Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 11, No. 3, 2019 Article ID IJIM-1273, 9 pages
Research Article

On the Approximate Solving of Dual Fuzzy Matrix Equations

Mohammad Asghari [*], Parastoo. Reihani [†], Mohsen Asghari [‡§], Shahriar Farahmand Rad [¶]

Received Date: 2018-09-02 Revised Date: 2019-03-07 Accepted Date: 2019-03-07

Abstract

In 2014, Gong et al. [Information Sciences 266 (2014) 112-133] proposed a simple method for solving dual fuzzy matrix equations in LR form. Later, Kaur et al, [Information Sciences 418-419 (2017) 184-185] showed that there is a technical flaw in their method and it is valid only for certain types of LR dual fuzzy matrix equations. The main aim of this paper is to eliminate this technical flaw and also correct some numerical results obtained by Gong et al.

Keywords : Fuzzy number; LR fuzzy number; Dual fuzzy matrix equation.

1 Introduction

Recently, Gong et al. [6] introduced a simple method of solving dual fuzzy matrix equation

$$A\widetilde{x} + \widetilde{B} = C\widetilde{x} + \widetilde{D}, \qquad (1.1)$$

where $A = (a_{ij})_{m\times n}$, $C = (c_{ij})_{m\times n}$ are arbitrary crisp matrices and $\widetilde{B} = (\widetilde{b_{ij}})_{m\times p}$, $\widetilde{D} = (\widetilde{d_{ij}})_{m\times p}$ and also the unknown matrix $\widetilde{x} = (\widetilde{x_{ij}})_{n\times p}$ are LR fuzzy number matrices. In their method, the LR dual fuzzy matrix equation (1.1) is converted into two classical matrix equations by using of the arithmetic operations on LR fuzzy numbers (see Theorem 3.1 of [6]). Then, the strong (weak) minimal LR fuzzy solutions of Eq. (1.1) are obtained via solving these two classical matrix equations by means of the generalized inverses of matrices [2, 3]. Unfortunately, Kaur and Kumar [7] showed that there is a technical flaw in the method proposed in [6], due to the following two reasons:

(i) if $\widetilde{x}$ is an LR fuzzy number and λ is a negative real number, then $\lambda\widetilde{x}$ will be an RL fuzzy number.

(ii) In general, based on the arithmetic operations on LR fuzzy numbers [1, 6, 7], an RL fuzzy number can not be added to an LR fuzzy number.

In fact, Gong et al. proved the basic theorem of their method (Theorem 3.1 of [6]) by considering the mathematical incorrect assumption that RL fuzzy numbers can be added to LR fuzzy numbers (see [7]). For this reason, Kaur and Kumar [7] stated that the method proposed in [6] is valid only if either A and C are non-negative real matrices or $L(x) = R(x)$ for all $x \in [0, 1]$.

[*]Department of Mathematics, Payame Noor University, Tehran, Iran.
[†]Department of Mathematics, Payame Noor University, Tehran, Iran.
[‡]Corresponding author. asghari2004@yahoo.com, Tel: +98(911)1292550.
[§]Department of Mathematics, Faculty of Sciences, Golestan University, Gorgan, Iran.
[¶]Department of Mathematics, Payame Noor University, Tehran, Iran.

In this paper, we eliminate the above technical flaw via correcting the proof of Theorem 3.1 of [6] by using the concept of r-cuts of fuzzy numbers. Moreover, we can conclude that the method proposed by Gong et al. [6] can be valid for any arbitrary LR dual fuzzy matrix equations. The important point is that if either A and C are not non-negative real matrices or $L(x) \neq R(x)$ for some or all values of x, then the method of [6] gives the **approximate** minimal LR fuzzy solution, which it can be strong or weak. Otherwise, the method of [6] gives the **accurate** minimal LR fuzzy solution for Eq. (1.1), similarly it can be strong or weak.

Furthermore, in this paper, it is shown that some numerical results obtained in [6] are incorrect. Here, we represent a modified version of them.

This paper is organized as follows. In Section 2, some basic definitions and concepts are given briefly. In Section 3, we present a correct proof of Theorem 3.1 of [6]. In Section 4 we modify some numerical results obtained in [6]. Section 5 ends this paper with conclusion.

2 Preliminaries

Definition 2.1 *[10, 6] A fuzzy number is a fuzzy set like $\widetilde{u} : \mathbb{R} \to [0, 1]$ which satisfies:*

(1) *$\widetilde{u}$ is upper semicontinuous,*

(2) *$\widetilde{u}$ is fuzzy convex, i.e., $\widetilde{u}(\lambda x + (1-\lambda)y) \geqslant \min\{\widetilde{u}(x), \widetilde{u}(y)\}$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$.*

(3) *$\widetilde{u}$ is normal, i.e., there exists $x_0 \in \mathbb{R}$ such that $\widetilde{u}(x_0) = 1$,*

(4) *$\overline{\{x \in \mathbb{R} | \widetilde{u}(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.*

It is known that, for all $r \in (0, 1]$, r-cuts of fuzzy number $\widetilde{u}$ are defined as $[\widetilde{u}]_r = \{x \in \mathbb{R} : \widetilde{u}(x) \geqslant r\}$ and also for $r = 0$ we have $[\widetilde{u}]_0 = \overline{\{x \in \mathbb{R} : \widetilde{u}(x) > 0\}}$. Then, from (1)-(4) it follows that $[\widetilde{u}]_r$ is a bounded closed interval for each $r \in [0, 1]$ [4]. In this paper, we denote the r-cuts of fuzzy number $\widetilde{u}$ as $[\widetilde{u}]_r = [\underline{u}(r), \overline{u}(r)]$, for each $r \in [0, 1]$.

Remark 2.1 *[8] Let*

$$\{[\underline{u}(r), \overline{u}(r)] \ : \ 0 \leqslant r \leqslant 1\},$$

be a given family of non-empty sets in $\mathbb{R}$. Then it represents the r-cuts of a fuzzy number $\widetilde{u}$ if and only if:

1) *The functions $\underline{u}$ and $\overline{u}$ be bounded over $[0, 1]$ and $\underline{u}(r) \leqslant \overline{u}(r)$ for each $r \in [0, 1]$.*

2) *The functions $\underline{u}$ and $\overline{u}$ be non-decreasing and non-increasing over $[0, 1]$, respectively.*

3) *The functions $\underline{u}$ and $\overline{u}$ be left-continuous over $[0, 1]$.*

Definition 2.2 *[9] Two fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$ are said to be equal, if and only if $[\widetilde{u}]_r = [\widetilde{v}]_r$, i.e., $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$, for each $r \in [0, 1]$.*

Definition 2.3 *For arbitrary two fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$, and $\lambda \in \mathbb{R}$, r-cuts of the sum $\widetilde{u} + \widetilde{v}$ and the product $\lambda \cdot \widetilde{u}$ are defined based on interval arithmetic as*

$$\begin{aligned} [\widetilde{u} + \widetilde{v}]_r &= [\widetilde{u}]_r + [\widetilde{v}]_r \\ &= \{x + y : x \in [\widetilde{u}]_r, \ y \in [\widetilde{v}]_r\} \\ &= [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)], \end{aligned} \quad (2.2)$$

$$\begin{aligned} [\lambda \cdot \widetilde{u}]_r &= \lambda \cdot [\widetilde{u}]_r \\ &= \{\lambda x : x \in [\widetilde{u}]_r\} \\ &= \begin{cases} [\lambda \underline{u}(r), \lambda \overline{u}(r)], & \lambda \geqslant 0, \\ [\lambda \overline{u}(r), \lambda \underline{u}(r)], & \lambda < 0. \end{cases} \end{aligned} \quad (2.3)$$

Now, we define a special case of fuzzy numbers, namely "LR fuzzy number". It should be noted that in throughout the paper, similar to the papers [6, 7], we consider LR fuzzy numbers that have a single mean value.

Definition 2.4 *[5, 6] A fuzzy number $\widetilde{u}$ is called an LR fuzzy number and denoted as $\widetilde{u} = (u, \alpha, \beta)_{LR}$ if*

$$\widetilde{u}(x) = \begin{cases} L(\frac{u-x}{\alpha}), & x \leqslant u, \quad \alpha > 0, \\ R(\frac{x-u}{\beta}), & u \leqslant x, \quad \beta > 0, \end{cases}$$

where u is the mean value of $\widetilde{u}$, and α and β are left and right spreads, respectively. Also, the functions $L(\cdot)$ and $R(\cdot)$ are called left and right shape functions, and fulfilling the following conditions:

(1) $L(x) = L(-x)$ *and* $R(x) = R(-x)$,

(2) $L(0) = R(0) = 1$ *and* $L(1) = R(1) = 0$,

(3) $L(\cdot)$ *and* $R(\cdot)$ *are non-increasing from* $[0, \infty)$ *to* $[0, 1]$.

In the following, we present the formulas for addition and scalar multiplication of LR fuzzy numbers, which was designed by Dubois et al. [5].

Definition 2.5 *For arbitrary two LR fuzzy numbers $\widetilde{u} = (u, \alpha, \beta)$ and $\widetilde{v} = (v, \gamma, \delta)$, and $\lambda \in \mathbb{R}$, have we*

$$\begin{aligned} \widetilde{u} + \widetilde{v} &= (u, \alpha, \beta)_{LR} + (v, \gamma, \delta)_{LR} \\ &= (u + v, \alpha + \gamma, \beta + \delta)_{LR}, \end{aligned}$$

$$\begin{aligned} \lambda \cdot \widetilde{u} &= \lambda \cdot (u, \alpha, \beta)_{LR} \\ &= \begin{cases} (\lambda u, \lambda \alpha, \lambda \beta)_{LR}, & \lambda \geqslant 0, \\ (\lambda u, -\lambda \beta, -\lambda \alpha)_{RL}, & \lambda < 0. \end{cases} \end{aligned}$$

Remark 2.2 *By the formulas presented in Definition 2.5, it is clear that an LR fuzzy number can not be added to an RL fuzzy number. In fact, in this case we have*

$$(u, \alpha, \beta)_{LR} + (v, \gamma, \delta)_{RL} = (u + v, \alpha + \gamma, \beta + \delta)_{L'R'},$$

where the shape functions $L'(\cdot)$ and $R'(\cdot)$ satisfy the conditions (1)-(4) of Definition 2.4. These shape functions are obtained from the shape functions $L(\cdot)$ and $R(\cdot)$. In general, it is difficult to obtain the shape functions $L'(\cdot)$ and $R'(\cdot)$.

Remark 2.3 *Let us consider arbitrary LR fuzzy number $\widetilde{u} = (u, \alpha, \beta)_{LR}$ with the r-cuts $[\widetilde{u}]_r = [\underline{u}(r), \overline{u}(r)]$. Then, we will have*

$$\underline{u}(0) = u - \alpha, \qquad \overline{u}(0) = u + \beta,$$

$$\underline{u}(1) = \overline{u}(1) = u.$$

In continuation, we assume that the reader is familiar with some basic definitions, for example: LR fuzzy matrix, generalized inverses of matrix, dual fuzzy matrix equation and strong (weak) minimal LR fuzzy solution (for more details, see [6]). In the next section, we present a correct proof of Theorem 3.1 of [6] based on the r-cuts of fuzzy numbers.

3 Eliminating of technical flaw

The method proposed by Gong et al. [6] was based on Theorem 3.1 of their paper. In this theorem, they used arithmetic operation on LR fuzzy number and converted the dual fuzzy matrix equation into two crisp matrix equations. Unfortunately, they proved Theorem 3.1 of their paper by considering the mathematical incorrect assumption that LR fuzzy numbers can be added to RL fuzzy numbers by the formulas presented in Definition 2.5 [7]. In this section, we modify the proof of Theorem 3.1 of [6] and conclude that the method of [6] can be valid for any arbitrary dual fuzzy matrix equations.

Theorem 3.1 *(Theorem 3.1 of [6]) Suppose that in Eq. (1.1) we have $A = (a_{ij})_{m\times n}$, $C = (c_{ij})_{m\times n}$, $\widetilde{B} = (\widetilde{b_{ij}})_{m\times p}$ where $\widetilde{b_{ij}} = \left(b_{ij}, b^l_{ij}, b^r_{ij}\right)_{LR}$, $\widetilde{D} = (\widetilde{d_{ij}})_{m\times p}$ where $\widetilde{d_{ij}} = \left(d_{ij}, d^l_{ij}, d^r_{ij}\right)_{LR}$, and $\widetilde{x} = (\widetilde{x_{ij}})_{n\times p}$ where $\widetilde{x_{ij}} = \left(x_{ij}, x^l_{ij}, x^r_{ij}\right)_{LR}$. Then, the LR dual fuzzy matrix equation (1.1) can be extended into two classical matrix equations as follows:*

$$Gx = F, \tag{3.4}$$

where

$$\begin{aligned} G &= A - C \\ &= \begin{pmatrix} a_{11} - c_{11} & \cdots & a_{1n} - c_{1n} \\ a_{21} - c_{21} & \cdots & a_{2n} - c_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} - c_{m1} & \cdots & a_{mn} - c_{mn} \end{pmatrix}, \end{aligned}$$

$$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix},$$

$$\begin{aligned} F &= D - B \\ &= \begin{pmatrix} d_{11} - b_{11} & \cdots & d_{1n} - b_{1p} \\ d_{21} - b_{21} & \cdots & d_{2n} - b_{2p} \\ \vdots & \vdots & \vdots \\ d_{m1} - b_{m1} & \cdots & d_{mn} - b_{mp} \end{pmatrix}, \end{aligned}$$

and

$$SW = Y, \tag{3.5}$$

where $S = T - Q$ is a $2m \times 2n$ matrix, i.e.

$$S = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1,2n} \\ s_{21} & s_{22} & \cdots & s_{2,2n} \\ \vdots & \vdots & \vdots & \vdots \\ s_{2m,1} & s_{2m,2} & \cdots & s_{2m,2n} \end{pmatrix}$$

$$= \begin{pmatrix} t_{11} - q_{11} & \cdots & t_{1,2n} - q_{1,2n} \\ t_{21} - q_{21} & \cdots & t_{2,2n} - q_{2,2n} \\ \vdots & \vdots & \vdots \\ t_{2m,1} - q_{2m,1} & \cdots & t_{2m,,2n} - q_{2m,,2n} \end{pmatrix},$$

in which t_{ij} and q_{ij}, $1 \leqslant i \leqslant 2m$, $1 \leqslant j \leqslant 2n$ are determined as follows:

if $a_{kh} \geqslant 0 \Rightarrow t_{kh} = t_{m+k,n+h} = a_{kh}$,

if $a_{kh} < 0 \Rightarrow t_{k,n+h} = t_{m+k,h} = -a_{kh}$,

if $c_{kh} \geqslant 0 \Rightarrow q_{kh} = q_{m+k,n+h} = c_{kh}$,

if $c_{kh} < 0 \Rightarrow q_{k,n+h} = q_{m+k,h} = -c_{kh}$,

where $1 \leqslant k \leqslant m$ and $1 \leqslant h \leqslant n$ and also any t_{ij} and q_{ij} which are not determined by the above items are zero, $1 \leqslant i \leqslant 2m$, $1 \leqslant j \leqslant 2n$. Moreover,

$$W = \begin{pmatrix} x^l \\ x^r \end{pmatrix}, \qquad Y = \begin{pmatrix} Y^l \\ Y^r \end{pmatrix},$$

where

$$x^l = \begin{pmatrix} x^l_{11} & x^l_{12} & \cdots & x^l_{1p} \\ x^l_{21} & x^l_{22} & \cdots & x^l_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ x^l_{n1} & x^l_{n2} & \cdots & x^l_{np} \end{pmatrix},$$

$$x^r = \begin{pmatrix} x^r_{11} & x^r_{12} & \cdots & x^r_{1p} \\ x^r_{21} & x^r_{22} & \cdots & x^r_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ x^r_{n1} & x^r_{n2} & \cdots & x^r_{np} \end{pmatrix},$$

$$Y^l = \begin{pmatrix} d^l_{11} - b^l_{11} & \cdots & d^l_{1n} - b^l_{1p} \\ d^l_{21} - b^l_{21} & \cdots & d^l_{2n} - b^l_{2p} \\ \vdots & \vdots & \vdots \\ d^l_{m1} - b^l_{m1} & \cdots & d^l_{mn} - b^l_{mp} \end{pmatrix},$$

$$Y^r = \begin{pmatrix} d^r_{11} - b^r_{11} & \cdots & d^r_{1n} - b^r_{1p} \\ d^r_{21} - b^r_{21} & \cdots & d^r_{2n} - b^r_{2p} \\ \vdots & \vdots & \vdots \\ d^r_{m1} - b^r_{m1} & \cdots & d^r_{mn} - b^r_{mp} \end{pmatrix}.$$

Proof. Unlike the paper of Gong et al. [6], we use the concept of r-cuts of fuzzy numbers to prove this theorem. Firstly, we represent the component form of Eq. (1.1) for any $j = 1, 2, \ldots, p$ and $i = 1, 2, \ldots, m$ as follows

$$\sum_{k=1}^{n} a_{ik}\, \widetilde{x_{kj}} + \widetilde{b_{ij}} = \sum_{k=1}^{n} c_{ik}\, \widetilde{x_{kj}} + \widetilde{d_{ij}}. \tag{3.6}$$

From Definition 2.2, for any $r \in [0, 1]$, we conclude

$$\sum_{k=1}^{n} a_{ik}\, [\widetilde{x_{kj}}]_r + [\widetilde{b_{ij}}]_r = \sum_{k=1}^{n} c_{ik}\, [\widetilde{x_{kj}}]_r + [\widetilde{d_{ij}}]_r, \tag{3.7}$$

or

$$\sum_{k=1}^{n} a_{ik}\, [\underline{x_{kj}}(r), \overline{x_{kj}}(r)] + [\underline{b_{ij}}(r), \overline{b_{ij}}(r)] =$$

$$\sum_{k=1}^{n} a_{ik} \sum_{k=1}^{n} c_{ik}\, [\underline{x_{kj}}(r), \overline{x_{kj}}(r)] + [\underline{d_{ij}}(r), \overline{d_{ij}}(r)]. \tag{3.8}$$

By using of Definition 2.3, we can obtain the following equations from Eq. (3.8)

$$\sum_{k \in U_i^+} a_{ik}\, \underline{x_{kj}}(r) + \sum_{k \in U_i^-} a_{ik}\, \overline{x_{kj}}(r) + \underline{b_{ij}}(r) =$$

$$\sum_{k \in V_i^+} c_{ik}\, \underline{x_{kj}}(r) + \sum_{k \in V_i^-} c_{ik}\, \overline{x_{kj}}(r) + \underline{d_{ij}}(r), \tag{3.9}$$

and

$$\sum_{k \in U_i^+} a_{ik}\, \overline{x_{kj}}(r) + \sum_{k \in U_i^-} a_{ik}\, \underline{x_{kj}}(r) + \overline{b_{ij}}(r) =$$

$$\sum_{k \in V_i^+} c_{ik}\, \overline{x_{kj}}(r) + \sum_{k \in V_i^-} c_{ik}\, \underline{x_{kj}}(r) + \overline{d_{ij}}(r), \tag{3.10}$$

where

$$U_i^+ = \{k : a_{ik} \geqslant 0\},$$

$$U_i^- = \{k : a_{ik} < 0\},$$

$$V_i^+ = \{k : c_{ik} \geqslant 0\},$$

and

$$V_i^+ = \{k : a_{ik} < 0\}.$$

From Eqs. (3.9) and (3.10) we have respectively

$$\left(\sum_{k\in U_i^+} a_{ik}\,\underline{x_{kj}}(r) - \sum_{k\in V_i^+} c_{ik}\,\underline{x_{kj}}(r)\right) + \left(\sum_{k\in U_i^-} a_{ik}\,\overline{x_{kj}}(r) - \sum_{k\in V_i^-} c_{ik}\,\overline{x_{kj}}(r)\right) = \underline{d_{ij}}(r) - \underline{b_{ij}}(r), \tag{3.11}$$

and

$$\left(\sum_{k\in U_i^+} a_{ik}\,\overline{x_{kj}}(r) - \sum_{k\in V_i^+} c_{ik}\,\overline{x_{kj}}(r)\right) + \left(\sum_{k\in U_i^-} a_{ik}\,\underline{x_{kj}}(r) - \sum_{k\in V_i^-} c_{ik}\,\underline{x_{kj}}(r)\right) = \overline{d_{ij}}(r) - \overline{b_{ij}}(r), \tag{3.12}$$

for any $j = 1, 2, \ldots, p$ and $i = 1, 2, \ldots, m$. Now, we define the $m \times n$ matrices A^+, A^-, C^+ and C^- as follows

$$(A^+)_{ij} = \begin{cases} a_{ij}, & a_{ij} \geqslant 0, \\ 0, & a_{ij} < 0, \end{cases}$$

$$(A^-)_{ij} = \begin{cases} 0, & a_{ij} \geqslant 0, \\ a_{ij}, & a_{ij} < 0, \end{cases} \tag{3.13}$$

$$(C^+)_{ij} = \begin{cases} c_{ij}, & c_{ij} \geqslant 0, \\ 0, & c_{ij} < 0, \end{cases}$$

$$(C^-)_{ij} = \begin{cases} 0, & c_{ij} \geqslant 0, \\ c_{ij}, & c_{ij} < 0, \end{cases} \tag{3.14}$$

for any $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Then, we have

$$A = A^+ + A^-, \qquad C = C^+ + C^-, \tag{3.15}$$

and

$$T = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}, \quad Q = \begin{pmatrix} C^+ & -C^- \\ -C^- & C^+ \end{pmatrix}. \tag{3.16}$$

Also, we define the $n \times p$ matrices $\underline{x}(r)$, $\overline{x}(r)$ and the $m \times p$ matrices $\underline{D}(r)$, $\overline{D}(r)$, $\underline{B}(r)$ and $\overline{B}(r)$ as follows

$$\underline{x}(r) = \begin{pmatrix} \underline{x_{11}}(r) & \cdots & \underline{x_{1p}}(r) \\ \vdots & \vdots & \cdots \\ \underline{x_{n1}}(r) & \cdots & \underline{x_{np}}(r) \end{pmatrix},$$

$$\overline{x}(r) = \begin{pmatrix} \overline{x_{11}}(r) & \cdots & \overline{x_{1p}}(r) \\ \vdots & \vdots & \cdots \\ \overline{x_{n1}}(r) & \cdots & \overline{x_{np}}(r) \end{pmatrix} \tag{3.17}$$

$$\underline{D}(r) = \begin{pmatrix} \underline{d_{11}}(r) & \cdots & \underline{d_{1p}}(r) \\ \vdots & \vdots & \cdots \\ \underline{d_{n1}}(r) & \cdots & \underline{d_{np}}(r) \end{pmatrix},$$

$$\overline{D}(r) = \begin{pmatrix} \overline{d_{11}}(r) & \cdots & \overline{d_{1p}}(r) \\ \vdots & \vdots & \cdots \\ \overline{d_{n1}}(r) & \cdots & \overline{d_{np}}(r) \end{pmatrix} \tag{3.18}$$

and

$$\underline{B}(r) = \begin{pmatrix} \underline{b_{11}}(r) & \cdots & \underline{b_{1p}}(r) \\ \vdots & \vdots & \cdots \\ \underline{b_{n1}}(r) & \cdots & \underline{b_{np}}(r) \end{pmatrix},$$

$$\overline{B}(r) = \begin{pmatrix} \overline{b_{11}}(r) & \cdots & \overline{b_{1p}}(r) \\ \vdots & \vdots & \cdots \\ \overline{b_{n1}}(r) & \cdots & \overline{b_{np}}(r) \end{pmatrix}. \tag{3.19}$$

By Eqs. (3.13)-(3.19), we can represent the matrix form of Eqs.(3.11) and (3.12) as follows

$$\begin{cases} (A^+ - C^+)\,\underline{x}(r) + (A^- - C^-)\,\overline{x}(r) \\ \qquad = \underline{D}(r) - \underline{B}(r), \\ (A^- - C^-)\,\underline{x}(r) + (A^+ - C^+)\,\overline{x}(r) \\ \qquad = \overline{D}(r) - \overline{B}(r), \end{cases} \tag{3.20}$$

for any $r \in [0, 1]$. Whenever in Eq. (3.20), we set $r = 1$, then by means of Eq. (3.15) we obtain

$$Ax - Cx = D - B, \tag{3.21}$$

because by Remark 2.3 we have

$$\underline{x}(1) = \overline{x}(1) = x,$$

$$\underline{D}(1) = \overline{D}(1) = D,$$

and

$$\underline{B}(1) = \overline{B}(1) = B.$$

From Eq. (3.21) we conclude

$$(A - C)x = D - B. \tag{3.22}$$

This complete the proof of Eq. (3.4).

On the other hand, if in Eq. (3.20) we set $r = 0$, then we obtain

$$\begin{cases} (A^+ - C^+)(x - x^l) + (A^- - C^-)(x + x^r) \\ \qquad = (D - D^l) - (B - B^l), \\ \\ (A^- - C^-)(x - x^l) + (A^+ - C^+)(x + x^r) \\ \qquad = (D + D^r) - (B + B^r), \end{cases} \tag{3.23}$$

because by Remark 2.3 we have

$$\underline{x}(0) = x - x^l, \quad \overline{x}(0) = x + x^r,$$
$$\underline{D}(0) = D - D^l, \quad \overline{D}(0) = D + D^r,$$

and

$$\underline{B}(0) = B - B^l, \quad \overline{B}(0) = B + B^r.$$

From Eqs. (3.21) and (3.23) we obtain

$$\begin{cases} (A^+ - C^+)\,x^l + (C^- - A^-)\,x^r = D^l - B^l, \\ (C^- - A^-)\,x^l + (A^+ - C^+)\,x^r = D^r - B^r. \end{cases} \tag{3.24}$$

According to the assumptions of theorem and Eq. (3.16), we can rewrite Eq. (3.24) as follows

$$\begin{bmatrix} (A^+ - C^+) & (C^- - A^-) \\ (C^- - A^-) & (A^+ - C^+) \end{bmatrix} \begin{pmatrix} x^l \\ x^r \end{pmatrix} = \begin{pmatrix} Y^l \\ Y^r \end{pmatrix}, \tag{3.25}$$

or

$$(T - Q)W = Y. \tag{3.26}$$

Therefore, the proof is completed. □

Remark 3.1 *Based on the method proposed by Gong et al. [6], we first solve the matrix equation (3.4) by the generalized inverses of matrices [2, 3] and obtain the minimal solution $x = (x_{ij})_{n\times p}$. In the next step, we solve the matrix equation (3.5) via the generalized inverses of matrices and obtain the minimal solution $x^l = (x^l_{ij})_{n\times p}$ and $x^r = (x^r_{ij})_{n\times p}$.*

If $x^l \geqslant 0$ and $x^r \geqslant 0$, we call $\widetilde{x} = (x, x^l, x^r)_{LR}$ is a strong minimal LR fuzzy solution of Eq. (1.1). Otherwise, the $\widetilde{x} = (x, x^l, x^r)_{LR}$ is said to a weak minimal LR fuzzy solution of Eq. (1.1) given by

- if $x^l_{ij} < 0$ and $x^r_{ij} > 0$, then:
$$\widetilde{x_{ij}} = \Big(x_{ij}, 0, \max\{-x^l_{ij}, x^r_{ij}\}\Big)_{LR},$$
- if $x^l_{ij} > 0$ and $x^r_{ij} < 0$, then:
$$\widetilde{x_{ij}} = \Big(x_{ij}, \max\{x^l_{ij}, -x^r_{ij}\}, 0\Big)_{LR},$$
- if $x^l_{ij} < 0$ and $x^r_{ij} < 0$, then:
$$\widetilde{x_{ij}} = \Big(x_{ij}, -x^r_{ij}, -x^l_{ij}\}\Big)_{LR},$$

Remark 3.2 *In general, it can be easily investigated that the strong solutions obtained by Gong et al. [6] may not satisfy the LR dual fuzzy matrix equation (1.1), unless either A and C be non-negative real matrices or $L(x) = R(x)$ for all $x \in [0, 1]$.*

For this reason, if either A and C are not non-negative real matrices or $L(x) \neq R(x)$ for some or all values of $x \in [0, 1]$, then we can consider the obtained solution by Gong et al. as an approximate solution for the LR dual fuzzy matrix equation (1.1).

In the next section, we will show that some numerical results obtained in [6] are incorrect. For these cases, we will obtain the correct strong (weak) minimal LR solutions by the method of Gong et al. [6].

4 Correction to some examples

Example 4.1 *(Example 4.1 of [6] , Section 4, pp. 124) Consider the following fuzzy linear system (LR dual fuzzy matrix equation)*

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \widetilde{x_1} \\ \widetilde{x_2} \end{pmatrix} + \begin{pmatrix} (2,1,1)_{LR} \\ (4,1,2)_{LR} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{x_1} \\ \widetilde{x_2} \end{pmatrix} + \begin{pmatrix} (5,2,1)_{LR} \\ (3,2,3)_{LR} \end{pmatrix}. \tag{4.27}$$

According to Theorem 3.1, we have

$$G = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad F = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -2 & 2 & 0 & 3 \\ -1 & 1 & 1 & 0 \\ 0 & 3 & -2 & 2 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Since that G is nonsingular and S is singular, we obtain

$$\begin{aligned} x &= G^{-1}F \\ &= \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2.5 \\ 2 \end{pmatrix}, \end{aligned}$$

and

$$X = S^{\dagger}Y = \begin{pmatrix} 0.8750 \\ 0.2500 \\ 0.6250 \\ 0.7500 \end{pmatrix},$$

where

$$S^{\dagger} = \begin{pmatrix} 1.375 & -0.375 & 1.125 & -0.125 \\ 0.250 & -0.250 & 0.750 & 0.250 \\ 1.125 & -0.125 & 1.375 & -0.375 \\ 0.750 & 0.250 & 0.250 & -0.250 \end{pmatrix}.$$

Since $X = \begin{pmatrix} X^l \\ X^r \end{pmatrix} \geqslant 0$, then the obtained solution is a strong minimal LR fuzzy solution given by

$$\widetilde{X} = \begin{pmatrix} (2.5, 0.8750, 0.625)_{LR} \\ (2, 0.2500, 0.7500)_{LR} \end{pmatrix}.$$

Of course, it is clear that since the matrices A and C are not non-negative, then the above strong solution does not satisfy Eq. (4.27), unless $L(x) = R(x)$ for any $x \in [0, 1]$. Therefore, if $L(x) \neq R(x)$ for some or all value of $x \in [0, 1]$, we can consider the above obtained solution as an approximate solution.

But, Gong et al. in [6], asserted that Eq. (4.27) has a weak minimal LR fuzzy solution as

$$\widetilde{X} = \begin{pmatrix} (-0.5, 0.5000, 0.0000)_{LR} \\ (-2, 0.9000, 0.2500)_{LR} \end{pmatrix}.$$

Example 4.2 *(Example 4.2 of [6] , Section 4, pp. 125) Consider the following fuzzy linear system (LR dual fuzzy matrix equation)*

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{x_1} \\ \widetilde{x_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{x_1} \\ \widetilde{x_2} \end{pmatrix} + \begin{pmatrix} (2,2,1)_{LR} \\ (3,1,1)_{LR} \\ (1,1,1)_{LR} \end{pmatrix}. \quad (4.28)$$

By Theorem 3.1, we have

$$G = \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 1 \end{pmatrix}, \qquad F = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \; Y = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

By the concept of the generalized inverses of matrices, we obtain

$$\begin{aligned} x &= G^{\dagger}F \\ &= \begin{pmatrix} 0.5000 & 0 & -0.5000 \\ 0.1667 & -0.3333 & 0.1667 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}, \end{aligned}$$

and

$$X = S^{\dagger}Y = \begin{pmatrix} 0.5000 \\ 1.3333 \\ 0 \\ 1.1667 \end{pmatrix}.$$

Therefore, in this case we obtain the following strong minimal LR fuzzy solution

$$\widetilde{X} = \begin{pmatrix} (0.5, 0.5, 0)_{LR} \\ (-0.5, 1.3333, 1.1667)_{LR} \end{pmatrix}.$$

But, Gong et al. in [6] obtained the following solution as a strong minimal LR fuzzy solution.

$$\widetilde{X} = \begin{pmatrix} (0.5, 0.5, 0)_{LR} \\ (0.1667, 1.3333, 1.0000)_{LR} \end{pmatrix}.$$

Example 4.3 *(Example 4.3 of [6] , Section 4, pp. 127) Consider the following fuzzy linear system (LR dual fuzzy matrix equation)*

$$\begin{pmatrix} \widetilde{x_1} \\ \widetilde{x_2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{x_1} \\ \widetilde{x_2} \end{pmatrix} + \begin{pmatrix} (2,2,1)_{LR} \\ (3,1,1)_{LR} \end{pmatrix}. \quad (4.29)$$

For the above equation, we have

$$G = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \qquad F = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \; Y = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

By the concept of the generalized inverses of matrices, we obtain

$$\begin{aligned} x &= G^{\dagger}F \\ &= \begin{pmatrix} -1.0000 & -1.0000 \\ -1.0000 & -0.0000 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -5.0000 \\ -2.0000 \end{pmatrix}, \end{aligned}$$

and

$$X = S^{\dagger}Y = \begin{pmatrix} -3.0000 \\ -2.0000 \\ -2.0000 \\ -1.0000 \end{pmatrix}.$$

Since the left and right spreads are negative, then by Remark 3.1, we obtain a weak minimal LR fuzzy solution as following

$$\widetilde{X} = \begin{pmatrix} (-5.0000, 2.0000, 3.0000)_{LR} \\ (-2.0000, 1.0000, 2.0000)_{LR} \end{pmatrix}.$$

Unfortunately, Gong et al. [6] obtained the following solution as a weak minimal LR fuzzy solution for Eq. (4.29).

$$\widetilde{X} = \begin{pmatrix} (-5.0000, 1.0000, 1.0000)_{LR} \\ (-2.0000, 0.0000, 1.0000)_{LR} \end{pmatrix}.$$

Example 4.4 *(Example 5.2 of [6] , Section 5, pp. 129) Consider the following LR dual fuzzy matrix equation*

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{x_{11}} & \widetilde{x_{12}} \\ \widetilde{x_{21}} & \widetilde{x_{22}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{x_{11}} & \widetilde{x_{12}} \\ \widetilde{x_{21}} & \widetilde{x_{22}} \end{pmatrix} + \begin{pmatrix} (2,2,1)_{LR} & (3,3,1)_{LR} \\ (3,1,1)_{LR} & (2,1,2)_{LR} \\ (1,1,1)_{LR} & (3,2,1)_{LR} \end{pmatrix}. \quad (4.30)$$

For the above equation, we have

$$G = \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 2 & 3 \\ 3 & 2 \\ 1 & 3 \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

$$Y = \begin{pmatrix} 2 & 3 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

By the concept of the generalized inverses of matrices, we obtain

$$x = G^{\dagger}F = \begin{pmatrix} 0.5000 & 0 \\ -0.5000 & 0.3333 \end{pmatrix},$$

where

$$G^{\dagger} = \begin{pmatrix} 0.5000 & 0 & -0.5000 \\ 0.1667 & -0.3333 & 0.1667 \end{pmatrix},$$

and

$$X = S^{\dagger}Y = \begin{pmatrix} 0.5000 & 0.5000 \\ 1.3333 & 2.1667 \\ 0 & 0 \\ 1.1667 & 1.3333 \end{pmatrix}.$$

Since the left and right spreads are non-negative, then we obtain a strong minimal LR fuzzy solution as following

$$\widetilde{X} = \begin{pmatrix} (0.50, 0.50, 0.00)_{LR} & (0.00, 0.50, 0.00)_{LR} \\ (-0.50, 1.33, 1.17)_{LR} & (0.33, 2.17, 1.33)_{LR} \end{pmatrix}.$$

In this example, since the matrix A is not non-negative, then the above strong solution can be considered as an approximate solution, unless $L(x) = R(x)$ for any $x \in [0, 1]$.

Unfortunately, in [6] Gong et al. obtained the following solution as a strong minimal LR fuzzy solution for Eq. (4.30).

$$\widetilde{X} = \begin{pmatrix} (0.50, 0.50, 0.00)_{LR} & (0.00, 0.50, 0.00)_{LR} \\ (0.17, 1.33, 1.00)_{LR} & (0.67, 2.17, 1.33)_{LR} \end{pmatrix}.$$

5 Conclusion

In this paper, by modifying the proof of Theorem 3.1 of [6], we showed that the method proposed by Gong et al. [6] can be approximately valid for all dual fuzzy matrix equations. This means that if either A and B be non-negative or $L(x) = R(x)$ for all $x \in [0, 1]$, then the solution obtained by Gong et al.'s method is an accurate solution. Otherwise it can be considered as an approximate solution. Also, we brought corrections to some examples presented in [6].

References

[1] T. Allahviranloo, R. Nuraei, M. Ghanbari, E. Haghi, A.A. Hosseinzadeh, A new metric for *L-R* fuzzy numbers and its application in fuzzy linear systems,*Soft Comput.* 16 (2012) 1743-1754.

[2] A. Ben-Israel, T. N. E. Greville, Generalized Inverses: Theory, second ed., *Springer-Verlag, New York,* 2003.

[3] A. Berman, R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, *Academic Press, New York,* 1979.

[4] W. Cong-Xin, M. Ming, Embedding problem of fuzzy number space: Part I, *Fuzzy Sets and Systems* 44 (1991) 33-38.

[5] D. Dubois, H. Prade, Operations on fuzzy numbers, *J. Syst. Sci.* 9 (1978) 613-626.

[6] Z. Gong, X. Guo, K. Liu, Approximate solution of dual fuzzy matrix equations, *Inf. Sci.* 266 (2014) 112-133.

[7] J. J. Kaur, A. Kumar, A note on 'approximate solution of dual fuzzy matrix equations', *Inf. Sci.* 19 (2017) 184-185.

[8] R. Nuraei, T. Allahviranloo, M. Ghanbari, Finding an inner estimation of the solution set of a fuzzy linear system, *Applied Mathematical Modelling* 37 (2013) 5148-5161.

[9] S. Seikkala, On the fuzzy initial value problem, *Fuzzy Sets and Systems* 24 (1987) 319-330.

[10] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, *Inform. Sci.* 8 (1975) 199-249.

Mohammad Asghari Larimi is a Ph.D. student in mathematics from Payame Noor University, Tehran, Iran. He has born in the Mazandaran, Ghaemshahr city in 1975. Got B.Sc. degree in applied mathematics from Payam Noor University in Behshahr and M. Sc. degree in mathematics from Khayyam University of Mashhad.

Parastoo Reihani Ardabili is an assistant professor in the department of mathematics at Payame Noor University, Tehran, Iran. Dr. Reihani received her Ph.D. in applied mathematics from the Iran University of Science and Technology. Her methodological research focuses on applied mathematics.

Mohsen Asghari-Larimi is an assistant professor in Applied Mathematics; Mathematics group in Golestan University, in Iran. Got B.Sc. in mathematics from Esfahan University and M.Sc. degrees in mathematics, fuzzy hyperstructures field from Mazandaran University and Ph.D. degree in applied mathematics, number Theory field from Limoges University in France.

Shahriar Farahmand-Rad is an assistant professor in Applied Mathematics in Payame Noor University, Tehran, Iran. Got B.Sc. in mathematics from Tehran University and M.Sc. degrees in mathematics, Ergodic Theory field from Tehran University and Ph.D. degree in applied mathematics, Combinatorial Optimization field from Amirkabir University in Iran.