-1,w}} \sim e_{k-1}e_{k-1,w}. \quad (3.32)$$

For $s=3$ the above equation(3.28) assumes the form (keeping in the mind $t_0=x_{k-1}$, $t_1=w_{k-1}$, $t_2=x_k$, $t_3=w_k$)

$$f(t)-N_3(t)=\frac{f^{(4)}(\alpha)}{4!}(t-w_k)(t-x_k)(t-w_{k-1})(t-x_{k-1}). \quad (3.33)$$

Using above result and with respect to t and putting $t=w_k$, we get

$$f'(w_k)-N_3'(w_k)=\frac{f^{(4)}(\alpha)}{4!}(w_k-x_k)(w_k-w_{k-1})(w_k-x_{k-1}). \quad (3.34)$$

Now

$$w_k-x_k=(w_k-\alpha)-(x_k-\alpha)=e_{k,w}-e_k.$$

Similarly

$$w_k-x_{k-1}=e_{k,w}-e_{k-1},$$
$$w_k-w_{k-1}=e_{k,w}-e_{k-1,w}.$$

Substituting these relations in Eq. (3.34) and simplifying we get

$$N_3'(w_k)=f'(w_k)-\frac{f^{(4)}(\alpha)}{4!}(w_k-x_k)(w_k-w_{k-1})(w_k-x_{k-1}) =f'(\alpha)(1+2c_2e_{k,w}+3c_3e_{k,w}^2+...)-\frac{f^{(4)}(\alpha)}{4!}(e_{k,w}-e_k)(e_{k,w}-e_{k-1,w})(e_{k,w}-e_{k-1}) \sim f'(\alpha)(1+2c_2e_{k,w}+c_4e_ke_{k-1,w}e_{k-1}). \quad (3.35)$$

Also,

$$N_3''(w_k)=f''(w_k)-\frac{2f^{(4)}(\alpha)}{4!}[(w_k-x_k)(w_k-w_{k-1})+(w_k-x_k)(w_k-x_{k-1})+(w_k-x_{k-1})(w_k-w_{k-1})] \sim f''(\alpha)(1+\frac{3c_3}{2c_2}e_{k,w}-\frac{c_4}{c_2}e_{k-1,w}e_{k-1}) \quad (3.36)$$

By dividing the relation (3.36) to (3.35), a result is obtained

$$\frac{N_3''(w_k)}{2N_3'(w_k)}\sim\frac{1}{2}\frac{f''(\alpha)(1+\frac{3c_3}{2c_2}e_{k,w}-\frac{c_4}{c_2}e_{k-1,w}e_{k-1})}{f'(\alpha)(1+2c_2e_{k,w}+c_4e_ke_{k-1,w}e_{k-1})} \sim\frac{f''(\alpha)}{2f'(\alpha)}(1-\frac{c_4}{c_2}e_{k-1,w}e_{k-1})\sim e_{k-1,w}e_{k-1}. \quad (3.37)$$

Then

$$c_2+p_k=\frac{f''(\alpha)}{2f'(\alpha)}\sim-\frac{N_3''(w_k)}{2N_3'(w_k)} \sim\frac{c_4}{c_2}e_{k-1,w}e_{k-1}\sim e_{k-1,w}e_{k-1} \quad (3.38)$$

Proof of relationships $(1+\gamma_kf'(\alpha))\sim(p_k+c_2)\sim e_{k-1}e_{k-1,w}e_{k-1,y}$, *for Eq.* (2.19).

For $s=3$ the above equation(3.28) assumes the form (keeping in the mind $t_0=x_{k-1}$, $t_1=w_{k-1}$, $t_2=y_{k-1}$, $t_3=x_k$)

$$f(t)-N_3(t)=\frac{f^{(4)}(\alpha)}{4!}(t-x_k)(t-y_{k-1})(t-w_{k-1})(t-x_{k-1}). \quad (3.39)$$

Using above result and with respect to t and putting $t = x_k$, we get

$$f'(x_k) - N_3'(x_k) = \frac{f^{(4)}(\alpha)}{4!}(x_k - x_{k-1})(x_k - w_{k-1})(x_k - y_{k-1}). \quad (3.40)$$

Now

$$x_k - x_{k-1} = (x_k - \alpha) - (x_{k-1} - \alpha) = e_k - e_{k-1}.$$

Similarly

$$x_k - w_{k-1} = e_k - e_{k-1,w},$$
$$x_k - y_{k-1} = e_k - e_{k-1,y}.$$

Substituting these relations in Eq. (3.40) and simplifying we get

$$N_3'(x_k) = f'(\alpha)(1 + 2c_2e_k + 3c_3e_k^2 + ...) - \frac{f^{(4)}(\alpha)}{4!}(e_k - e_{k-1})(e_k - x_{k-1,w})(e_k - x_{k-1,y}) \sim f'(\alpha)(1 + 2c_2e_k + c_4e_{k-1}e_{k-1,w}e_{k-1,y}). \quad (3.41)$$

And thus

$$1 + f'(\alpha)\gamma_k \sim 1 - \frac{1}{1 + 2c_2e_k + c_4e_{k-1}e_{k-1,w}e_{k-1,y}} \sim e_{k-1}e_{k-1,w}e_{k-1,y}. \quad (3.42)$$

For $s = 4$ the above equation(3.28) assumes the form (keeping in the mind $t_0 = x_{k-1}$, $t_1 = w_{k-1}$, $t_2 = y_{k-1}$, $t_3 = x_k$ $t_4 = w_k$).

$$f(t) - N_4(t) = \frac{f^{(5)}(\alpha)}{5!}(t - w_k)(t - x_k)(t - w_{k-1})(t - x_{k-1}). \quad (3.43)$$

Using above result and with respect to t and putting $t = w_k$, we get

$$f'(w_k) - N_4'(w_k) = \frac{f^{(5)}(\alpha)}{5!}(w_k - x_k)(w_k - y_{k-1})(w_k - w_{k-1})(w_k - x_{k-1}). \quad (3.44)$$

Now

$$w_k - x_k = (w_k - \alpha) - (x_k - \alpha) = e_{k,w} - e_k.$$

Similarly

$$w_k - x_{k-1} = e_{k,w} - e_{k-1},$$
$$w_k - w_{k-1} = e_{k,w} - e_{k-1,w},$$
$$w_k - y_{k-1} = e_{k,w} - e_{k-1,y}.$$

Substituting these relations in Eq. (3.44) and simplifying we get

$$N_4'(w_k) = f'(w_k) - \frac{f^{(5)}(\alpha)}{5!}(w_k - x_k)(w_k - y_{k-1})(w_k - w_{k-1})(w_k - x_{k-1}) = f'(\alpha)(1 + 2c_2e_{k,w} + 3c_3e_{k,w}^2 + ...) - \frac{f^{(5)}(\alpha)}{5!}(e_{k,w} - e_k)(e_{k,w} - e_{k-1,y})(e_{k,w} - e_{k-1,w})(e_{k,w} - e_{k-1}) \sim f'(\alpha)(1 + 2c_2e_{k,w} - c_5e_ke_{k-1,y}e_{k-1,w}e_{k-1}). \quad (3.45)$$

Also,

$$N_4''(w_k) = f''(w_k) - \frac{2f^{(5)}(\alpha)}{5!}[(w_k - x_k)(w_k - y_{k-1})(w_k - x_{k-1}) + (w_k - x_k)(w_k - y_{k-1})(w_k - w_{k-1}) + (w_k - x_k)(w_k - w_{k-1})(w_k - x_{k-1}) + (w_k - y_{k-1})(w_k - w_{k-1})(w_k - x_{k-1})] \sim f''(\alpha)(1 + \frac{3c_3}{2c_2}e_{k,w} + \frac{c_5}{c_2}e_{k-1,y}e_{k-1,w}e_{k-1}). \quad (3.46)$$

By dividing the relation (3.46) to (3.45), a result is obtained

$$\frac{N_4''(w_k)}{2N_4'(w_k)} \sim \frac{1}{2}\frac{f''(\alpha)(1 + \frac{3c_3}{2c_2}e_{k,w} + \frac{c_5}{c_2}e_{k-1,y}e_{k-1,w}e_{k-1})}{f'(\alpha)(1 + 2c_2e_{k,w} - c_5e_ke_{k-1,y}e_{k-1,w}e_{k-1})} \sim \frac{f''(\alpha)}{2f'(\alpha)}(1 + \frac{c_5}{c_2}e_{k-1,y}e_{k-1,w}e_{k-1}). \quad (3.47)$$

Then

$$c_2 + p_k = \frac{f''(\alpha)}{2f'(\alpha)} \sim -\frac{N_4''(w_k)}{2N_4'(w_k)} \sim \frac{c_5}{c_2}e_{k-1,y}e_{k-1,w}e_{k-1} \sim e_{k-1,y}e_{k-1,w}e_{k-1} \quad (3.48)$$

Proof of relationships $(1 + \gamma_kf'(\alpha)) \sim (p_k + c_2) \sim e_{k-1}e_{k-1,w}e_{k-1,y}e_{k-1,z}$, for Eq. (2.20).
For $s = 4$ the above equation(3.28) assumes the form (keeping in the mind $t_0 = x_{k-1}$, $t_1 = w_{k-1}$, $t_2 = y_{k-1}$, $t_3 = z_{k-1}$ $t_4 = x_k$)

$$f(t) - N_4(t) = \frac{f^{(5)}(\alpha)}{5!}(t - x_k)(t - z_{k-1})(t - y_{k-1})(t - w_{k-1})(t - x_{k-1}). \quad (3.49)$$

Using above result and with respect to t and putting $t = x_k$, we get

$$f'(x_k) - N_4'(x_k) = \frac{f^{(5)}(\alpha)}{5!}(x_k - x_{k-1})$$
$$(x_k - w_{k-1})(x_k - y_{k-1})(x_k - z_{k-1}) \qquad (3.50)$$

Now

$$x_k - x_{k-1} = (x_k - \alpha) - (x_{k-1} - \alpha) = e_k - e_{k-1}.$$

Similarly

$$x_k - w_{k-1} = e_k - e_{k-1,w},$$
$$x_k - y_{k-1} = e_k - e_{k-1,y},$$
$$x_k - z_{k-1} = e_k - e_{k-1,z}.$$

Substituting these relations in Eq. (3.50) and simplifying we get

$$N_4'(x_k) = f'(\alpha)(1 + 2c_2e_k + 3c_3e_k^2 + ...)-$$
$$\frac{f^{(5)}(\alpha)}{5!}(e_k - e_{k-1})(e_k - x_{k-1,w})(e_k - x_{k-1,y})$$
$$(e_k - x_{k-1,z}) \sim f'(\alpha)(1 + 2c_2e_k-$$
$$c_5e_{k-1}e_{k-1,w}e_{k-1,y}e_{k-1,z}) \qquad (3.51)$$

And thus

$$1 + f'(\alpha)\gamma_k \sim 1-$$
$$\frac{1}{1 + 2c_2e_k - c_5e_{k-1}e_{k-1,w}e_{k-1,y}e_{k-1,z}}$$
$$\sim e_{k-1}e_{k-1,w}e_{k-1,y}e_{k-1,z}. \qquad (3.52)$$

For $s = 5$ the above equation(3.28) assumes the form (keeping in the mind $t_0 = x_{k-1}$, $t_1 = w_{k-1}$, $t_2 = y_{k-1}$, $t_3 = z_{k-1}$, $t_4 = x_k$, $t_5 = w_k$)

$$f(t) - N_5(t) = \frac{f^{(6)}(\alpha)}{6!}(t - y_k)(t - w_k)$$
$$(t - x_k)(t - w_{k-1})(t - x_{k-1}). \qquad (3.53)$$

Using above result and with respect to t and putting $t = w_k$, we get

$$f'(w_k) - N_5'(w_k) = \frac{f^{(6)}(\alpha)}{6!}(w_k - x_k)(w_k-$$
$$z_{k-1})(w_k - y_{k-1})(w_k - w_{k-1})(w_k - x_{k-1}). \qquad (3.54)$$

Now

$$w_k - x_k = (w_k - \alpha) - (x_k - \alpha) = e_{k,w} - e_k.$$

Similarly

$$w_k - x_{k-1} = e_{k,w} - e_{k-1},$$
$$w_k - w_{k-1} = e_{k,w} - e_{k-1,w},$$
$$w_k - y_{k-1} = e_{k,w} - e_{k-1,y},$$
$$w_k - z_{k-1} = e_{k,w} - e_{k-1,z}.$$

Substituting these relations in Eq. (3.54) and simplifying we get

$$N_5'(w_k) = f'(w_k) - \frac{f^{(6)}(\alpha)}{6!}(w_k - x_k)$$
$$(w_k - z_{k-1})(w_k - y_{k-1})(w_k - w_{k-1})(w_k - x_{k-1})$$
$$= f'(\alpha)(1 + 2c_2e_{k,w} + 3c_3e_{k,w}^2 + ...) - \frac{f^{(6)}(\alpha)}{6!}$$
$$(e_{k,w} - e_k)(e_{k,w} - e_{k-1,z})(e_{k,w} - e_{k-1,y})$$
$$(e_{k,w} - e_{k-1,w})(e_{k,w} - e_{k-1}) \sim f'(\alpha)$$
$$(1 + 2c_2e_{k,w} + c_6e_ke_{k-1,z}e_{k-1,y}e_{k-1,w}e_{k-1}). \qquad (3.55)$$

Also,

$$N_5''(w_k) = f''(w_k) - \frac{2f^{(6)}(\alpha)}{6!}[(w_k - x_k)$$
$$(w_k - z_{k-1})(w_k - y_{k-1})(w_k - w_{k-1})$$
$$+ (w_k - x_k)(w_k - z_{k-1})(w_k - y_{k-1})$$
$$(w_k - x_{k-1}) + (w_k - x_k)(w_k - z_{k-1})$$
$$(w_k - w_{k-1})(w_k - x_{k-1}) + (w_k - x_k)$$
$$(w_k - y_{k-1})(w_k - w_{k-1})(w_k - x_{k-1})+$$
$$(w_k - z_{k-1})(w_k - y_{k-1})(w_k - w_{k-1})(w_k - x_{k-1})]$$
$$\sim f''(\alpha)(1 + \frac{3c_4}{2c_2}e_{k,w} + \frac{c_6}{c_2}e_{k-1,z}e_{k-1,y}e_{k-1,w}e_{k-1}). \qquad (3.56)$$

By dividing the relation (3.56) to (3.55), a result is obtained

$$\frac{N_5''(w_k)}{2N_5'(w_k)} \sim \frac{1}{2}$$
$$\frac{f''(\alpha)(1 + \frac{3c_4}{2c_2}e_{k,w} + \frac{c_6}{c_2}e_{k-1,z}e_{k-1,y}e_{k-1,w}e_{k-1})}{f'(\alpha)(1 + 2c_2e_{k,w} + c_6e_ke_{k-1,z}e_{k-1,y}e_{k-1,w}e_{k-1})}$$
$$\sim \frac{f''(\alpha)}{2f'(\alpha)}(1 + \frac{c_6}{c_2}e_{k-1,z}e_{k-1,y}e_{k-1,w}e_{k-1}). \qquad (3.57)$$

Then

$$c_2 + p_k = \frac{f''(\alpha)}{2f'(\alpha)} \sim -\frac{N_5''(w_k)}{2N_5'(w_k)} \sim \frac{c_6}{c_2} e_{k-1,z} e_{k-1,y} e_{k-1,w} e_{k-1} \sim e_{k-1,z} e_{k-1,y} e_{k-1,w} e_{k-1}. \quad (3.58)$$

Similarly, other results can be proved.□

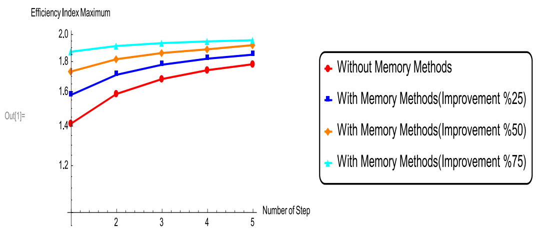

Figure 1: Comparison methods.

Theorem 3.1 *Let the function f be sufficiently differentiable in a neighborhood of its simple zero α. If an initial approximation x_0 is sufficiently close to α. Then, R-order of convergence of the one-, two-, three- and four-step methods (3.22), (3.23), (2.20), (3.25) and (3.26) with memory is 3.5, 7, 14, 28 and $3.5 * 2^{n-1}$ respectively.*

Proof: First, we assume that the R-order of convergence of sequence x_k, w_k, y_k, z_k and u_k are at least r, r_1, r_2, r_3, and, r_4, respectively. Hence

$$e_{k+1} \sim e_k^r \sim (e_{k-1}^r)^r \sim e_{k-1}^{r^2}, \quad (3.59)$$

and

$$e_{k,w} \sim e_k^{r_1} \sim (e_{k-1}^r)^{r_1} \sim e_{k-1}^{rr_1}, \quad (3.60)$$

similarly

$$e_{k,y} \sim e_k^{r_2} \sim (e_{k-1}^r)^{r_2} \sim e_{k-1}^{rr_2}, \quad (3.61)$$

$$e_{k,z} \sim e_k^{r_3} \sim (e_{k-1}^r)^{r_3} \sim e_{k-1}^{rr_3}, \quad (3.62)$$

$$e_{k,u} \sim e_k^{r_4} \sim (e_{k-1}^r)^{r_4} \sim e_{k-1}^{rr_4}, \quad (3.63)$$

Now, we will prove the results or each method separately as follows:

(i) Modified scheme for Eq. (3.22)
It can be derived that

$$e_{k,w} \sim (1 + f'(\alpha)\gamma_k)e_k, \quad (3.64)$$

$$e_{k+1} \sim (1 + f'(\alpha)\gamma_k)(p_k + c_2)e_k^2. \quad (3.65)$$

Using the results of lemma (3.1) in the Eqs. (3.64) and (3.65), we obtain

$$e_{k,w} \sim e_{k-1}^{r_1+r+1}, \quad (3.66)$$

$$e_{k+1} \sim e_{k-1}^{2(r_1+r+1)}. \quad (3.67)$$

Now comparing the equal powers of e_{k-1} in (3.66) and (3.67), we get the following nonlinear system

$$\begin{cases} rr_1 - r_1 - r - 1 = 0, \\ r^2 - 2(r_1 + 1) - 2r = 0. \end{cases}$$

After solving these equations, we get $r_1 = \frac{1}{4}(3+\sqrt{17}) \simeq 1.78$, $r = \frac{1}{2}(3+\sqrt{17}) \simeq 3.56$. It confirms the convergence of method (3.22). It is the same method that was reviewed by Dzunic in 2013 (DM) [7].

(ii) Modified method II. For method (3.23), it can be derived that

$$e_{k,w} \sim (1 + f'(\alpha)\gamma_k)e_k, \quad (3.68)$$

$$e_{k,y} \sim (1 + f'(\alpha)\gamma_k)(p_k + c_2)e_k^2, \quad (3.69)$$

$$e_{k+1} \sim (1 + f'(\alpha)\gamma_k)^2(p_k + c_2)e_k^4. \quad (3.70)$$

Using the results of lemma (3.1) and the relationships (3.68)-(3.70), we have

$$e_{k,w} \sim e_{k-1}^{r_2+r_1+1+r}, \quad (3.71)$$

$$e_{k,y} \sim e_{k-1}^{2(r_2+r_1+1+r)}, \quad (3.72)$$

$$e_{k+1} \sim e_{k-1}^{3(r_2+r_1+1)+4r}. \quad (3.73)$$

Now comparing the equal powers of e_k in (3.71)-(3.73), we get the following nonlinear system

$$\begin{cases} rr_1 - r_2 - r_1 - 1 - r = 0, \\ rr_2 - 2(r_2 + r_1 + 1) - 2r = 0, \\ r^2 - 3(r_2 + r_1 + 1) - 4r = 0. \end{cases}$$

This system has the solutions $r_1 = 2$, $r_2 = 4$, $r = 7$. It confirms the convergence of method (3.23). The method was also reviewed by Cordero et. al in 2015 (CLBTM) [5].

Table 1: Comparison of the absolute errors and COC of proposed methods by other methods

$f_1(x) = x^3 + 4x^2 - 10, \alpha = 1.3652, x_0 = 1, \gamma_0 = p_0 = 0.1$					
Methods	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$	*COC*	*EI*
ZLHM2 (2.2)	0.27996(0)	0.79062(−1)	0.75362(−2)	2.0000	1.41421
ZLHM4 (2.3)	0.22980(−1)	0.30178(−4)	0.30013(−4)	4.0000	1.58740
ZLHM8 (2.4)	0.28396(−3)	0.30013(−4)	0.30013(−4)	8.0000	1.68179
ZLHM16 (2.5)	0.36520(0)	0.30045(−4)	0.30013(−4)	16.0000	1.74110
TM (2.12)	0.27996(0)	0.64692(−2)	0.32600(−4)	2.4693	1.57140
TLAM3 (3.16),[30]	0.36520(0)	0.27996(0)	0.28623(−2)	3.0000	1.73205
TLAM6 (3.17),[30]	0.36520(0)	0.22980(−1)	0.30013(−4)	6.0000	1.81712
TLAM12 (3.18),[30]	0.36520(0)	0.28396(−3)	0.30013(−4)	12.0000	1.86121
TLAM24 (3.19),[30]	0.36520(0)	0.30045(−4)	0.30013(−4)	24.0000	1.88818
DM (3.22)	03623(0)	0.36337(0)	0.31043(−3)	3.7314	1.93168
CLBTM (3.23)	0.36523(0)	0.38342(−1)	0.10130(−12)	7.0000	1.91293
TKM14 (3.24)	0.36523(0)	0.70101(−3)	0.18334(−28)	14.0000	1.93434

(iii) Modified scheme for Eq. (3.24), it can be derived that

$$e_{k,w} \sim (1 + f'(\alpha)\gamma_k)e_k, \tag{3.74}$$

$$e_{k,y} \sim (1 + f'(\alpha)\gamma_k)(p_k + c_2)e_k^2, \tag{3.75}$$

$$e_{k,z} \sim (1 + f'(\alpha)\gamma_k)^2(p_k + c_2)e_k^4, \tag{3.76}$$

$$e_{k+1} \sim (1 + f'(\alpha)\gamma_k)^4(p_k + c_2)^2 e_k^8, \tag{3.77}$$

Using the results of lemma (3.1) in the Eqs. (3.74)-(3.77), we conclude

$$e_{k,w} \sim e_{k-1}^{r_3+r_2+r_1+1+r}, \tag{3.78}$$

$$e_{k,y} \sim e_{k-1}^{2(r_3+r_2+r_1+1)+2r}, \tag{3.79}$$

$$e_{k,z} \sim e_{k-1}^{3(r_3+r_2+r_1+1)+4r}, \tag{3.80}$$

$$e_{k+1} \sim e_{k-1}^{6(r_3+r_2+r_1+1)+8r}. \tag{3.81}$$

Now comparing the equal powers of e_k in (3.78)-(3.81), we get the following nonlinear system

$$\begin{cases} rr_1 - r_3 - r_2 - r_1 - 1 - r = 0, \\ rr_2 - 2(r_3 + r_2 + r_1 + 1) - 2r = 0, \\ rr_3 - 3(r_3 + r_2 + r_1 + 1) - 4r = 0, \\ r^2 - 6(r_3 + r_2 + r_1 + 1) - 8r = 0. \end{cases}$$

After solving these equations, we get $r_1 = 2$, $r_2 = 4$, $r_3 = 7$, $r = 14$. It confirms the convergence of method (3.24).

(iv) Modified scheme for Eq. (3.25), it can be derived that

$$e_{k,w} \sim (1 + f'(\alpha)\gamma_k)e_k, \tag{3.82}$$

$$e_{k,y} \sim (1 + f'(\alpha)\gamma_k)(p_k + c_2)e_k^2, \tag{3.83}$$

$$e_{k,z} \sim (1 + f'(\alpha)\gamma_k)^2(p_k + c_2)e_k^4, \tag{3.84}$$

$$e_{k,u} \sim (1 + f'(\alpha)\gamma_k)^4(p_k + c_2)^2 e_k^8, \tag{3.85}$$

$$e_{k+1} \sim (1 + f'(\alpha)\gamma_k)^8(p_k + c_2)^4 e_k^{16}. \tag{3.86}$$

Using the results of lemma (3.1) in the Eqs. (3.82)-(3.86), we deduce

$$e_{k,w} \sim e_{k-1}^{r_4+r_3+r_2+r_1+1+r}, \tag{3.87}$$

$$e_{k,y} \sim e_{k-1}^{2(r_4+r_3+r_2+r_1+1)+2r}, \tag{3.88}$$

$$e_{k,z} \sim e_{k-1}^{3(r_4+r_3+r_2+r_1+1)+4r}, \tag{3.89}$$

$$e_{k,u} \sim e_{k-1}^{6(r_4+r_3+r_2+r_1+1)+8r}, \tag{3.90}$$

$$e_{k+1} \sim e_{k-1}^{12(r_4+r_3+r_2+r_1+1)+16r}. \tag{3.91}$$

Table 1. Coninue.

$f_2(x) = \log(1+x^2) + e^{x^2-3x}\sin x, \alpha = 0, x_0 = 0.5, \gamma_0 = p_0 = 0.1$					
ZLHM2 (2.2)	0.50000(0)	0.42599(−1)	0.36497(−2)	2.0000	1.41421
ZLHM4 (2.3)	0.50000(0)	0.20643(−6)	0.17972(−24)	4.0000	1.58740
ZLHM8 (2.4)	0.50000(0)	0.17931(−3)	0.40313(−28)	8.0000	1.68179
ZLHM16 (2.5)	0.50000(0)	0.42775(−6)	0.58997(−92)	15.0000	1.71877
TM (2.12)	0.42599(−1)	0.11207(−2)	0.20649(−6)	2.4196	1.55551
TLAM3 (3.16),[30]	0.50000(0)	0.42599(−1)	0.17284(−2)	3.0000	1.73205
TLAM6 (3.17),[30]	0.50000(0)	0.20643(−1)	0.33649(−10)	6.0000	1.81712
TLAM12 (3.18),[30]	0.50000(0)	0.17931(−3)	0.12661(−41)	12.0000	1.86121
TLAM24 (3.19),[30]	0.35000(0)	0.34372(−11)	0.56903(−272)	24.0000	1.88818
DM (3.22)	0.50000(0)	0.64108(−1)	0.12721(−2)	3.5505	1.88428
CLBTM (3.23)	0.50000(0)	0.29521(−1)	0.14548(−9)	7.0000	1.91293
TKM14 (3.24)	0.50000(0)	0.42197(−3)	0.16707(−42)	14.0000	1.93434
$f_3(x) = x\log(1+x\sin(x)) + e^{-1+x^2+x\cos(x)}\sin\pi x, \alpha = 0, x_0 = 0.6, \gamma_0 = p_0 = 0.1$					
ZLHM2 (2.2)	0.60000(0)	0.47811(0)	0.92238(−1)	2.0000	1.41421
ZLHM4 (2.3)	0.60000(0)	0.23476(0)	0.11818(−2)	4.0000	1.58740
ZLHM8 (2.4)	0.60000(0)	0.35452(−1)	0.46141(−12)	8.0000	1.68179
ZLHM16 (2.5)	0.60000(0)	0.12209(−2)	0.23588(−44)	16.0000	1.74110
TM (2.12)	0.47811(0)	0.56230(−1)	0.12602(−2)	2.4825	1.57560
TLAM3 (3.16),[30]	0.60000(0)	0.47811(0)	0.69702(−3)	3.0000	1.73205
TLAM6 (3.17),[30]	0.60000(0)	0.23476(0)	0.82605(−5)	6.0000	1.81712
TLAM12 (3.18),[30]	0.60000(0)	0.35452(−1)	0.92939(−19)	12.0000	1.86121
TLAM24 (3.19),[30]	0.60000(0)	0.77939(−3)	0.17667(−74)	24.0000	1.88818
DM (3.22)	0.60000(0)	0.36450(0)	0.54166(−1)	3.6076	1.89937
CLBTM (3.23)	0.60000(0)	0.18448(0)	0.50259(−5)	7.0000	1.91293
TKM14 (3.24)	0.36520(0)	0.18799(−1)	0.25194(−25)	14.0000	1.93434

Now comparing the equal powers of e_{k-1} in (3.87)-(3.91) we get the following nonlinear system

$$\begin{cases} rr_1 - r_4 - r_3 - r_2 - r_1 - 1 - r = 0, \\ rr_2 - 2(r_4 + r_3 + r_2 + r_1 + 1) - 2r = 0, \\ rr_3 - 3(r_4 + r_3 + r_2 + r_1 + 1) - 4r = 0, \\ rr_4 - 6(r_4 + r_3 + r_2 + r_1 + 1) - 8r = 0, \\ r^2 - 12(r_4 + r_3 + r_2 + r_1 + 1) - 16r = 0. \end{cases}$$

After solving these equations, we get $r_1 = 2$, $r_2 = 4$, $r_3 = 7$, $r_4 = 14$, $r = 28$. It confirms the convergence of method (3.25).

(v) Modified scheme for Eq. (3.26), it can be derived that

$$e_{-1,y} \sim (1 + f'(\alpha)\gamma_k)e_{0,y}, \tag{3.92}$$

$$e_{1,y} \sim (1 + f'(\alpha)\gamma_k)(p_k + c_2)e_{0,y}^2, \tag{3.93}$$

$$e_{2,y} \sim (1 + f'(\alpha)\gamma_k)^2(p_k + c_2)e_{0,y}^4, \tag{3.94}$$

$$e_{k-1,y} \sim (1 + f'(\alpha)\gamma_k)^{2^{k-1}}(p_k + c_2)^{2^{k-2}}e_{0,y}^{2^{k-1}}, \tag{3.95}$$

$$e_{k,y} \sim (1 + f'(\alpha)\gamma_k)^{2^{k-1}}(p_k + c_2)^{2^{k-2}}e_{0,y}^{2^k}. \tag{3.96}$$

Table 2: Comparison improvment of convergence order the proposed method with other schemes

with memory methods	number of steps	optimal order	p	percentage increase
CLBTM[5]	2	4.000	7.000	%75
DM[7]	1	2.000	3.500	%75
DPM[8]	1	2.000	3.000	%50
DPPM[9]	3	8.000	11.000	%37.5
EM[10]	3	8.000	12.000	%50
KKBM[14]	2	4.000	7.000	%75
LMMJM [17]	3	4.000	6.000	%50
LSSAKM[18]	3	8.000	12.000	%50
LMNKSM[19]	3	8.000	12.000	%50
LSGAM[20]	3	8.000	12.000	%50
LTM[21]	3	8.000	12.000	%50
SSSM[26]	3	8.000	12.000	%50
TrM[32]	1	2.000	2.410	%20.5
WM[33]	2	4.000	4.235	%5.75
TKM14(3.24)	3	8.000	14.000	%75
TKM28(3.25)	4	16.000	28.000	%75

Table 3: Comparison efficiency index of proposed method by with and without memory methods

without memory methods	EF	p	EI	with memory methods	EF	p	EI
without memory methods	EF	p	EI	with memory methods	EF	p	EI
AM[1]	3	3.000	1.44225	CLKTM[3]	3	6.000	1.81712
CTM[4]	3	4.000	1.58740	CLBTM[5]	3	7.000	1.91293
DHM[6]	3	3.000	1.44225	DM[7]	3	3.5600	1.88680
FGDM[12]	4	7.000	1.62658	DPM [8]	2	3.000	1.73205
GKM[13]	5	16.00	1.74110	DPPM[9]	4	11.000	1.82116
OM[23]	3	4.000	1.58740	EM[10]	4	12.000	1.86121
RWBM[25]	3	4.000	1.58740	KKBM[14]	3	7.000	1.91293
SSSM[26]	4	8.000	1.68179	LAM [17]	3	6.000	1.81712
SM[22]	2	2.000	1.41421	LSSAKM [18]	4	12.000	1.86121
ZLHM[34]	2	2.000	1.41421	LSGAM[20]	4	12.000	1.86121
ZLHM[34]	3	4.000	1.44225	SSGM[27]	4	12.000	1.86121
ZLHM[34]	4	8.000	1.58740	SLTKM[28]	4	12.000	1.86121
ZLHM[34]	5	16.000	1.74110	WM[33]	3	4.235	1.61790
ACCTM[2]	4	8.000	1.58740	TKM14(3.24)	4	14.000	1.93434
LMMWM[16]	5	16.000	1.74110	TKM28(3.25)	5	28.000	1.94729

Using the results of lemma (3.1) in the Eqs. (3.92)-(3.96), we deduce

$$e_{-1,0} \sim e_{k-1}^{r_k+r_{k-1}+r_{k-2}+\cdots+r_1+1+r}, \tag{3.97}$$

$$e_{1,y} \sim e_{k-1}^{2(r_k+r_{k-1}+r_{k-2}+\cdots+r_1+1)+2r}, \tag{3.98}$$

$$e_{2,y} \sim e_{k-1}^{3(r_k+r_{k-1}+r_{k-2}+\cdots+r_1+1)+4r}, \tag{3.99}$$

$$\vdots$$

$$e_{k-1,y} \sim e_{k-1}^{3*2^{k-2}(r_k+r_{k-1}+r_{k-2}+\cdots+r_1+1)+2^{k-1}r}, \tag{3.100}$$

$$e_{k,y} \sim e_{k-1}^{3*2^{k-1}(r_k+r_{k-1}+r_{k-2}+\cdots+r_1+1)+2^{k}r}. \tag{3.101}$$

Now comparing the equal powers of e_{k-1} in (3.97)-(3.101) we get the following nonlinear system

$$\begin{cases} rr_1-(r_k+r_{k-1}+r_{k-2}+\cdots+ \\ r_1+1)-r=0, \\ rr_2-2(r_k+r_{k-1}+r_{k-2}+\cdots+ \\ r_1+1)-2r=0, \\ rr_3-3(r_k+r_{k-1}+r_{k-2}+\cdots+ \\ r_1+1)-4r=0, \\ \vdots \\ rr_{k-1}-3*2^{k-2}(r_k+r_{k-1}+r_{k-2}+ \\ \cdots+r_1+1)-2^{k-1}r=0, \\ r_k^2-3*2^{k-1}(r_k+r_{k-1}+r_{k-2}+\cdots+ \\ r_1+1)-2^{k}r=0. \end{cases}$$

This system has the solutions $r_1=2$, $r_2=4$, $r_3=7$, $\cdots$, $r_{k-1}=3.5*2^{k-1}$, $r_k=3.5*2^{k}$. It confirms the convergence of method (3.26). □

Remark 3.1 *As can be easily seen that the improvement the order of convergence from 2 to 3.56 (75% of an improvement) is attained without any additional functional evaluations. Therefore, the efficiency index of the proposed method (3.22) is* $3.56^{1/2}=1.88680$.

Remark 3.2 *Improvement of the order of convergence from 4 to 7 shows an increase of 75% and this operation is achieved without any additional assessment. Therefore, the efficiency index of the proposed method (3.23) is* $7^{1/3}=1.91293$.

Remark 3.3 *Increasing the order of convergence from 8 to 14 without imposing a new assessment of functions indicates a 75% improvement in the efficiency index of the proposed method. Therefore, the efficiency index of the proposed method (3.24) is* $14^{1/4}=1.93434$.

Remark 3.4 *It can be easily seen that the improvement of the order of convergence from 16 to 28 (75% of an improvement) is attained without any additional functional evaluations, which points to very high computational efficiency of the proposed method. Therefore, the efficiency index of the proposed method (3.25) is* $28^{1/5}=1.94729$.

Remark 3.5 *It should be noted that if we only approximate parameter* γ *using the Newton's interpolation polynomial, we will achieve a 50% improvement in convergence order. the efficiency index of the proposed family with memory is* $EI=3^{1/2}=1.732050808$, $EI=6^{1/3}=1.817120593$, $EI=12^{1/4}=1.861209718$, $EI=24^{1/5}=1.888175023$ *as mentioned in [30].*

4 Numerical examples

The iterative methods without memory (2.2)-(2.5) and with memory methods (3.22)-(3.26) are used to solve nonlinear functions $f_k(x)=0$, $(k=1,2,3)$ and the computation results. The errors are determined $|x_k-\alpha|$ of approximations to the sought zeros, produced by the different methods at the first three iterations are given in Table 1 where, $a(-b)$ stands for $a\times10^{-b}$. Also, for each test function, the initial estimation values and the last value of the computational order of convergence, COC [4] computed by the expression

$$COC=\frac{\log|(x_{n+1}-x_n)/(x_n-x_{n-1})|}{\log|(x_n-x_{n-1})/(x_{n-1}-x_{n-2})|}, \tag{4.102}$$

and also the efficiency index (EI) are included. The package Mathematica 10, with 1000 arbitrary precision arithmetic, has been used in computations. The exact value of the simple root and its initial approximation are α and x_0 respectively. For a good comparison, we choose test functions from recent studies (e.g.[31]): $f_1(x)=x^3+4x^2-10$, $\alpha=1.3652$, $x_0=1$.

$f_2(x) = \log(1+x^2) + e^{-3x+x^2}\sin(x)$, $\alpha = 0$, $x_0 = 0.5$.
$f_3(x) = x\log(1 + x\sin(x)) + e^{-1+x^2+x\cos(x)}\sin\pi x$, $\alpha = 0$, $x_0 = 0.6$

5 Conclusion

In this paper, we have obtained a new class of with memory methods. The orders of convergence of the new methods with memory are $3.5, 7, 14$ and 28. Also, the Newton's interpolatory polynomials of 2 till 5 degrees are applied for constructing considerably faster methods employing information from the current and previous iteration without any additional evaluations of the function. The results show that this new methods is useful to find an acceptable approximation of the exact solution. The significant increase in the convergence speed of the proposed methods is attained without additional functional evaluations, which points to a very high computationally efficiency. In other words, the efficiency index of the proposed family with memory is $3.56^{1/2} = 1.886800000$, $7^{1/3} = 1.912931183$, $14^{1/4} = 1.934336420$, $28^{1/5} = 1.947294361$, which is much better than methods described in mentioned references and single-step optimal methods up to five steps without memory having efficiency indexes $2^{1/2} \simeq 1.414213562$, $4^{1/3} \simeq 1.587401052$, $8^{1/4} \simeq 1.681792831$, $16^{1/5} \simeq 1.741101127$, $32^{1/6} \simeq 1.781797436$, respectively.

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